Exercises for Chapter 6

A. Use the method of proof by contradiction to prove the following statements. (In each case you should also think about how a direct or contrapositive proof would work. You will find in most cases that proof by contradiction is easier.)

1. Suppose $n \in \mathbb{Z}$. If n is odd, then n^2 is odd.

2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

3. Prove that $\sqrt[3]{2}$ is irrational.

4. Prove that $\sqrt{6}$ is irrational.

5. Prove that $\sqrt{3}$ is irrational.

6. If $a,b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

7. If $a,b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.

8. Suppose $a,b,c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

9. Suppose $a,b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

10. There exist no integers a and b for which $21a + 30b = 1$.

11. There exist no integers a and b for which $18a + 6b = 1$.

12. For every positive rational number x, there is a positive rational number y for which $y < x$.

13. For every $x \in [\pi/2, \pi]$, $\sin x - \cos x \geq 1$.

14. If A and B are sets, then $A \cap (B - A) = \emptyset$.

15. If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.

16. If a and b are positive real numbers, then $a + b \leq 2\sqrt{ab}$.

17. For every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$.

18. Suppose $a,b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd.

B. Prove the following statements using any method from chapters 4, 5 or 6.

19. The product of any five consecutive integers is divisible by 120. (For example, the product of 3,4,5,6 and 7 is 2520, and $2520 = 120 \cdot 21$.)

20. We say that a point $P = (x, y)$ in the Cartesian plane is **rational** if both x and y are rational. More precisely, P is rational if $P = (x, y) \in \mathbb{Q}^2$. An equation $F(x, y) = 0$ is said to have a **rational point** if there exists $x_0, y_0 \in \mathbb{Q}$ such that $F(x_0, y_0) = 0$. For example, the curve $x^2 + y^2 - 1 = 0$ has rational point $(x_0, y_0) = (1, 0)$. Show that the curve $x^2 + y^2 - 3 = 0$ has no rational points.

21. Exercise 20 involved showing that there are no rational points on the curve $x^2 + y^2 - 3 = 0$. Use this fact to show that $\sqrt{3}$ is irrational.

22. Explain why $x^2 + y^2 - 3 = 0$ not having any rational solutions (Exercise 20) implies $x^2 + y^2 - 3^k = 0$ has no rational solutions for k an odd, positive integer.

23. Use the above result to prove that $\sqrt{3^k}$ is irrational for all odd, positive k.