Several ways to a Berwald manifold – and some steps beyond*

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Abstract

After summarizing some necessary preliminaries and tools, including Berwald derivative and Lie derivative in pull-back formalism, we present several equivalent conditions, each of which characterizes Berwald manifolds among Finsler manifolds. These range from Berwald's classical definition to the existence of a torsion-free covariant derivative on the base manifold compatible with the Finsler function, the vanishing of the h-Berwald differential of the Cartan tensor and Aikou's characterization of Berwald manifolds. Finally, we study some implications of V. Matveev's observation according to which quadratic convexity may be omitted from the definition of a Berwald manifold. These include, among others, a generalization of Z. I. Szabó's well-known metrization theorem, and also leads to a natural generalization of Berwald manifolds, to *Berwald – Matveev manifolds*.

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1 Introduction

Positive definite Berwald manifolds constitute the conceptually simplest and the best understood class of Finsler manifolds. Their conceptual simplicity is due to the fact that Berwald manifolds are affinely connected manifolds at the same time, whose parallelism structure is related to the normed vector space structure of the tangent spaces in the most natural manner: parallel translations are norm preserving. Berwald himself called a Finsler manifold an 'affinely connected space' if

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(B) *the connection parameters arising from the geodesic equation are independent of the direction arguments.*

It turns out that the affine connection of a Berwald manifold is the Levi-Civita connection of a Riemannian metric on the base manifold. This key observation of Z. I. Szabó is the starting point of his structure theorem on Berwald manifolds [19].

Our decision to write a comprehensive survey concerning Berwald's condition (B) was strongly motivated by some e-mails between Vladimir Matveev and the first author. A quotation from a letter of Matveev:

'I always thought that a Finsler manifold is Berwald *if and only if* there exists a torsion-free affine connection whose transport preserves the Finsler function F. Is the statement correct? If yes, do you have a reference where it is written?

Of course I understand that a Berwald metric (in a standard definition) does have the above property: indeed, in this case the Chern connection is actually an affine connection, and it preserves the Finsler function F. Thus, my question is essentially whether the existence of an affine connection preserving the Finsler function implies that the metric is Berwald. . . ' He also mentioned that the question had appeared in a discussion with Marc Troyanov.

Our answer was affirmative. We did not find, however, any reference where the statement was formulated explicitly and proved in a simple and self-contained way.

To our request Matveev also sketched a possible proof, found by him and Troyanov. Although their argumentation was not elaborated in every detail, we found it nice and original. We thought, however, that it used rather heavy tools from Riemannian geometry to a quite simple problem, and depended too strongly on the assumption of positive definiteness.

Since the question is natural and important, we believed it useful to present a proof which is as self-contained as possible, and which uses only the simplest tools of Finsler geometry (and connection theory). Moreover, besides the property formulated by Matveev, we present a number of other properties that characterize Berwald manifolds among Finsler manifolds. Some of them are folklore, or can be ferreted out from the literature (see, e.g., Refs. [2, 22]); we collected them here, however, so that they would be more easily accessible. On the other hand, in several problems, it is advantageous to have an appropriate version of Berwaldian property (B).

We wrote this paper not only, or not primarily, for Finsler geometers, and we hope that anyone with a basic knowledge of differential geometry can fairly easily read it and will find it indeed useful. So in sections 2–4 we collect the most necessary preparatory material concerning sprays, Ehresmann connections, (nonlinear) parallel translations, and some basic facts on Finsler functions. In section 5 we formulate and prove several equivalents of (B); the first of them is just a more precise reformulation of the relevant property. In section 6 we present a detailed, index-free proof of a further nice and important characterization of Berwald manifolds among Finsler manifolds, discovered by T. Aikou [1]. To do this, we need an appropriate concept of 'Lie derivative along the tangent bundle projection', which we briefly explain here; for more details we refer to [10] and 2.39 in [20].

In section 7 we leave the realm of classical Berwald manifolds. In references [15, 16, 17] Matveev and his collaborators drew attention to the fact that the metrization of the affine connection of a Berwald manifold discovered by Z. I. Szabó may be carried out in a more general setting. Namely, the quadratic convexity of a Finsler function assured by our conditions (F1)–(F4) in section 4, may be weakened substantially. This observation leads to the less restrictive notion of *Berwald – Matveev manifold*. We devote the greater part of the concluding section to the averaged metric construction explained first in [16] (and later also in [15]), and to a completion of the proof of Theorem 1 in Matveev's paper [15]. In our arguments we utilize a trick which we learnt from Matveev during an after-lunch conversation in Debrecen, April 2011. Finally, we exhibit a further method to associate a Riemannian metric to a Berwald – Matveev manifold in a natural way, applying Loewner ellipsoids.

2 Notations and definitions

2.1. Throughout the paper, M will be an n-dimensional ($n \geq 1$) smooth manifold whose underlying topological space is Hausdorff, second countable and connected. $C^\infty(M)$ is the real algebra of smooth functions on M.

2.2. The tangent bundle of M will be denoted by $\tau : TM \to M$. Analogously, $\tau_{TM} : TTM \to TM$ will stand for the tangent bundle of the tangent manifold TM. The shorthand for these vector bundles will be τ and τ_{TM}, respectively. The vector fields on M form a $C^\infty(M)$-module, which will be denoted by $\mathfrak{X}(M)$. $o \in \mathfrak{X}(M)$ is the zero vector field on M. The *deleted bundle* for τ is the fibre bundle $\mathring{\tau} : \mathring{T}M \to M$, where $\mathring{T}M := TM \setminus o(M)$, $\mathring{\tau} := \tau \restriction \mathring{T}M$. The Lie bracket $[X, Y]$ of $X, Y \in \mathfrak{X}(M)$ is the unique vector field on M satisfying

$$[X, Y](f) = X(Yf) - Y(Xf), \quad f \in C^\infty(M).$$

2.3. If $\varphi : M \to N$ is a smooth mapping between smooth manifolds, its derivative is the bundle map $\varphi_* : TM \to TN$ whose restriction $(\varphi_*)_p$ to a tangent space T_pM $(p \in M)$ is given by

$$(\varphi_*)_p(v)(h) := v(h \circ \varphi); \quad v \in T_pM,\ h \in C^\infty(N).$$

Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are φ-related if $\varphi_* \circ X = Y \circ \varphi$; in this case we write $X \underset{\varphi}{\sim} Y$. A vector field ξ on TM is called *projectable* if there exists a vector field X on M such that $\xi \underset{\tau}{\sim} X$. In particular, ξ is said to be *vertical* if $\xi \underset{\tau}{\sim} o$. The vertical vector fields form a (finitely generated) module, denoted by $\mathfrak{X}^{\mathsf{v}}(TM)$, over the ring $C^\infty(TM)$, which is also a subalgebra of the real Lie algebra $\mathfrak{X}(TM)$.

2.4. The *vertical lift* of a function $f \in C^\infty(M)$ in $C^\infty(TM)$ is $f^{\mathsf{v}} := f \circ \tau$, the *complete lift* of f is $f^{\mathsf{c}} \in C^\infty(TM)$ defined by

$$f^{\mathsf{c}}(v) := v(f), \text{ if } v \in TM.$$

There exists a canonical vertical vector field C on TM such that

$$Cf^{\mathsf{c}} := f^{\mathsf{c}}, \text{ for all } f \in C^\infty(M).$$

C is said to be the *Liouville vector field* (or *radial vector field*) on TM.

2.5. Let X be a vector field on M. The *vertical lift* $X^{\mathsf{v}} \in \mathfrak{X}^{\mathsf{v}}(TM)$ of X is the unique vertical vector field on TM such that

$$X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}} \text{ for all } f \in C^\infty(M).$$

The *complete lift* $X^{\mathsf{c}} \in \mathfrak{X}(TM)$ of X is the unique vector field on TM such that

$$X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}, \quad X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}} \text{ for all } f \in C^\infty(M).$$

If $(X_i)_{i=1}^n$ is a local frame for TM, then $\left(X_i^{\mathsf{v}}, X_i^{\mathsf{c}}\right)_{i=1}^n$ is a local frame for TTM, therefore

> *in order to define a tensor field on TM, it is sufficient to specify its action on vertical and complete lifts of vector fields on M.*

Thus there exists a unique type $(1,1)$ tensor field $\mathbf{J}$ on TM such that

$$\mathbf{J}X^{\mathsf{v}} = 0, \quad \mathbf{J}X^{\mathsf{c}} = X^{\mathsf{v}} \text{ for all } X \in \mathfrak{X}(M).$$

$\mathbf{J}$ is said to be the *vertical endomorphism* of $\mathfrak{X}(TM)$ (or of TTM).

2.6. d denotes the operator of exterior derivative, defined on a function $f \in C^\infty(M)$ and on a 1-form $\omega \in \mathfrak{X}^*(M)$ by

$$df(X) := Xf \quad \textit{and} \quad d\omega(X,Y) := X\omega(Y) - Y\omega(X) - \omega\big([X,Y]\big)$$

$\big(X, Y \in \mathfrak{X}(M)\big)$.

The substitution operator i_X, associated to a vector field $X \in \mathfrak{X}(M)$, acts on a type $(0,k)$ or $(1,k)$ $(k \in \mathbb{N} \setminus \{0\})$ tensor field A on M by

$$i_X A(X_1, \ldots, X_{k-1}) := A(X, X_1, \ldots, X_{k-1}).$$

2.7. To any type $(1,1)$ tensor field $A \in \mathcal{T}^1_1(M) \cong \operatorname{End}\big(\mathfrak{X}(M)\big)$ we associate a vertical vector field $\overline{A} \in \mathfrak{X}^{\mathsf{v}}(TM)$, by prescribing its action on the complete lifts of smooth functions on M by

$$\big(\overline{A} f^{\mathsf{c}}\big)(v) := A_{\tau(v)}(v)f; \quad f \in C^\infty(M),\ v \in TM.$$

Then we have $\big[C, \overline{A}\,\big] = 0$.

2.8. By a *semispray* for M we mean a mapping

$$S : TM \longrightarrow TTM$$

satisfying the following conditions:

(S_1) $\tau_{TM} \circ S = 1_{TM}$.

(S_2) S is of class C^1 on TM, smooth on $\overset{\circ}{T}M$.

(S_3) $\mathbf{J}S = C$.

A semispray is called a *spray* if it also satisfies

(S_4) $[C, S] = S$.

If a spray is of class C^2 (and hence smooth) on TM, we speak of an *affine spray.*

2.9. If $(\mathcal{U}, u) = \big(\mathcal{U}, (u^i)_{i=1}^n\big)$ is a chart on M, then

$$\big(\tau^{-1}(\mathcal{U}), (x^i, y^i)_{i=1}^n\big), \quad x^i := (u^i)^{\mathsf{v}},\ y^i := (u^i)^{\mathsf{c}}$$

is a chart on TM, called the *chart induced by* $(\mathcal{U}, u)$. In our (not too frequent) coordinate calculations *Einstein's summation convention* will be applied: any index occurring twice, once up, once down, is summed over.

To conclude this section, we present the coordinate expressions of some objects introduced above.

(i) If $\xi \in \mathfrak{X}^{\mathsf{v}}(TM)$, then

$$\xi \restriction \tau^{-1}(\mathcal{U}) = \xi^{n+i} \frac{\partial}{\partial y^i}, \quad \xi^{n+i} \in C^\infty\big(\tau^{-1}(\mathcal{U})\big).$$

(ii) In the induced coordinates, the Liouville vector field takes the form

$$C \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial y^i}.$$

(iii) If $f \in C^\infty(\mathcal{U})$, its complete lift is

$$f^{\mathsf{c}} = y^i \left(\frac{\partial f}{\partial u^i} \circ \tau \right) = (u^i)^{\mathsf{c}} \left(\frac{\partial f}{\partial u^i} \right)^{\mathsf{v}}.$$

(iv) If $X \in \mathfrak{X}(M)$, $X \restriction \mathcal{U} = X^i \frac{\partial}{\partial u^i}$, then

$$X^{\mathsf{v}} \restriction \tau^{-1}(\mathcal{U}) = \big(X^i \circ \tau\big) \frac{\partial}{\partial y^i},$$
$$X^{\mathsf{c}} \restriction \tau^{-1}(\mathcal{U}) = \big(X^i \circ \tau\big) \frac{\partial}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^j} \circ \tau \right) \frac{\partial}{\partial y^i}.$$

(v) If $A \in \mathcal{T}^1_1(M)$, $A\left(\frac{\partial}{\partial u^j}\right) = A^i_j \frac{\partial}{\partial u^i}$ $(j \in \{1, \dots, n\})$, then

$$\overline{A} \restriction \tau^{-1}(\mathcal{U}) = y^j \left(A^i_j \circ \tau \right) \frac{\partial}{\partial y^i}.$$

(vi) If $S : TM \to TTM$ is a semispray, then

$$S \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where the functions $G^i : \tau^{-1}(\mathcal{U}) \to \mathbb{R}$ are of class C^1, and smooth on $\mathring{\tau}^{-1}(\mathcal{U})$.

3 Ehresmann connections and parallel translations

3.1. Consider the vector bundle

$$\pi : TM \times_M TM \to TM,$$

where $TM \times_M TM := \{(u,v) \in TM \times TM \mid \tau(u) = \tau(v)\}$, and π is the restriction of the canonical projection $\mathrm{pr}_1 : TM \times TM \to TM$, $(u,v) \mapsto u$ onto $TM \times_M TM$.

In terms of the theory of bundles, π is just the pull-back of the tangent bundle $\tau : TM \to M$ by τ. The $C^\infty(TM)$-module of sections of π will be denoted by $\mathrm{Sec}(\pi)$. For any vector field X on M, the mapping

$$\widehat{X} : v \in TM \longmapsto \widehat{X}(v) := \big(v, X(\tau(v))\big) \in TM \times_M TM$$

is a section of π, called a *basic section*. Basic sections generate the module $\mathrm{Sec}(\pi)$ in the sense that locally any section in $\mathrm{Sec}(\pi)$ can be obtained as a $C^\infty(TM)$-linear combination of basic sections. In particular, if $\big(\mathcal{U}, (u^i)_{i=1}^n\big)$ is a chart of M, then $\left(\widehat{\frac{\partial}{\partial u^i}}\right)_{i=1}^n$ is a frame of $TM \times_M TM$ over $\tau^{-1}(\mathcal{U})$.

We have a canonical $C^\infty(TM)$-linear isomorphism

$$\mathrm{vl} : \mathrm{Sec}(\pi) \to \mathfrak{X}^{\mathsf{v}}(TM),$$

called the *vertical lift*, given on the basic sections by

$$\mathrm{vl}\big(\widehat{X}\big) := X^{\mathsf{v}},\ X \in \mathfrak{X}(M).$$

We shall also need the pull-back of the tangent bundle $\tau : TM \to M$ by the mapping $\mathring{\tau} : \mathring{T}M \to M$; this is the vector bundle

$$\mathring{\pi} : \mathring{T}M \times_M TM \to \mathring{T}M.$$

We write $\mathrm{Sec}(\mathring{\pi})$ for the $C^\infty(\mathring{T}M)$-module of its sections. As before, any vector field on M determines a basic section in $\mathrm{Sec}(\mathring{\pi})$. For the $C^\infty(\mathring{T}M)$-module $\mathsf{T}^k_l(\mathrm{Sec}(\mathring{\pi}))$ of type $(k,l) \in \mathbb{N} \times \mathbb{N}$ tensors over the module $\mathrm{Sec}(\mathring{\pi})$ we apply the abbreviation $\mathcal{T}^k_l(\mathring{\pi})$, with the agreement $\mathcal{T}^0_0(\mathring{\pi}) := C^\infty(\mathring{T}M)$. Elements of $\mathcal{T}^k_l(\mathring{\pi})$ are also mentioned as *tensors* (of type (k,l)) *along* $\mathring{\tau}$; they have a reasonable and natural pointwise interpretation.

We note finally that it will be useful to extend the derivative $\tau_* : TTM \to TM$ of τ into a mapping

$$\widetilde{\tau}_* : TTM \longrightarrow TM \times_M TM$$

given by

$$\widetilde{\tau}_*(w) := \big(v, (\tau_*)_v(w)\big),\ \textit{if}\ w \in T_vTM.$$

3.2. By an *Ehresmann connection in TM* we mean a mapping

$$\mathcal{H} : TM \times_M TM \to TTM$$

satisfying the following conditions:

(C1) $\mathcal{H}$ is fibre-preserving and fibrewise linear, i.e., for each $v \in TM$, $w, w_1, w_2 \in T_{\tau(v)}M$, $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\mathcal{H}(v, w) \in T_vTM,$$

and

$$\mathcal{H}(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 \mathcal{H}(v, w_1) + \lambda_2 \mathcal{H}(v, w_2).$$

(C2) $\widetilde{\tau}_* \circ \mathcal{H} = 1_{TM \times_M TM}$, or equivalently, $(\tau_*)_v\big(\mathcal{H}(v, w)\big) = w$, *for all* $v \in TM$, $w \in T_{\tau(v)}M$.

(C3) $\mathcal{H}$ is smooth over $\overset{\circ}{T}M \times_M TM$.

(C4) For each $p \in M$ and $v \in T_pM$, $\mathcal{H}(o(p), v) = (o_*)_p(v)$.

If

$$\mathcal{H}_v := \mathcal{H} \restriction \{v\} \times T_{\tau(v)}M \quad (v \in TM),$$

then by (C1), (C2) and (C4) $\mathcal{H}_v$ is an injective linear mapping of $T_{\tau(v)}M \cong \{v\} \times T_{\tau(v)}M$ into T_vTM, therefore

$$H_vTM := \operatorname{Im}(\mathcal{H}_v)$$

is an n-dimensional subspace of T_vTM, called the *horizontal subspace* of T_vTM with respect to $\mathcal{H}$. If

$$V_vTM := \operatorname{Ker}(\tau_*)_v$$

is the (canonical) vertical subspace of T_vTM, then we have the direct decomposition

$$T_vTM = V_vTM \oplus H_vTM.$$

The mapping

$$\mathbf{h} := \mathcal{H} \circ \widetilde{\tau}_* : TTM \longrightarrow TTM$$

is a projection operator: fibrewise linear and $\mathbf{h}^2 = \mathbf{h}$. $\mathbf{h}$ is called the *horizontal projection*, $\mathbf{v} := 1_{TTM} - \mathbf{h}$ is called the *vertical projection* associated to $\mathcal{H}$.

The *horizontal lift* of a vector field X on M with respect to $\mathcal{H}$ (or the $\mathcal{H}$-*horizontal lift*, briefly the horizontal lift of X) is the vector field $X^{\mathrm{h}} \in \mathfrak{X}\big(\mathring{T}M\big)$ defined by

$$X^{\mathrm{h}}(v) := \mathcal{H}(v, X(\tau(v))), \; v \in \mathring{T}M.$$

If $(X_i)_{i=1}^n$ is a local frame of TM, then $(X_i^{\mathsf{v}}, X_i^{\mathrm{h}})_{i=1}^n$ is a local frame of $T\mathring{T}M$ (cf. 2.5). We define a $C^\infty(\mathring{T}M)$-linear mapping

$$\mathcal{V} : \mathfrak{X}(\mathring{T}M) \longrightarrow \mathrm{Sec}(\mathring{\pi}),$$

called the *vertical mapping* associated to $\mathcal{H}$, specifying its action on the vertical and horizontal lifts of vector fields on M by

$$\mathcal{V}(X^{\mathsf{v}}) = \widehat{X}, \quad \mathcal{V}(X^{\mathrm{h}}) = 0; \quad X \in \mathfrak{X}(M).$$

The vertical projection and the vertical mapping are related by $\mathbf{v} = \mathrm{vl} \circ \mathcal{V}$.

An Ehresmann connection $\mathcal{H}$ is said to be *homogeneous* if $\big[X^{\mathrm{h}}, C\big] = 0$ for all $X \in \mathfrak{X}(M)$; *torsion-free* if $\big[X^{\mathrm{h}}, Y^{\mathsf{v}}\big] - \big[Y^{\mathrm{h}}, X^{\mathsf{v}}\big] - [X, Y]^{\mathsf{v}} = 0$ for all $X, Y \in \mathfrak{X}(M)$.

By a *linear connection* on TM (or, by an abuse of language, on M) we mean a homogeneous Ehresmann connection which is of class C^1 (and hence smooth) over $TM \times_M TM$. The motivation of this terminology will be clear from the coordinate description below.

Remark. We draw a definite distinction between an Ehresmann connection and a (possibly nonlinear) covariant derivative operator in Koszul's sense, although they are two sides of the same coin.

Coordinate description. Let $\mathcal{H}$ be an Ehresmann connection in TM. Specify a chart $\big(\mathcal{U}, (u^i)_{i=1}^n\big)$ of M, and consider the induced chart $\big(\tau^{-1}(\mathcal{U}), (x^i, y^i)_{i=1}^n\big)$ of TM. $\mathcal{H}$ determines unique functions

$$N_j^i : \tau^{-1}(\mathcal{U}) \to \mathbb{R}, \; i, j \in \{1, \dots, n\},$$

smooth on $\tau^{-1}(\mathcal{U}) \cap \mathring{T}M$, such that for each $j \in \{1, \dots, n\}$,

$$\left(\frac{\partial}{\partial u^j}\right)^{\mathrm{h}}(v) = \left(\frac{\partial}{\partial x^j}\right)_v - N_j^i(v)\left(\frac{\partial}{\partial y^i}\right)_v.$$

They are called the *Christoffel symbols* or the *connection parameters* for $\mathcal{H}$ with respect to the given charts. If $\mathcal{H}$ is homogeneous, the N_j^i's are positive-homogeneous of degree 1; if $\mathcal{H}$ is linear, they can be written in the form

$$N_j^i = \big(\Gamma_{jk}^i \circ \tau\big)\, y^k, \; \Gamma_{jk}^i \in C^\infty(\mathcal{U}),$$

i.e., they are linear functions on each tangent space.

For the vertical mapping associated to $\mathcal{H}$ we obtain

$$\mathcal{V}\left(\frac{\partial}{\partial x^j}\right) = \mathcal{V}\left(\left(\frac{\partial}{\partial u^j}\right)^{\mathrm{h}} + N^i_j\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{v}}\right) = N^i_j\widehat{\frac{\partial}{\partial u^i}},$$
$$\mathcal{V}\left(\frac{\partial}{\partial y^j}\right) = \widehat{\frac{\partial}{\partial u^j}} \quad \big(j \in \{1,\dots,n\}\big).$$

3.3. We recall two basic examples of constructing an Ehresmann connection.

(a) *Crampin's construction* [7]. Any semispray S for M induces a torsion-free Ehresmann connection such that

$$X^{\mathrm{h}} = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]) \textit{ for all } X \in \mathfrak{X}(M).$$

(b) *Ehresmann connection from a covariant derivative.* Let D be a covariant derivative operator on M. There exists a unique Ehresmann connection $\mathcal{H}_D$ in TM such that the horizontal lift X^{h_D} of a vector field on M with respect to $\mathcal{H}_D$ is given by

$$X^{\mathrm{h}_D} = X^{\mathsf{c}} - \overline{DX},$$

where $\overline{DX} \in \mathfrak{X}^{\mathsf{v}}(TM)$ is the vertical vector field constructed from the covariant differential DX, as described in 2.7. Since $[X^{\mathsf{c}}, C] = 0$ and $\big[\overline{DX}, C\big] = 0$, it follows that $\mathcal{H}_D$ is a homogeneous Ehresmann connection. If $\big(\mathcal{U}, (u^i)_{i=1}^n\big)$ is a chart on M, and the Christoffel symbols of D on $\mathcal{U}$ are $\Gamma^i_{jk} \in C^\infty(\mathcal{U})$, then

$$\left(\frac{\partial}{\partial u^k}\right)^{\mathrm{h}_D} = \frac{\partial}{\partial x^k} - \big(\Gamma^i_{jk} \circ \tau\big)\, y^j \frac{\partial}{\partial y^i},$$

so $\mathcal{H}_D$ is a linear connection. An easy calculation shows that

$$\big[X^{\mathsf{v}}, \overline{DY}\big] = (D_X Y)^{\mathsf{v}} \textit{ for all } X, Y \in \mathfrak{X}(M).$$

Thus

$$\begin{aligned}\big[X^{\mathrm{h}_D}, Y^{\mathsf{v}}\big] - \big[Y^{\mathrm{h}_D}, X^{\mathsf{v}}\big] - [X,Y]^{\mathsf{v}} &= [X^{\mathsf{c}}, Y^{\mathsf{v}}] - [Y^{\mathsf{c}}, X^{\mathsf{v}}] - [X,Y]^{\mathsf{v}} \\ - \big[\overline{DX}, Y^{\mathsf{v}}\big] + \big[\overline{DY}, X^{\mathsf{v}}\big] &= [X,Y]^{\mathsf{v}} - [Y,X]^{\mathsf{v}} - [X,Y]^{\mathsf{v}} \\ + (D_Y X)^{\mathsf{v}} - (D_X Y)^{\mathsf{v}} &= -(D_X Y - D_Y X - [X,Y])^{\mathsf{v}},\end{aligned}$$

therefore $\mathcal{H}_D$ is torsion-free if, and only if, D is torsion-free.

3.4. Let a *homogeneous* Ehresmann connection $\mathcal{H}$ be specified in TM. Let I be an open interval containing 0. Consider a (smooth) curve $\gamma : I \to M$ and a vector field $X : I \to TM$ along γ (then $\tau \circ X = \gamma$). X is said to be *parallel* along γ with respect to $\mathcal{H}$ ($\mathcal{H}$*-parallel*, or simply parallel) if

$$\dot{X}(t) = \mathcal{H}(X(t), \dot{\gamma}(t)) \textit{ for all } t \in I,$$

briefly, if $\dot{X} = \mathcal{H}(X, \dot{\gamma})$.

We note that if $\gamma : I \to M$ is an integral curve of a vector field $Z \in \mathfrak{X}(M)$, and $X : I \to TM$ is $\mathcal{H}$-parallel along γ, then X is an integral curve of Z^{h}, i.e., $\dot{X} = Z^{\mathrm{h}} \circ X$.

Indeed, at any point $t \in I$, we have on the one hand

$$\dot{X}(t) = \mathcal{H}\big(X(t), \dot{\gamma}(t)\big) = \mathcal{H}\big(X(t), Z(\gamma(t))\big);$$

on the other hand

$$Z^{\mathrm{h}}\big(X(t)\big) := \mathcal{H}\big(X(t), Z(\tau(X(t)))\big) = \mathcal{H}\big(X(t), Z(\gamma(t))\big).$$

The general existence and uniqueness theorem for solutions of homogeneous ODEs (see, e.g., [4]) guarantees that, for any tangent vector $v \in \mathring{T}_{\gamma(0)}M$, there exists a unique parallel vector field X along γ such that $X(0) = v$. If $t \in I$, the mapping

$$(P_\gamma)_0^t : \mathring{T}_{\gamma(0)}M \to \mathring{T}_{\gamma(t)}M,\ v \mapsto (P_\gamma)_0^t(v) := X(t)$$

is called *parallel translation along* γ from $p = \gamma(0)$ to $q = \gamma(t)$. $(P_\gamma)_0^t$ is a positive-homogeneous diffeomorphism from $\mathring{T}_pM$ to $\mathring{T}_qM$.

By an ($\mathcal{H}$-) *horizontal lift of a (smooth)* curve $\gamma : I \to M$ we mean a curve

$$\gamma^{\mathrm{h}} : I \longrightarrow \mathring{T}M$$

such that

$$\tau \circ \gamma^{\mathrm{h}} = \gamma \textit{ and } \dot{\gamma}^{\mathrm{h}}(t) \in H_{\gamma^{\mathrm{h}}(t)}\mathring{T}M \textit{ for each } t \in I.$$

Lemma 1. *If γ^{h} is an $\mathcal{H}$-horizontal lift of a curve $\gamma : I \to M$, then γ^{h} is parallel along γ, i.e.,*

$$\dot{\gamma}^{\mathrm{h}}(t) = \mathcal{H}\big(\gamma^{\mathrm{h}}(t), \dot{\gamma}(t)\big), \quad t \in I.$$

Proof. Since

$$\dot{\gamma}(t) = (\gamma_*)_t\left(\frac{d}{dr}\right)_t = \big((\tau \circ \gamma^{\mathrm{h}})_*\big)_t\left(\frac{d}{dr}\right)_t = (\tau_*)_{\gamma^{\mathrm{h}}(t)}\big((\gamma^{\mathrm{h}})_*\big)_t\left(\frac{d}{dr}\right)_t$$

$$= (\tau_*)_{\gamma^{\mathrm{h}}(t)}\dot{\gamma}^{\mathrm{h}}(t),$$

we obtain

$$\dot{\gamma}^{\mathrm{h}}(t) = \mathbf{h}\big(\dot{\gamma}^{\mathrm{h}}(t)\big) = \mathcal{H}\big(\widetilde{\tau}_*(\dot{\gamma}^{\mathrm{h}}(t))\big) = \mathcal{H}\big(\gamma^{\mathrm{h}}(t), (\tau_*)_{\gamma^{\mathrm{h}}(t)}\dot{\gamma}^{\mathrm{h}}(t)\big) = \mathcal{H}\big(\gamma^{\mathrm{h}}(t), \dot{\gamma}(t)\big).$$

□

Lemma 2. *Let X be a vector field on M, and let*

$$\varphi : W \subset \mathbb{R} \times M \longrightarrow M, \quad (t,p) \longmapsto \varphi(t,p)$$

be the flow generated by X. If $\mathcal{H}$ is an Ehresmann connection in TM, then the flow generated by the $\mathcal{H}$-horizontal lift X^{h} of X is the mapping

$$\varphi^{\mathrm{h}} : \widetilde{W} \longrightarrow \mathring{T}M, \; (t,v) \longmapsto \varphi^{\mathrm{h}}(t,v) := \varphi^{\mathrm{h}}_v(t),$$

where

$$\widetilde{W} := \big\{(t,v) \in \mathbb{R} \times \mathring{T}M \big| (t, \tau(v)) \in W\big\},$$

and for any fixed $v \in \mathring{T}M$, φ^{h}_v is the horizontal lift of the curve $\varphi_{\tau(v)}$ defined by $\varphi_{\tau(v)}(t) := \varphi(t, \tau(v))$, starting from v.

Proof. We have to check that for any fixed $v \in \mathring{T}M$,

$$\dot{\overline{\varphi^{\mathrm{h}}_v}} = X^{\mathrm{h}} \circ \varphi^{\mathrm{h}}_v.$$

If t is in the domain of φ^{h}_v, then

$$X^{\mathrm{h}}\big(\varphi^{\mathrm{h}}_v(t)\big) := \mathcal{H}\big(\varphi^{\mathrm{h}}_v(t), X \circ \tau(\varphi^{\mathrm{h}}_v(t))\big) = \mathcal{H}\big(\varphi^{\mathrm{h}}_v(t), X \circ \varphi_{\tau(v)}(t)\big)$$
$$= \mathcal{H}\big(\varphi^{\mathrm{h}}_v(t), \dot{\varphi}_{\tau(v)}(t)\big) \overset{\text{Lemma 1}}{=} \dot{\overline{\varphi^{\mathrm{h}}_v}}(t).$$

□

As in the classical theory of linear connections, a regular curve $\gamma : I \to M$ is said to be a *geodesic* of an Ehresmann connection $\mathcal{H}$ if its velocity field $\dot{\gamma}$ is $\mathcal{H}$-parallel, i.e.,

$$\ddot{\gamma} = \mathcal{H}(\dot{\gamma}, \dot{\gamma}).$$

In local coordinates, the equations of parallel vector fields become

(P) $$X^{i\prime} + \big(N^i_j \circ X\big)\gamma^{j\prime} = 0 \quad (i \in \{1, \dots, n\}),$$

if $X \restriction \gamma^{-1}(\mathcal{U}) = X^i\left(\frac{\partial}{\partial u^i} \circ \gamma\right)$, $\gamma^i := u^i \circ \gamma$, and, as above, the functions N^i_j are the Christoffel symbols for $\mathcal{H}$. In particular, the *geodesic equations* take the form

(G) $$\gamma^{i\prime\prime} + \big(N^i_j \circ \dot{\gamma}\big)\gamma^{j\prime} = 0, \quad i \in \{1, \dots, n\}.$$

We need the following simple observation.

Lemma 3. *If D is a torsion-free covariant derivative operator on M, and $\mathcal{H}_D$ is the Ehresmann connection induced by D, then a vector field $X : I \to TM$ along a curve $\gamma : I \to M$ is parallel with respect to D if, and only if, it is $\mathcal{H}_D$-parallel.*

Indeed, if the Christoffel symbols for D are the functions $\Gamma^i_{jk} \in C^\infty(\mathcal{U})$, then the Christoffel symbols for $\mathcal{H}_D$ are $N^i_j = \left(\Gamma^i_{jk} \circ \tau\right) y^k$, hence

$$N^i_j \circ X = \left(\left(\Gamma^i_{jk} \circ \tau\right) y^k\right) \circ X = \left(\Gamma^i_{jk} \circ \tau \circ X\right) \left(y^k \circ X\right) = \left(\Gamma^i_{jk} \circ \gamma\right) X^k,$$

so equations (P) become

$$X^{i\prime} + \left(\Gamma^i_{jk} \circ \gamma\right) \gamma^{j\prime} X^k = 0,\ i \in \{1, \dots, n\},$$

which are the familiar equations of parallelism with respect to a covariant derivative operator.

3.5. We say that an Ehresmann connection $\mathcal{H}$ in TM is *compatible* with a C^1-function $F : TM \to \mathbb{R}$ if $dF \circ \mathcal{H} = 0$, or, equivalently, if

$$X^{\mathrm{h}} F = 0 \textit{ for all } X \in \mathfrak{X}(M).$$

Lemma 4. *Assume that $\mathcal{H}$ is a homogeneous Ehresmann connection in TM, and let $F : TM \to \mathbb{R}$ be a C^1-function. The following are equivalent:*

(i) $\mathcal{H}$ is compatible with F.

(ii) For any $\mathcal{H}$-parallel vector field $X : I \to TM$ along a curve $\gamma : I \to M$, the function
$$F \circ X : I \to \mathbb{R}$$
is constant.

Proof. Let $\frac{d}{dr}$ be the canonical vector field on I ($r := 1_{\mathbb{R}}$). $F \circ X$ can be considered as a curve in $\mathbb{R}$; then for each $t \in \mathbb{R}$ we have

$$\begin{aligned}(F \circ X)'(t) = \dot{\overline{F \circ X}}(t) &= ((F \circ X)_*)_t \left(\frac{d}{dr}\right)_t \\ &= (F_*)_{X(t)} \circ (X_*)_t \left(\frac{d}{dr}\right)_t = (F_*)_{X(t)} \left(\dot{X}(t)\right) \\ &= (dF)_{X(t)}(\mathcal{H}(X(t)), \dot{\gamma}(t))) = dF \circ \mathcal{H}(X(t), \dot{\gamma}(t))\end{aligned}$$

identifying in our calculation the derivative $(F_*)_{X(t)}$ with the differential $(dF)_{X(t)}$, and taking into account the condition that X is parallel along γ. The relation so obtained implies immediately that $F \circ X$ *is constant if, and only if,* $dF \circ \mathcal{H} = 0$. □

Lemma 5. *Let $\overline{S}$ be a spray for M, and let $\overline{\mathcal{H}}$ be the Ehresmann connection induced by $\overline{S}$ (3.3(a)). For a C^1 function F on TM, the following are equivalent:*

(i) $\overline{\mathcal{H}}$ is compatible with F.

(ii) $\overline{S}F = 0$ and $X^{\mathsf{c}}F - \overline{S}(X^{\mathsf{v}}F) = 0$ for any vector field X on M.

Proof. Let $X^{\overline{\mathrm{h}}}$ be the horizontal lift of $X \in \mathfrak{X}(M)$ with respect to $\overline{\mathcal{H}}$. Then

$$2X^{\overline{\mathrm{h}}}F = X^{\mathsf{c}}F + X^{\mathsf{v}}(\overline{S}F) - \overline{S}(X^{\mathsf{v}}F).$$

Since $\overline{S}$ is a spray, it is horizontal with respect to $\overline{\mathcal{H}}$, so (i) implies (ii). The converse is immediate from the above formula. □

4 Basic facts on Finsler functions

4.1. A function $F : TM \to \mathbb{R}$ is said to be a *Finsler function* if

(F1) F is smooth on $\mathring{T}M$;

(F2) F is positive-homogeneous of degree 1, i.e., $F(\lambda v) = \lambda F(v)$ for all $v \in TM$ and positive real number λ;

(F3) $F(v) > 0$ if $v \in \mathring{T}M$;

(F4) the *metric tensor* $g : \mathrm{Sec}(\mathring{\pi}) \times \mathrm{Sec}(\mathring{\pi}) \to C^\infty(\mathring{T}M)$ of (M, F) given on basic sections by

$$g\big(\widehat{X}, \widehat{Y}\big) := \frac{1}{2}X^{\mathsf{v}}\big(Y^{\mathsf{v}}F^2\big) \quad (X, Y \in \mathfrak{X}(M))$$

is fibrewise nondegenerate.

A *Finsler manifold* is a manifold endowed with a Finsler function on its tangent manifold. More formally, a Finsler manifold is a pair (M, F) consisting of a manifold M and a Finsler function F on TM. For each $v \in TM$, $F(v)$ is called the *Finsler norm* of v. The function $E := \frac{1}{2}F^2$ is mentioned as the *energy function* of (M, F).

By conditions (F1) and (F2), F is continuous on TM and vanishes on $o(M)$. The energy function is positive-homogeneous of degree 2, and it is of class C^1 on TM (see [20], 3.11, Property 4). It may be shown that our requirements on a Finsler function also imply that the metric tensor is (fibrewise) positive definite [13, 14].

FACT 1. *Let (M, F) be a Finsler manifold. There exists a unique spray S for M, called the* canonical spray *of (M, F), defined to be zero on $o(M)$ and satisfying*

$$i_S d(dE \circ \mathbf{J}) = -dE$$

on $\mathring{T}M$.

FACT 2. *The torsion-free Ehresmann connection associated to the canonical spray of a Finsler manifold by Crampin's construction* (3.3(a)) *is* homogeneous *and* compatible with the Finsler function.

For a quite recent index-free proof of this fact we refer to [21].

FACT 3 (the uniqueness of the canonical connection)**.** *If a torsion-free, homogeneous Ehresmann connection is compatible with a Finsler function, then it is the canonical connection of the Finsler manifold.*

Since this result plays a key role in answering Matveev's query, by the suggestion of the referee, we present here a more or less complete proof.

Let $\mathcal{H}$ be the Ehresmann connection associated to the canonical spray S of a Finsler manifold (M, F), and let $\overline{\mathcal{H}}$ be a torsion-free, homogeneous Ehresmann connection compatible with F, and hence also with the energy function E. By a fundamental result of the theory of Ehresmann connections (see, e.g., [20], Theorem 2 and Corollary 6 in 3.3) there exists a spray $\overline{S}$ for M such that $\overline{\mathcal{H}}$ is generated by $\overline{S}$, i.e.,

$$X^{\bar{\mathsf{h}}} := \overline{\mathcal{H}}(\widehat{X}) = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, \overline{S}]) \text{ for all } X \in \mathfrak{X}(M).$$

Since $\overline{\mathcal{H}}$ is compatible with E, it follows by Lemma 5 that

$$X^{\mathsf{c}}E - \overline{S}(X^{\mathsf{v}}E) = 0, \text{ for all } X \in \mathfrak{X}(M).$$

However, this relation is just another formulation of the Euler−Lagrange equation

$$i_{\overline{S}} d(dE \circ \mathbf{J}) + dE = 0.$$

Indeed, this relation holds identically over $\mathfrak{X}^{\mathsf{v}}(TM)$, while for any vector field X on M we have

$$\begin{aligned}(i_{\overline{S}} d(dE \circ \mathbf{J}) + dE)(X^{\mathsf{c}}) &= \overline{S}(X^{\mathsf{v}}E) - X^{\mathsf{c}}(CE) - \mathbf{J}[\overline{S}, X^{\mathsf{c}}]E + X^{\mathsf{c}}E \\ &= -(X^{\mathsf{c}}E - \overline{S}(X^{\mathsf{v}}E))\end{aligned}$$

taking into account the fact that $[\overline{S}, X^{\mathsf{c}}]$ is vertical. Now the non-degeneracy of the 2-form $d(dE \circ \mathbf{J})$, assured by (F4), implies that $\overline{S} = S$, and hence $\overline{\mathcal{H}} = \mathcal{H}$.

We note that another recent proof of Fact 3 can be found in Ref. [23]. It is based on an idea of Z. I. Szabó, utilizing the so-called Rapcsák equation to conclude the uniqueness statement. Actually, there is no essential difference between the two argumentations.

Remark. If $\mathcal{H}$ is the canonical connection of a Finsler manifold, then, by Lemma 4, *the Finsler norm of a vector remains invariant under $\mathcal{H}$-parallel translations.*

4.2. Let (M, F) be a Finsler manifold with canonical spray S and canonical connection $\mathcal{H}$.

(a) By a *geodesic* of (M, F) we mean a geodesic of its canonical connection, or, equivalently, a regular curve $\gamma : I \to M$ whose velocity field is an integral curve of the canonical spray:

$$\ddot{\gamma} = S \circ \dot{\gamma}.$$

(b) $\mathcal{H}$ (as any Ehresmann connection) induces a covariant derivative operator

$$\nabla : \mathfrak{X}(\mathring{T}M) \times \operatorname{Sec}(\mathring{\pi}) \to \operatorname{Sec}(\mathring{\pi}),$$

called the *Berwald derivative* of (M, F), such that for all $X, Y \in \mathfrak{X}(M)$,

$$\operatorname{vl} \nabla_{X^{\mathrm{h}}} \widehat{Y} = \left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right], \ \nabla_{X^{\mathsf{v}}} \widehat{Y} = 0.$$

Here, the first relation can be written in the equivalent form

$$\nabla_{X^{\mathrm{h}}} \widehat{Y} = \mathcal{V}\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right].$$

(c) The *Berwald curvature* of (M, F) is the type (1,3) tensor $\mathbf{B}$ on the $C^\infty(\mathring{T}M)$-module $\operatorname{Sec}(\mathring{\pi})$ such that

$$\operatorname{vl} \mathbf{B}\left(\widehat{X}, \widehat{Y}, \widehat{Z}\right) = \left[X^{\mathsf{v}}, \left[Y^{\mathrm{h}}, Z^{\mathsf{v}}\right]\right]$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Lemma 6. *If ∇ is the Berwald derivative induced by a torsion-free Ehresmann connection $\mathcal{H}$, then for any vector fields X, Y on M,*

$$\nabla_{X^{\mathrm{h}}} \widehat{Y} - \nabla_{Y^{\mathrm{h}}} \widehat{X} = \widehat{[X, Y]}.$$

Proof. Applying the definition of ∇ and the torsion-freeness of $\mathcal{H}$,

$$\mathrm{vl}\left(\nabla_{X^{\mathrm{h}}}\widehat{Y} - \nabla_{Y^{\mathrm{h}}}\widehat{X}\right) = [X^{\mathrm{h}}, Y^{\mathsf{v}}] - [Y^{\mathrm{h}}, X^{\mathsf{v}}] = [X,Y]^{\mathsf{v}} = \mathrm{vl}\,\widehat{[X,Y]},$$

whence our claim. □

4.3. For the sake of readers who prefer the language of classical tensor calculus, we present here the coordinate expressions of some important objects introduced above.

Let (M,F) be a Finsler manifold. Choose a chart $\left(\mathcal{U},(u^i)_{i=1}^n\right)$ on M, and consider the induced chart $\left(\tau^{-1}(\mathcal{U}),(x^i,y^i)_{i=1}^n\right)$ on TM.

(i) The components of the metric tensor of (M,F) are the functions

$$g_{ij} := g\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},\ \ i,j \in \{1,\dots,n\}.$$

(ii) Over $\tau^{-1}(\mathcal{U})$, the canonical spray of (M,F) can be represented in the form

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where the *spray coefficients* are

$$G^i = \frac{1}{4} g^{ij}\left(\frac{\partial^2 F^2}{\partial x^r \partial y^j} y^r - \frac{\partial F^2}{\partial x^j}\right);\ \ (g^{ij}) := (g_{ij})^{-1}.$$

(iii) The Christoffel symbols of the canonical connection and the Berwald derivative are

$$G^i_j := \frac{\partial G^i}{\partial y^j} \quad \text{and} \quad G^i_{jk} := \frac{\partial G^i}{\partial y^j \partial y^k},$$

respectively. Then

$$\left(\frac{\partial}{\partial u^j}\right)^{\mathrm{h}} = \frac{\partial}{\partial x^j} - G^i_j \frac{\partial}{\partial y^i},\quad \nabla_{\left(\frac{\partial}{\partial u^j}\right)^{\mathrm{h}}} \widehat{\frac{\partial}{\partial u^k}} = G^i_{jk} \widehat{\frac{\partial}{\partial u^i}}.$$

The components of the Berwald curvature are given by

$$\mathbf{B}\left(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}}, \widehat{\frac{\partial}{\partial u^l}}\right) = G^i_{jkl} \widehat{\frac{\partial}{\partial u^i}},\quad G^i_{jkl} := \frac{\partial G^i_{jk}}{\partial y^l},$$

from which it is clear that $\mathbf{B}$ is totally symmetric.

5 Berwald manifolds

5.1. In this subsection we present eight of the announced equivalents of the Berwaldian property (B). Note that condition (B1) is just its restatement in more precise terms.

Proposition 7. *Let (M, F) be a Finsler manifold. The following conditions are equivalent:*

(B1) *(M, F) is an 'affinely connected space' in Berwald's sense [6], that is, the Christoffel symbols G^i_{jk} of the Berwald derivative 'depend only on the position'.*

(B2) *The Berwald curvature of (M, F) vanishes.*

(B3) *There exists a torsion-free covariant derivative operator D on M such that for any two vector fields X, Y on M we have*

$$(D_X Y)^{\mathsf{v}} := [X^{\mathrm{h}}, Y^{\mathsf{v}}],$$

where the horizontal lift is taken with respect to the canonical connection of (M, F).

(B4) *The Berwald derivative ∇ of (M, F) is* h-basic *in the sense that there exists a covariant derivative operator D on M such that*

$$\mathrm{vl}\,\nabla_{X^{\mathrm{h}}}\widehat{Y} = (D_X Y)^{\mathsf{v}} \quad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

(B5) *The Lie bracket $\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right]$ is a vertical lift for any vector fields X, Y on M.*

(B6) *The canonical spray of (M, F) is an affine spray.*

(B7) *The canonical connection of (M, F) is a linear connection.*

(B8) *There exists a torsion-free covariant derivative operator D on M such that the parallel translations with respect to D preserve the Finsler norms of tangent vectors to M.*

(B9) *There exists a torsion-free covariant derivative operator D on M such that the geodesics of D coincide with the geodesics of (M, F) as parametrized curves.*

Proof. We organize our reasoning according to the following scheme:

$$
\begin{array}{ccccccccc}
 & & \text{(B1)} \Longleftrightarrow \text{(B2)} & \Longleftrightarrow & \text{(B5)} & \Longleftarrow & \text{(B7)} & & \\
 & & & \nearrow\!\!\!\nearrow & \Downarrow & & \Uparrow & \nwarrow\!\!\!\nwarrow & \\
 & & \text{(B4)} & \Longleftarrow & \text{(B3)} & \Longrightarrow & \text{(B8)} & & \text{(B6)} \; . \\
 & & & & & \searrow\!\!\!\searrow & & \nearrow\!\!\!\nearrow & \\
 & & & & & & \text{(B9)} & &
\end{array}
$$

(B1) $\Longleftrightarrow$ (B2) This is obvious, since, as we have just seen, the components of the Berwald curvature are the functions $\frac{\partial G^i_{jk}}{\partial y^l}$.

(B2)$\Longrightarrow$(B5) If $\mathbf{B} = 0$, then, by 4.2(c), for any vector fields X, Y, Z on M, we have

$$\left[\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right], Z^{\mathsf{v}}\right] = 0.$$

Since $X^{\mathrm{h}} \underset{\tau}{\sim} X$, $Y^{\mathsf{v}} \underset{\tau}{\sim} 0$, $\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right]$ is vertical. This vertical vector field commutes with any vertically lifted vector field, which implies easily that $\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right]$ is itself a vertical lift.

(B5)$\Longrightarrow$(B2) If $\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right]$ is a vertical lift for each $X, Y \in \mathfrak{X}(M)$, then for any vector field Z on M,

$$0 = \left[Z^{\mathsf{v}}, \left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right]\right] = \mathrm{vl}\,\mathbf{B}\big(\widehat{Z}, \widehat{X}, \widehat{Y}\big) = \mathrm{vl}\,\mathbf{B}\big(\widehat{X}, \widehat{Y}, \widehat{Z}\big),$$

hence $\mathbf{B} = 0$.

(B5)$\Longrightarrow$(B3) For any vector fields X, Y on M, let

$$(D_X Y)^{\mathsf{v}} := \left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right].$$

Then the mapping

$$D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ (X, Y) \mapsto D_X Y$$

is well-defined. From the nice properties of the Lie bracket $\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right]$ (see the Theorem in section 3 in Crampin's paper [8], or verify it immediately) it follows that D is a covariant derivative operator on M with vanishing torsion.

$$\left.\begin{array}{l}\text{(B3)} \Longrightarrow \text{(B4)} \\ \text{(B4)} \Longrightarrow \text{(B5)}\end{array}\right\} \text{These are obvious since } \mathrm{vl}\,\nabla_{X^{\mathrm{h}}}\widehat{Y} = [X^{\mathrm{h}}, Y^{\mathsf{v}}].$$

(B7)$\Longrightarrow$(B5) We prove this by a simple coordinate calculation, using the local apparatus introduced in 4.3.

Let X and Y be vector fields on M,

$$X \restriction \mathcal{U} = X^i \frac{\partial}{\partial u^i},\ Y \restriction \mathcal{U} = Y^i \frac{\partial}{\partial u^i}.$$

Then, over $\tau^{-1}(\mathcal{U})$,

$$X^{\mathrm{h}} = \left(X^j \frac{\partial}{\partial u^j}\right)^{\mathrm{h}} = \left(X^j \circ \tau\right) \left(\frac{\partial}{\partial u^j}\right)^{\mathrm{h}} = \left(X^j \circ \tau\right) \left(\frac{\partial}{\partial x^j} - G^i_j \frac{\partial}{\partial y^i}\right),$$

where the functions G^i_j are the Christoffel symbols of the canonical connection of (M, F). By its linearity,

$$G^i_j = \left(\Gamma^i_{jl} \circ \tau\right) y^l, \ \Gamma^i_{jl} \in C^\infty(\mathcal{U}),$$

hence

$$G^i_{jk} = \frac{\partial G^i_j}{\partial y^k} = \Gamma^i_{jk} \circ \tau.$$

Thus

$$\begin{aligned}
\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right] &= \left(X^j \circ \tau\right) \left[\frac{\partial}{\partial x^j} - G^i_j \frac{\partial}{\partial y^i}, Y^{\mathsf{v}}\right] \\
&= \left(X^j \circ \tau\right) \left(\left[\frac{\partial}{\partial x^j}, \left(Y^i \circ \tau\right) \frac{\partial}{\partial y^i}\right] + \left(Y^{\mathsf{v}} G^i_j\right) \frac{\partial}{\partial y^i}\right) \\
&= \left(X^j \circ \tau\right) \left(\frac{\partial Y^i}{\partial u^j} \circ \tau + \left(Y^k \circ \tau\right) G^i_{jk}\right) \frac{\partial}{\partial y^i} \\
&= \left(\left(X^j \frac{\partial Y^i}{\partial u^j} + X^j Y^k \Gamma^i_{jk}\right) \frac{\partial}{\partial u^i}\right)^{\mathsf{v}},
\end{aligned}$$

which proves that $\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right]$ is a vertical lift.

(B8)$\Longrightarrow$(B7) As we saw in 3.3(b), the Ehresmann connection $\mathcal{H}_D$ determined by D is torsion-free, homogeneous and linear. Lemmas 3, 4 guarantee that $\mathcal{H}_D$ is compatible with F, therefore

$$\mathcal{H}_D = \mathcal{H} := \textit{the canonical connection of } (M, F)$$

by Fact 3 in 4.1, proving that $\mathcal{H}$ is a linear connection. (Wee see that the heart of our answer to Matveev's query is the uniqueness of the canonical connection of a Finsler manifold!)

(B3)$\Longrightarrow$(B8) Again consider the Ehresmann connection $\mathcal{H}_D$. By our condition and the torsion-freeness of $\mathcal{H}_D$, for any vector fields X, Y on M we have

$$\begin{aligned}
\left[X^{\mathrm{h}}, Y^{\mathsf{v}}\right] &= (D_X Y)^{\mathsf{v}} = \left[X^{\mathsf{v}}, \overline{DY}\right] = [X^{\mathsf{v}}, Y^{\mathsf{c}}] - \left[X^{\mathsf{v}}, Y^{\mathrm{h}_D}\right] \\
&= [X, Y]^{\mathsf{v}} - \left[X^{\mathsf{v}}, Y^{\mathrm{h}_D}\right] = \left[X^{\mathrm{h}_D}, Y^{\mathsf{v}}\right].
\end{aligned}$$

Thus, $X^{\mathrm{h}} - X^{\mathrm{h}_D}$ is a vertical vector field which commutes with each vertical lift, so it is itself a vertical lift. Therefore, it is positive-homogeneous of degree 0 and degree 1 at the same time, which is possible only if $X^{\mathrm{h}} - X^{\mathrm{h}_D} = 0$. We conclude that $\mathcal{H}_D = \mathcal{H}$, and $\mathcal{H}_D$ is compatible with the Finsler function F. By Lemma 3, $\mathcal{H}_D$ generates the same parallelism as D, so, in view of Lemma 4, the parallel translations with respect to D preserve the Finsler norms of the tangent vectors. This shows that the covariant derivative D in (B3) satisfies the requirement of (B8).

(B3)$\Longrightarrow$(B9) Due to the preceding argumentation, we have already known that $\mathcal{H}_D = \mathcal{H}$. Since, by Lemma 3 again, the $\mathcal{H}_D$-geodesics are the same parametrized curves as the D-geodesics, (B9) is indeed a consequence of (B3).

(B9)$\Longrightarrow$(B6) Let S be the canonical spray of (M, F) and $\mathcal{H}_D$ the Ehresmann connection determined by the given covariant derivative. It can be checked immediately that the mapping

$$S_D : TM \to TTM,\ v \mapsto S_D(v) := \mathcal{H}_D(v, v)$$

is an affine spray. By our condition, it follows that for a regular curve $\gamma : I \to M$,

$$\ddot{\gamma} = S \circ \dot{\gamma} \textit{ if, and only if, } \ddot{\gamma} = S_D \circ \dot{\gamma}.$$

Since any vector $v \in \mathring{T}M$ is the initial velocity of an

$$S\textit{-geodesic} = S_D\textit{-geodesic},$$

we conclude that $S = S_D$, and hence S is an affine spray.

(B6)$\Longrightarrow$(B7) Let S be the canonical spray of (M, F). If S is an affine spray, then its spray coefficients $G^i : \tau^{-1}(\mathcal{U}) \to \mathbb{R}$ are of class C^2. Since these functions are positive-homogeneous of degree 2, it follows that fibrewise they are quadratic forms. So there exist smooth functions Γ^i_{jk} on $\mathcal{U}$ such that

$$G^i = \frac{1}{2}\left(\Gamma^i_{kl} \circ \tau\right) y^k y^l \textit{ and } \Gamma^i_{kl} = \Gamma^i_{lk} \quad (i, k, l \in \{1, \dots, n\}).$$

Then the Christoffel symbols of the canonical connection $\mathcal{H}$ of (M, F) are the smooth functions

$$G^i_j := \frac{\partial G^i}{\partial y^j} = \left(\Gamma^i_{jk} \circ \tau\right) y^k,$$

hence $\mathcal{H}$ is a linear connection. This proves the implication and concludes the proof of the Proposition. □

If one, and hence each, of conditions (B1)–(B9) is satisfied, (M, F) is said to be a *Berwald manifold*. The torsion-free covariant derivative D appearing in (B3), (B4), (B8), (B9) is clearly unique; it will be called the *base covariant derivative* of the Berwald manifold.

5.2. Our list of equivalent conditions for a Finsler manifold to be a Berwald manifold is quite abundant, but does not exhaust all of the known and useful characterizations. In this subsection we deal with two other conditions, and a third one will be explained in detail in the next section.

(a) By the *exponential map* of a Finsler manifold (M, F) we mean the exponential map for its canonical spray S: if for any vector v in TM, γ_v denotes the maximal geodesic of S with initial velocity v, and

$$\widetilde{TM} := \{v \in TM | \gamma_v(1) \text{ is defined}\},$$

then $\widetilde{TM}$ is an open neighbourhood of the image $o(M)$ of the zero vector field on M, and

$$\exp : \widetilde{TM} \longrightarrow M, \quad v \longmapsto \exp(v) := \gamma_v(1).$$

It is well-known that exp is smooth on $\widetilde{TM} \setminus o(M)$ and C^1 on $\widetilde{TM}$. If (M, F) is a Berwald manifold, then it follows from (B6) that its exponential map is smooth on its domain. Conversely, *if* exp *is* C^2 *on the entire* $\widetilde{TM}$*, then* (M, F) *is a Berwald manifold.* A possible proof of this assertion is sketched in [2], Exercise 5.3.5.

(b) To formulate the next version of the Berwaldian property, we need some preparations. Keeping the introduced notations, let a Finsler manifold (M, F) be given. If $A \in \mathcal{T}^0_k(\mathring{\pi})$ or $A \in \mathcal{T}^1_k(\mathring{\pi})$, we define the (*canonical*) *vertical differential* $\nabla^{\mathsf{v}} A$ and the *h-Berwald differential* $\nabla^{\mathrm{h}} A$ of A as the type $(0, k+1)$, resp. $(1, k+1)$ tensors along $\mathring{\tau}$ given by

$$\nabla^{\mathsf{v}} A(\widetilde{X}, \widetilde{Y}_1, \ldots, \widetilde{Y}_k) := (\nabla_{\mathrm{vl}\,\widetilde{X}} A)(\widetilde{Y}_1, \ldots \widetilde{Y}_k)$$

and

$$\nabla^{\mathrm{h}} A(\widetilde{X}, \widetilde{Y}_1, \ldots, \widetilde{Y}_k) := (\nabla_{\mathcal{H}\widetilde{X}} A)(\widetilde{Y}_1, \ldots \widetilde{Y}_k),$$

$(\widetilde{X}, \widetilde{Y}_1, \ldots, \widetilde{Y}_k \in \mathrm{Sec}(\mathring{\pi}))$. If, in particular, $A := F \in C^\infty(\mathring{T}M)$, then

$$\nabla^{\mathsf{v}} F(\widetilde{X}) = (\mathrm{vl}\,\widetilde{X})F, \quad \nabla^{\mathrm{h}} F(\widetilde{X}) = (\mathcal{H}\widetilde{X})F.$$

Notice that in terms of vertical differentials, the metric tensor of (M, F) can be given by the formula $g = \nabla^{\mathsf{v}} \nabla^{\mathsf{v}} E$.

By the (*lowered*) *Cartan tensor* of (M, F) we mean the type $(0,3)$ tensor

$$\mathcal{C}_\flat := \frac{1}{2}\nabla^{\mathsf{v}} g = \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} E$$

along $\mathring{\tau}$. It is easy to check that $\mathcal{C}_\flat$ vanishes if, and only if, (M, F) reduces to a Riemannian manifold, i.e., there exists a Riemannian metric g_M on M such that

$$g(\widehat{X}, \widehat{Y}) = g_M(X, Y) \circ \tau; \quad X, Y \in \mathfrak{X}(M).$$

FACT 4. $\nabla^{\mathsf{v}} E \circ \mathbf{B} = \nabla^{\mathrm{h}} g$.

This is just an index-free reformulation of a familiar relation in classical Finsler geometry, and it can easily be checked. For a proof in our style we refer to [3].

FACT 5. *Let S be the canonical spray of (M, F). Then*

$$\nabla_S \mathcal{C}_\flat = -\frac{1}{2}\nabla^{\mathrm{h}} g.$$

This is also a key observation, whose proof is quite tricky; see [9], or 3.11, Property 10/(ii) in [20]. A simplified version of the latter can be found in [23], Prop. 7.4.

Now we calculate the h-Berwald differential of $\mathcal{C}_\flat$. Applying the relevant Ricci identity to exchange the ∇^{h} and the ∇^{v} differentiations, and taking into account Fact 4, for any vector fields X, Y, Z, U on M we obtain

$$\begin{aligned}2\nabla^{\mathrm{h}}\mathcal{C}_\flat(\widehat{X}, \widehat{Y}, \widehat{Z}, \widehat{U}) &= \nabla^{\mathrm{h}}\nabla^{\mathsf{v}} g(\widehat{X}, \widehat{Y}, \widehat{Z}, \widehat{U}) \\ &= \nabla^{\mathsf{v}}(\nabla^{\mathsf{v}} E \circ \mathbf{B})(\widehat{Y}, \widehat{X}, \widehat{Z}, \widehat{U}) + g\big(\mathbf{B}(\widehat{X}, \widehat{Y}, \widehat{Z}), \widehat{U}\big) \\ &+ g\big(\mathbf{B}(\widehat{X}, \widehat{Y}, \widehat{U}), \widehat{Z}\big),\end{aligned}$$

from which it follows that $\mathbf{B} = 0$ implies $\nabla^{\mathrm{h}}\mathcal{C}_\flat = 0$.

Conversely, suppose that $\nabla^{\mathrm{h}}\mathcal{C}_\flat = 0$. Then

$$0 = \nabla_{\mathcal{H}\delta}\mathcal{C}_\flat = \nabla_S \mathcal{C}_\flat \overset{\text{Fact 5}}{=} -\frac{1}{2}\nabla^{\mathrm{h}} g,$$

i.e., $\nabla^{\mathrm{h}} g = 0$. Thus the first term on the right-hand side of the above relation vanishes, and, using the total symmetry of $\mathbf{B}$, we find that

$$\begin{aligned}0 &= 2\nabla^{\mathrm{h}}\mathcal{C}_\flat(\widehat{X}, \widehat{Y}, \widehat{Z}, \widehat{U}) + 2\nabla^{\mathrm{h}}\mathcal{C}_\flat(\widehat{X}, \widehat{Z}, \widehat{U}, \widehat{Y}) - 2\nabla^{\mathrm{h}}\mathcal{C}_\flat(\widehat{X}, \widehat{U}, \widehat{Y}, \widehat{Z}) \\ &= g\big(\mathbf{B}(\widehat{X}, \widehat{Y}, \widehat{Z}), \widehat{U}\big) + g\big(\mathbf{B}(\widehat{X}, \widehat{Y}, \widehat{U}), \widehat{Z}\big) + g\big(\mathbf{B}(\widehat{X}, \widehat{Z}, \widehat{U}), \widehat{Y}\big) \\ &+ g\big(\mathbf{B}(\widehat{X}, \widehat{Z}, \widehat{Y}), \widehat{U}\big) - g\big(\mathbf{B}(\widehat{X}, \widehat{U}, \widehat{Y}), \widehat{Z}\big) - g\big(\mathbf{B}(\widehat{X}, \widehat{U}, \widehat{Z}), \widehat{Y}\big) \\ &= 2g\big(\mathbf{B}(\widehat{X}, \widehat{Y}, \widehat{Z}), \widehat{U}\big).\end{aligned}$$

This implies the vanishing of $\mathbf{B}$.

Thus we have proved:

Proposition 8. *A Finsler manifold is a Berwald manifold if, and only if, the h-Berwald differential of its Cartan tensor vanishes.*

Actually, we verified this result only for the type $(0,3)$ *lowered* Cartan tensor. It can easily be checked, however, that it remains true also for the type $(1,2)$ *Cartan tensor* metrically equivalent to $\mathcal{C}_\flat$.

An analogous characterization of Berwald manifolds in terms of the h-Cartan, or, what is the same, the h-Chern – Rund derivative is well-known, see, e.g., [2, 18, 22]. To our surprise, the formulation in terms of the h-Berwald differential seems to be new.

6 Aikou's characterization of Berwald manifolds

Let ξ be a *projectable* vector field on TM. We define a *Lie derivative operator* $\mathcal{L}_\xi$ on the tensor algebra of the $C^\infty(\mathring{T}M)$-module $\mathrm{Sec}(\mathring{\pi})$, by prescribing its action

$$\text{on } \textit{functions} \text{ by } \mathcal{L}_\xi f := \xi f,\ f \in C^\infty(\mathring{T}M);$$
$$\text{on } \textit{sections} \text{ by } \mathcal{L}_\xi \widetilde{Y} := \mathrm{vl}^{-1}[\xi, \mathrm{vl}\,\widetilde{Y}],\ \widetilde{Y} \in \mathrm{Sec}(\mathring{\pi}),$$

and by extending it to the whole tensor algebra in such a way that $\mathcal{L}_\xi$ satisfies the product rule of tensor derivations. Since ξ is projectable, the Lie bracket $[\xi, \mathrm{vl}\,\widetilde{Y}]$ is a vertical vector field, so the definition of $\mathcal{L}_\xi \widetilde{Y}$ is legitimate. If $\mathbf{v}$ is the vertical projection associated to an Ehresmann connection in TM then $[\xi, \mathrm{vl}\,\widetilde{Y}] = \mathbf{v}[\xi, \mathrm{vl}\,\widetilde{Y}] = \mathrm{vl} \circ \mathcal{V}[\xi, \mathrm{vl}\,\widetilde{Y}]$, hence

$$\mathcal{L}_\xi \widetilde{Y} = \mathcal{V}[\xi, \mathrm{vl}\,\widetilde{Y}].$$

We shall find this formula useful in what follows. In particular, for any vector fields X, Y on M we get

$$\mathcal{L}_{X^{\mathsf{c}}} \widehat{Y} = \mathcal{V}[X^{\mathsf{c}}, Y^{\mathsf{v}}] = \mathcal{V}[X, Y]^{\mathsf{v}} = \widehat{[X, Y]} = \widehat{\mathcal{L}_X Y},$$

so the Lie derivative just introduced is a natural extension of the 'ordinary Lie derivative' on the base manifold. (For simplicity, we do not make notational distinction between the two types of Lie derivatives.)

The Lie derivative of a section of $\mathring{\pi}$ with respect to the horizontal lift of a vector field on M also has a nice dynamic interpretation.

Lemma 9. *Let $\mathcal{H}$ be an Ehresmann connection in TM, and let X^{h} be the $\mathcal{H}$-horizontal lift of $X \in \mathfrak{X}(M)$. Then for any section $\widetilde{Y} \in \mathrm{Sec}(\mathring{\pi})$ and tangent vector $u \in \mathring{T}M$ we have*

$$\begin{aligned}\big(\mathcal{L}_{X^{\mathrm{h}}}\widetilde{Y}\big)(u) &= \lim_{t\to 0}\frac{(\varphi^{\mathrm{h}}_{-t})_*\widetilde{Y}\big(\varphi^{\mathrm{h}}_t(u)\big) - \widetilde{Y}(u)}{t}\\ &= \big(t \longmapsto (\varphi^{\mathrm{h}}_{-t})_*\widetilde{Y}(\varphi^{\mathrm{h}}_t(u))\big)'(0),\end{aligned}$$

where $\varphi : W \subset \mathbb{R}\times M \to M$ is the flow generated by X, φ^{h} is the extension of φ described in Lemma 2 in 3.4, and $\varphi^{\mathrm{h}}_t(u) := \varphi^{\mathrm{h}}(t,u)$ if (t,u) is in the domain of φ^{h}, and t is fixed.

Proof. (1) φ^{h}_t is a diffeomorphism between two open subsets of $\mathring{T}M$. However, for any vector u in the domain of φ^{h}_t, we may consider the derivative $((\varphi^{\mathrm{h}}_t)_*)_u$ as a mapping from T_pM onto $T_{\varphi_t(p)}M$, where $p := \tau(u)$, identifying φ^{h}_t with its restriction to $\mathring{T}_pM$, and identifying also the tangent spaces of a tangent space to M with the tangent space itself. This interpretation of the derivative of φ^{h}_t will be applied automatically in what follows.

(2) Since, by Lemma 2, φ^{h} is the flow generated by X^{h}, our claim is an easy consequence of the dynamic interpretation of the ordinary Lie derivative:

$$\begin{aligned}\big(\mathcal{L}_{X^{\mathrm{h}}}\widetilde{Y}\big)(u) :&= \mathcal{V}[X^{\mathrm{h}}, \mathrm{vl}\,\widetilde{Y}](u)\\ &= \mathcal{V}\lim_{t\to 0}\frac{(\varphi^{\mathrm{h}}_{-t})_*(\mathrm{vl}\,\widetilde{Y})(\varphi^{\mathrm{h}}_t(u)) - (\mathrm{vl}\,\widetilde{Y})(u)}{t}\\ &= \lim_{t\to 0}\frac{(\varphi^{\mathrm{h}}_{-t})_*\widetilde{Y}(\varphi^{\mathrm{h}}_t(u)) - \widetilde{Y}(u)}{t},\end{aligned}$$

taking into account the linearity of vl and the obvious relation
$\mathcal{V}\circ \mathrm{vl} = 1_{\mathrm{Sec}(\mathring{\pi})}$. □

Lemma 10. *Hypothesis and notation as above. If*

$$b : \mathrm{Sec}(\mathring{\pi}) \times \mathrm{Sec}(\mathring{\pi}) \to C^\infty(\mathring{T}M)$$

is a type $(0,2)$ tensor, then

$$\mathcal{L}_{X^{\mathrm{h}}} b = \lim_{t\to 0}\frac{(\varphi^{\mathrm{h}}_t)^* b - b}{t},$$

or, more precisely, for any vectors $u \in \mathring{T}M$; $v, w \in T_{\tau(u)}M$,

$$(*)\qquad \big(\mathcal{L}_{X^{\mathrm{h}}} b\big)_u(v,w) = \lim_{t\to 0}\frac{\big((\varphi^{\mathrm{h}}_t)^* b\big)_u(v,w) - b_u(v,w)}{t},$$

where $(\varphi^{\mathrm{h}}_t)^*$ denotes pull-back, given by

$$\big((\varphi^{\mathrm{h}}_t)^* b\big)_u(v,w) := b_{\varphi^{\mathrm{h}}_t(u)}\big(((\varphi^{\mathrm{h}}_t)_*)_u(v), ((\varphi^{\mathrm{h}}_t)_*)_u(w)\big).$$

Proof. Let, for brevity, $p := \tau(u)$. If $X(p) = 0$, then both sides of $(*)$ vanish. Otherwise, there is a positive real number ε and there are vector fields Y, Z on M such that

$$Y(\varphi_t(p)) = ((\varphi_t^{\mathrm{h}})_*)_u(v) \text{ and } Z(\varphi_t(p)) = ((\varphi_t^{\mathrm{h}})_*)_u(w),$$

whenever $|t| < \varepsilon$. Hence, identifying the basic sections $\widehat{Y}, \widehat{Z}$ with their 'principal parts' $Y \circ \tau, Z \circ \tau$, and applying the previous lemma, we obtain:

$$\begin{aligned}
\big(\mathcal{L}_{X^{\mathrm{h}}} b\big)_u(v,w) &= \big(\mathcal{L}_{X^{\mathrm{h}}} b\big)(\widehat{Y},\widehat{Z})(u) \\
&= \big(X^{\mathrm{h}} b(\widehat{Y},\widehat{Z}) - b(\mathcal{L}_{X^{\mathrm{h}}}\widehat{Y},\widehat{Z}) - b(\widehat{Y},\mathcal{L}_{X^{\mathrm{h}}}\widehat{Z})\big)(u) \\
&= \big(t \longmapsto b_{\varphi_t^{\mathrm{h}}(u)}\big(Y(\varphi_t(p)), Z(\varphi_t(p))\big)\big)'(0) \\
&- \big(t \longmapsto b_u\big((\varphi_{-t}^{\mathrm{h}})_* Y(\varphi_t(p)), w\big) + b_u\big(v, (\varphi_{-t}^{\mathrm{h}})_* Z(\varphi_t(p))\big)\big)'(0) \\
&= \big(t \longmapsto b_{\varphi_t^{\mathrm{h}}(u)}\big(((\varphi_t^{\mathrm{h}})_*)_u(v), ((\varphi_t^{\mathrm{h}})_*)_u(w)\big) - 2b_u(v,w)\big)'(0) \\
&= \big(t \longmapsto ((\varphi_t^{\mathrm{h}})^* b)_u(v,w)\big)'(0) \\
&= \lim_{t\to 0} \frac{((\varphi_t^{\mathrm{h}})^* b)_u(v,w) - b_u(v,w)}{t}.
\end{aligned}$$

□

Corollary 11. *If (M, F) is a Berwald manifold, $\mathcal{H}$ is its canonical connection, then the metric tensor of (M, F) is constant along the flow generated by any $\mathcal{H}$-horizontal lift Z^{h} of $Z \in \mathfrak{X}(M)$.*

Proof. By Lemma 10, it is enough to check that $\mathcal{L}_{Z^{\mathrm{h}}} g = 0$. Choosing two vector fields X, Y on M, we calculate:

$$\begin{aligned}
(\mathcal{L}_{Z^{\mathrm{h}}} g)(\widehat{X},\widehat{Y}) &= Z^{\mathrm{h}} g(\widehat{X},\widehat{Y}) - g(\mathcal{L}_{Z^{\mathrm{h}}}\widehat{X},\widehat{Y}) - g(\widehat{X},\mathcal{L}_{Z^{\mathrm{h}}}\widehat{Y}) \\
&= Z^{\mathrm{h}}(X^{\mathsf{v}} Y^{\mathsf{v}} E) - g(\mathcal{V}[Z^{\mathrm{h}}, X^{\mathsf{v}}], \widehat{Y}) - g(\widehat{X}, \mathcal{V}[Z^{\mathrm{h}}, Y^{\mathsf{v}}]) \\
&\overset{\text{(B3)}}{=} Z^{\mathrm{h}}(X^{\mathsf{v}} Y^{\mathsf{v}} E) - g\big(\widehat{D_Z X}, \widehat{Y}\big) - g\big(\widehat{X}, \widehat{D_Z Y}\big) \\
&= Z^{\mathrm{h}}(X^{\mathsf{v}} Y^{\mathsf{v}} E) - (D_Z X)^{\mathsf{v}}(Y^{\mathsf{v}} E) - X^{\mathsf{v}}((D_Z Y)^{\mathsf{v}} E) \\
&\overset{\text{(B3)}}{=} Z^{\mathrm{h}}(X^{\mathsf{v}} Y^{\mathsf{v}} E) - [Z^{\mathrm{h}}, X^{\mathsf{v}}](Y^{\mathsf{v}} E) - X^{\mathsf{v}}([Z^{\mathrm{h}}, Y^{\mathsf{v}}] E) \\
&= X^{\mathsf{v}} Y^{\mathsf{v}} (Z^{\mathrm{h}} E) = 0,
\end{aligned}$$

since $\mathcal{H}$ is compatible with E. □

Proposition 12 (T. Aikou [1]). *A Finsler manifold is a Berwald manifold if, and only if, there exists a Riemannian metric g_M on M such that for every vector field Z on M, $\mathcal{L}_{Z^{\mathrm{h}}} \widehat{g}_M = 0$. Here the horizontal lift is taken with* *respect to the canonical connection of (M, F), and $\widehat{g}_M$ is the natural lift of g_M into $\mathcal{T}^0_2(\mathring{\pi})$ given on basic sections by*

$$\widehat{g}_M(\widehat{X}, \widehat{Y}) := (g_M(X, Y))^{\mathsf{v}}; \quad X, Y \in \mathfrak{X}(M).$$

Proof. Sufficiency. Let X and Y be vector fields on M. Then, as above,

$$\begin{aligned}(\mathcal{L}_{Z^{\mathrm{h}}}\widehat{g}_M)(\widehat{X}, \widehat{Y}) &= Z^{\mathrm{h}}\widehat{g}_M(\widehat{X}, \widehat{Y}) - \widehat{g}_M\big(\mathcal{V}[Z^{\mathrm{h}}, X^{\mathsf{v}}], \widehat{Y}\big) - \widehat{g}_M\big(\widehat{X}, \mathcal{V}[Z^{\mathrm{h}}, Y^{\mathsf{v}}]\big) \\ &= Z^{\mathrm{h}}\big(g_M(X, Y)\big)^{\mathsf{v}} - \widehat{g}_M(\nabla_{Z^{\mathrm{h}}}\widehat{X}, \widehat{Y}) - \widehat{g}_M\big(\widehat{X}, \nabla_{Z^{\mathrm{h}}}\widehat{Y}\big).\end{aligned}$$

Here, as it can be seen at once,

$$Z^{\mathrm{h}}\big(g_M(X, Y)\big)^{\mathsf{v}} = \big(Z g_M(X, Y)\big)^{\mathsf{v}},$$

so by our condition $\mathcal{L}_{Z^{\mathrm{h}}}\widehat{g}_M = 0$ it follows that

$$\widehat{g}_M(\nabla_{Z^{\mathrm{h}}}\widehat{X}, \widehat{Y}) + \widehat{g}_M(\widehat{X}, \nabla_{Z^{\mathrm{h}}}\widehat{Y}) = \big(Z g_M(X, Y)\big)^{\mathsf{v}}.$$

Permuting Z, X and Y cyclically, we obtain

$$\begin{aligned}\widehat{g}_M(\nabla_{X^{\mathrm{h}}}\widehat{Y}, \widehat{Z}) + \widehat{g}_M(\widehat{Y}, \nabla_{X^{\mathrm{h}}}\widehat{Z}) &= \big(X g_M(Y, Z)\big)^{\mathsf{v}}, \\ \widehat{g}_M(\nabla_{Y^{\mathrm{h}}}\widehat{Z}, \widehat{X}) + \widehat{g}_M(\widehat{Z}, \nabla_{Y^{\mathrm{h}}}\widehat{X}) &= \big(Y g_M(Z, X)\big)^{\mathsf{v}}.\end{aligned}$$

Adding both sides of the last two relations and subtracting the preceding one, we find

$$\begin{aligned}&\widehat{g}_M\big(\nabla_{X^{\mathrm{h}}}\widehat{Y} + \nabla_{Y^{\mathrm{h}}}\widehat{X}, \widehat{Z}\big) + \widehat{g}_M\big(\nabla_{Y^{\mathrm{h}}}\widehat{Z} - \nabla_{Z^{\mathrm{h}}}\widehat{Y}, \widehat{X}\big) + \widehat{g}_M\big(\nabla_{X^{\mathrm{h}}}\widehat{Z} - \nabla_{Z^{\mathrm{h}}}\widehat{X}, \widehat{Y}\big) \\ &\quad = \big(X g_M(Y, Z) + Y g_M(Z, X) - Z g_M(X, Y)\big)^{\mathsf{v}}.\end{aligned}$$

By Lemma 6, the left-hand side of this relation can be written in the form

$$\widehat{g}_M\big(2\nabla_{X^{\mathrm{h}}}\widehat{Y} - \widehat{[X, Y]}, \widehat{Z}\big) + \widehat{g}_M\big(\widehat{[Y, Z]}, \widehat{X}\big) + \widehat{g}_M\big(\widehat{[X, Z]}, \widehat{Y}\big),$$

so we obtain

$$\begin{aligned}2\widehat{g}_M(\nabla_{X^{\mathrm{h}}}\widehat{Y}, \widehat{Z}) &= \big(X g_M(Y, Z) + Y g_M(Z, X) - Z g_M(X, Y)\big)^{\mathsf{v}} \\ &\quad + \big(- g_M(X, [Y, Z]) + g_M(Y, [Z, X]) + g_M(Z, [X, Y])\big)^{\mathsf{v}}.\end{aligned}$$

If D is the Levi-Civita derivative of (M, g_M), then, by the Koszul formula, the right-hand side of the last relation is just

$$2\big(g_M(D_X Y, Z)\big)^{\mathsf{v}} = 2\widehat{g}_M(\widehat{D_X Y}, \widehat{Z}),$$

hence

$$\widehat{g}_M\big(\nabla_{X^{\mathrm{h}}}\widehat{Y} - \widehat{D_XY}, \widehat{Z}\big) = 0 \textit{ for all } X, Y, Z \in \mathfrak{X}(M).$$

This implies that

$$\mathrm{vl}\,\nabla_{X^{\mathrm{h}}}\widehat{Y} = (D_XY)^{\mathsf{v}} \textit{ for all } X, Y \in \mathfrak{X}(M),$$

whence, by (B4), (M, F) is a Berwald manifold.

Necessity. Assume that (M, F) is a positive definite Berwald manifold. By a celebrated observation of Z. I. Szabó [19], there exists a Riemannian metric g_M on M whose Levi-Civita derivative is the base covariant derivative D of (M, F). (For an instructive, quite recent proof of this fact we refer to Vincze's paper [24]. In the next section we shall see that it works under more general assumptions.)

g_M satisfies our requirement: for any vector fields X, Y, Z on M we have

$$\begin{aligned}
(\mathcal{L}_{Z^{\mathrm{h}}}\widehat{g}_M)(\widehat{X}, \widehat{Y}) &= Z^{\mathrm{h}}\widehat{g}_M(\widehat{X}, \widehat{Y}) - \widehat{g}_M(\mathcal{L}_{Z^{\mathrm{h}}}\widehat{X}, \widehat{Y}) - \widehat{g}_M(\widehat{X}, \mathcal{L}_{Z^{\mathrm{h}}}\widehat{Y}) \\
&= \big(Zg_M(X, Y)\big)^{\mathsf{v}} - \widehat{g}_M(\mathcal{V}[Z^{\mathrm{h}}, X^{\mathsf{v}}], \widehat{Y}) - \widehat{g}_M(\widehat{X}, \mathcal{V}[Z^{\mathrm{h}}, Y^{\mathsf{v}}]) \\
&\overset{\text{(B3)}}{=} \big(Zg_M(X, Y) - g_M(D_ZX, Y) - g_M(X, D_ZY)\big)^{\mathsf{v}} \\
&= \big(Dg_M(Z, X, Y)\big)^{\mathsf{v}} = 0,
\end{aligned}$$

since D is a metric derivative on (M, g_M). □

7 On Matveev's generalization of Berwald manifolds

7.1. In what follows, by a *vector space* we shall mean a finite dimensional (but non-trivial) real vector space endowed with the canonical linear topology. Sometimes, tacitly, we also assume that the considered n-dimensional vector space V is a manifold whose smooth structure is defined by a linear bijection $V \to \mathbb{R}^n$.

We recall that in the context of these vector spaces we have a natural and efficient concept of differentiability of mappings. Namely, let V and W be vector spaces, and let $\mathrm{L}(V, W)$ be the vector space of linear mappings of V into W. Let $\mathcal{U}$ be an open subset of V. A mapping $\varphi : \mathcal{U} \to W$ is called *differentiable at a point* $p \in \mathcal{U}$ if for some $\varphi'(p) \in \mathrm{L}(V, W)$

$$\lim_{t\to 0}\frac{\varphi(p + tv) - \varphi(p)}{t} = \varphi'(p)(v), \quad v \in V.$$

$\varphi : \mathcal{U} \to W$ is *differentiable* if it is differentiable at every point of $\mathcal{U}$; then its *derivative* is the mapping

$$\varphi' : \mathcal{U} \longrightarrow \mathrm{L}(V, W), \quad p \longmapsto \varphi'(p).$$

φ is *twice differentiable* if φ' is differentiable; the derivative of φ' is a mapping

$$\varphi'' : \mathcal{U} \longrightarrow \mathrm{L}\big(V, \mathrm{L}(V, W)\big), \quad p \longmapsto \varphi''(p),$$

called the *second derivative* of φ. Here $\mathrm{L}(V, \mathrm{L}(V, W))$ may be canonically identified with the vector space $\mathrm{L}^2(V, W)$ of bilinear mappings $V \times V \to W$. For further fine details we refer to [13], Ch.1.

7.2. Let V and W be vector spaces, r a real number, and $\mathcal{U}$ a (nonempty) subset of V. A mapping $\varphi : \mathcal{U} \to W$ is said to be *positive-homogeneous of degree* r, briefly r^+*-homogeneous*, if for each positive real number λ and each $v \in \mathcal{U}$,

$$\lambda v \in \mathcal{U} \text{ and } \varphi(\lambda v) = \lambda^r \varphi(v).$$

If, in particular, $\mathcal{U}$ is an open subset of V with the property that $\lambda\mathcal{U} \subset \mathcal{U}$ for all positive $\lambda \in \mathbb{R}$, and $f : \mathcal{U} \to \mathbb{R}$ is a differentiable function, then, as it has been observed by Euler,

$$f \text{ is } r^+\text{-homogeneous if, and only if, } f'(v)(v) = rf(v) \text{ for all } v \in \mathcal{U}.$$

7.3. We canonically identify the tangent space T_pV of a vector space V at a point p with V. Then a (smooth) vector field on an open subset $\mathcal{U}$ of V is just a smooth mapping $X : \mathcal{U} \to V$. As in the general theory of manifolds, we denote by $\mathfrak{X}(\mathcal{U})$ the $C^\infty(\mathcal{U})$-module of vector fields on $\mathcal{U}$. $\mathfrak{X}(\mathcal{U})$ *is generated by the constant vector fields* of the form

$$X : \mathcal{U} \longrightarrow V, \quad p \longmapsto X(p) := v \text{ for all } p \in \mathcal{U}.$$

We denote by $Z_{\mathcal{U}}$ the *radial vector field*

$$\mathcal{U} \longrightarrow V, \quad Z_{\mathcal{U}}(p) := p.$$

It plays the same role as the Liouville vector field (2.4) in the general theory. For simplicity, the suffix $\mathcal{U}$ will be omitted.

If f is a differentiable function on $\mathcal{U}$ and $X \in \mathfrak{X}(\mathcal{U})$, then

$$(Xf)(p) = X(p)f = f'(p)(X(p)), \text{ for all } p \in \mathcal{U}.$$

In particular,

$$(Zf)(p) = f'(p)(p), \ p \in \mathcal{U}.$$

It follows that f *is* r^+*-homogeneous if, and only if, (*$\lambda\mathcal{U} \subset \mathcal{U}$ *for all positive* $\lambda \in \mathbb{R}$ *and)* $Zf = rf$.

Now let $f \in C^\infty(\mathcal{U})$; $X, Y \in \mathfrak{X}(\mathcal{U})$. Then at each point $p \in \mathcal{U}$,

$$X(Yf)(p) = f''(p)\big(X(p), Y(p)\big) + Y'(p)(X(p))(f)$$

therefore

$$[X, Y](p) = Y'(p)(X(p)) - X'(p)(Y(p)).$$

From this we see that *if* X *is a constant vector field on* $\mathcal{U}$ *and* $Z \in \mathfrak{X}(\mathcal{U})$ *is the radial vector field, then* $[X, Z] = X$.

7.4. Let V be a vector space, k a positive integer, and let $\mathrm{L}^k(V)$ denote the vector space of k-linear real-valued functions on V. If $\mathcal{U}$ is an open subset of V, then any k-linear function $A \in \mathrm{L}^k(V)$ may be interpreted as a type $(0, k)$ *tensor field* whose value A_p at a point $p \in \mathcal{U}$ is just A, i.e.,

$$A_p(v_1, \ldots, v_k) := A(v_1, \ldots, v_k); \quad (v_1, \ldots, v_k) \in V^k.$$

Equivalently, we may consider A as a $C^\infty(\mathcal{U})$-multilinear mapping $(\mathfrak{X}(\mathcal{U}))^k \to C^\infty(\mathcal{U})$, given by

$$A(X_1, \ldots, X_k)(p) := A(X_1(p), \ldots, X_k(p)); \quad X_1, \ldots, X_k \in \mathfrak{X}(\mathcal{U}),\ p \in \mathcal{U}.$$

Keeping these in mind, for any vector field X on $\mathcal{U}$ we may define the contracted tensor field $i_X A$ and the Lie derivative $\mathcal{L}_X A$ in the usual manner. If A is skew-symmetric, we may also speak of the exterior derivative dA.

Lemma 13. *Let* $\mathcal{U}$ *be an open subset of a vector space* V*, and let* $A \in \mathrm{L}^k(V)$ *be a* k*-form, considered as a type* $(0, k)$ *tensor field on* $\mathcal{U}$*. Then* $\mathcal{L}_Z A = kA$*, where* $Z \in \mathfrak{X}(\mathcal{U})$ *is the radial vector field.*

Proof. It is enough to check the relation for constant vector fields $X_1, \ldots, X_k$ in $\mathcal{U}$. Then

$$\begin{aligned}(\mathcal{L}_Z A)(X_1, \ldots, X_k) : &= Z\big(A(X_1, \ldots, X_k)\big) - \sum_{i=1}^{k} A(X_1, \ldots, [Z, X_i], \ldots, X_k) \\ &= \sum_{i=1}^{k} A(X_1, \ldots, X_k) = kA(X_1, \ldots, X_k)\end{aligned}$$

since the function $A(X_1, \ldots, X_k)$ is constant, while $[Z, X_i] = -X_i$ $(i \in \{1, \ldots, k\})$ as we have seen in 7.3. □

7.5. By a *pre-Finsler norm* on a vector space V we mean a function $f : V \to \mathbb{R}$ such that

(i) f is of class C^2 on $V \setminus \{0\}$;

(ii) f is 1^+-homogeneous.

Then $\psi := \frac{1}{2} f^2$ is the *energy* associated to f. A pre-Finsler norm $f : V \to \mathbb{R}$ is said to be a *gauge* if

(iii) $f(v) > 0$ for all $v \in V \setminus \{0\}$;

(iv) f is subadditive, i.e., $f(v + w) \leq f(v) + f(w)$ *for all* $v, w \in V$.

Notice that the energy ψ is 2^+-homogeneous and differentiable at 0, with derivative $0 \in V^* := \mathrm{L}(V, \mathbb{R})$. It follows also immediately that a gauge $f : V \to \mathbb{R}$ is a convex function:

$$f\big((1-t)v + tw\big) \leq (1-t)f(v) + tf(w)$$

for all $v, w \in V$ and $t \in [0, 1]$. Thus, by condition (i), the second derivatives

$$f''(u) : V \times V \longrightarrow \mathbb{R}, \quad u \in V \setminus \{0\}$$

are positive semidefinite.

A vector space equipped with a pre-Finsler norm or with a gauge will be called a *pre-Finsler* or a *gauge vector space*, respectively.

7.6. Let (V, f) be an n-dimensional gauge vector space, $n \geq 2$. Then

$$B := \big\{v \in V \big| f(v) \leq 1\big\} \textit{ and } S := \big\{v \in V \big| f(v) = 1\big\}$$

are the *f-unit ball* and the *f-unit sphere* of (V, f), respectively. The tangent space $T_a S$ of S at a point $a \in S$ may be identified with the $(n-1)$-dimensional subspace $\mathrm{Ker}(f'(a))$ of V. Notice that $a \notin T_a S$, since

$$f'(a)(a) = f(a) > 0$$

by condition (ii) and (iii) in 7.5.

Let an orientation of V be given, and let $\Omega : V^n \to \mathbb{R}$ be the unique n-form such that $\int_B \Omega = 1$. Then the $(n-1)$-form ω on S given by

$$\omega_a(v_2, \dots, v_n) := \Omega(a, v_2, \dots, v_n), \quad a \in S;\ v_i \in T_a S \subset V,\ i \in \{2, \dots, n\}$$

orients S; this is the orientation of S induced by the orientation of V (cf. [12] 3.21, Ex. 2). Equivalently, ω may simply be defined by the formula

$$\omega := i_Z \Omega \restriction S.$$

The following observation is a slight generalization of Lemma 2 of Vincze's paper [24], with essentially the same proof.

Lemma 14. *Let (V, f) be a gauge vector space of dimension $n \geq 2$. If $h : V \to \mathbb{R}$ is a 0^+-homogeneous function, of class C^1 outside the zero, then*

$$\int_B h = \frac{1}{n} \int_S h.$$

Proof. By a slight abuse of notation, we are going to use Stokes' formula. Since $Zh = i_Z dh = 0$, and, by Lemma 13, $\Omega = \frac{1}{n}\mathcal{L}_Z\Omega$, we obtain

$$\begin{aligned}\int_B h :&= \int_B h\Omega = \frac{1}{n}\int_B \mathcal{L}_Z h\Omega = \frac{1}{n}\int_B (i_Z \circ d + d \circ i_Z)h\Omega \\ &= \frac{1}{n}\int_B d(i_Z h\Omega) = \frac{1}{n}\int_S i_Z h\Omega = \frac{1}{n}\int_S h\omega =: \frac{1}{n}\int_S h.\end{aligned}$$

□

7.7. We shall now associate a Euclidean structure to a gauge, by the averaged metric construction of Matveev et al. [16].

Lemma 15. *Let (V, f) be a gauge vector space with energy function $\psi = \frac{1}{2}f^2$. If for each $v, w \in V$,*

$$b(v, w) := \int_S \big(u \longmapsto \psi''(u)(v, w)\big)\omega,$$

then b is a positive definite scalar product on V.

Proof. Bilinearity and symmetry of b are obvious, since for all $u \in V \setminus \{0\}$, the second derivative $\psi''(u) : V \times V \to \mathbb{R}$ has these properties. In addition, $\psi''(u)$ is positive semidefinite, since ψ is also a convex function. If $v \in V \setminus \{0\}$ and $a := \frac{1}{f(v)}v$, then $a \in S$ and

$$\begin{aligned}\psi''(a)(v, v) &= \psi''\left(\frac{1}{f(v)}v\right)(v, v) = \psi''(v)(v, v) \\ &= \psi'(v)(v) = 2\psi(v) = f(v)^2 > 0,\end{aligned}$$

taking into account that ψ'', ψ' and ψ are 0^+-, 1^+- and 2^+-homogeneous, respectively, and applying repeatedly Euler's relation (7.2). Thus the function

$$u \in S \longmapsto \psi''(u)(v, v) \in \mathbb{R}$$

with fixed $v \in V \setminus \{0\}$, is positive at the point $a \in S$, therefore, by continuity, it is positive also in a neighbourhood of a.

This proves the positive definiteness of b. □

7.8. We recall that a *rough section* in a vector bundle $\pi : N \to M$ is any mapping $s : M \to N$ such that $\pi \circ s = 1_M$. In what follows, we shall use the term 'rough tensor field' in this sense.

Now suppose that D is a covariant derivative operator on the manifold M. We say that a rough tensor field A of type $(0, k)$ on M is *invariant by* D*-parallel translation*, if for any two points p, q in M and curve segment γ from p to q, for the parallel translation $P_\gamma : T_pM \to T_qM$ along γ we have $P_\gamma^* A_q = A_p$, where

$$(P_\gamma^* A_q)(v_1, \dots, v_k) := A_q\big(P_\gamma(v_1), \dots, P_\gamma(v_k)\big);\ v_i \in T_pM,\ i \in \{1, \dots, k\}.$$

Lemma 16. *Let D be a covariant derivative on M. If a rough covariant tensor field on M is invariant by D-parallel translation, then it is actually smooth.*

Proof. The property in question is local, so it is enough to show the desired smoothness in a neighbourhood of an arbitrarily chosen point of M. So let $p \in M$, and let $\mathcal{U}$ be a normal neighbourhood of p. If $(e_i)_{i=1}^n$ is a basis of T_pM, then it can be extended to a frame $(E_i)_{i=1}^n$ for TM over $\mathcal{U}$ by parallel translations along geodesics starting from p. Differential equation theory (smoothness of ODE solutions) guarantees that the vector fields E_i are smooth. Now, if a rough covariant tensor field A on M is invariant by D-parallel translation, then the components of A with respect to $(E_i)_{i=1}^n$ are constant; hence A is smooth over $\mathcal{U}$. □

7.9. We say that a function $F : TM \to \mathbb{R}$ is a *pre-Finsler function*, resp. a *gauge function* for M, if it is of class C^2 on $\overset{\circ}{T}M$ and $F_p := F \upharpoonright T_pM$ is a pre-Finsler norm, resp. a gauge for each $p \in M$. As in the Finslerian case, the function $E := \frac{1}{2}F^2$ is called the *energy function* associated to the pre-Finsler function (or gauge function) F. A manifold equipped with a pre-Finsler function (resp. a gauge function) is said to be a *pre-Finsler manifold* (resp. a *gauge manifold*). A gauge manifold (M, F) becomes a Finsler manifold, if F is smooth on $\overset{\circ}{T}M$, and for each $p \in M$, $u \in T_pM \setminus \{0\}$

$$(F_p)''(u)(v, v) = 0 \textit{ implies } v \in \operatorname{span}(u);$$

see [13], Prop 4.5.

7.10. Now we are in a position to introduce the main actor of this chapter, and to formulate and prove Matveev's generalization of Szabó's theorem on Riemann metrizability of the base covariant derivative of a Berwald manifold.

We say that a triplet (M, F, D) is a *Berwald – Matveev manifold* if F is a gauge function for M and D is a torsion-free covariant derivative on M which is *compatible with* F in the sense that the parallel translations with respect to D preserve the F-norms of tangent vectors to M. The next Proposition assures that this compatibility condition determines the covariant derivative D uniquely.

Proposition 17. *Let* (M, F, D) *be a Berwald – Matveev manifold of dimension* n, $n \geq 2$. *For every point* $p \in M$, *let*

$F_p := F \restriction T_pM$; $E_p := E \restriction T_pM$, $E = \frac{1}{2}F^2$;

$B_p \subset T_pM$ *the unit* F_p*-ball;*

$S_p \subset T_pM$ *the unit* F_p*-sphere;*

$Z_p : T_pM \to T_pM$, $v \mapsto Z_p(v) := v$ *the radial vector field on* T_pM;

Ω_p *the unique volume form on* T_pM *such that* $\int_{B_p} \Omega_p = 1$;

$\omega_p := i_{Z_p}\Omega_p \restriction S_p$ *the induced volume form on* S_p.

Define a type $(0, 2)$ *rough tensor field* g_M *on* M *by prescribing its value* $(g_M)_p$ *at a point* $p \in M$ *according to Lemma 15, that is, by the rule*

$$(g_M)_p(v, w) := \int_{S_p} \big(u \longmapsto (E_p)''(u)(v, w)\big)\omega_p \ ; \quad v, w \in T_pM.$$

Then g_M *is a (positive definite) Riemannian metric on* M *whose Levi-Civita derivative is* D.

Proof. By Lemma 15, $(g_M)_p$ is a positive definite scalar product on T_pM for each $p \in M$. So it is enough to check that g_M is invariant by D-parallel translation: then Lemma 16 implies that g_M is a Riemannian metric on M, and it follows at once that the Levi-Civita derivative of (M, g_M) is just D. Let $p, q \in M$, and let $\gamma : [0, 1] \to M$ be a curve segment connecting p with q, i.e. $\gamma(0) = p$, $\gamma(1) = q$. Consider the parallel translation $P_\gamma : T_pM \to T_qM$ along γ.

Claim 1. $P_\gamma^*\Omega_q = \pm\Omega_p$.

To see this, let us first note that P_γ preserves the F-norms of the tangent vectors to M by our compatibility condition, so it is a diffeomorphism of B_p onto B_q. Thus, applying the 'change of variables' formula for the integral of differential forms, we obtain

$$\int_{B_p} P_\gamma^*\Omega_q = \pm \int_{B_q} \Omega_q = \pm 1.$$

This implies (by the uniqueness of Ω_p) the desired relation.

Claim 2. *For fixed $v, w \in T_pM$, let*

$$\varphi(z) := (E_q)''(z)(P_\gamma(v), P_\gamma(w)), \quad z \in T_qM \setminus \{0\}.$$

Then φ is 0^+-homogeneous and

$$\varphi \circ P_\gamma(u) = (E_p)''(u)(v, w), \quad u \in T_pM \setminus \{0\}.$$

The first assertion is obvious, since $(E_q)''$ is 0^+-homogeneous. The formula for $\varphi \circ P_\gamma$ may be checked immediately, using the definition of the second derivative of E_q at $P_\gamma(u)$ and taking into account that P_γ is a linear mapping.

Claim 3. $P_\gamma^*(g_M)_q = (g_M)_p$.

Let $v, w \in T_pM$. Then

$$\begin{aligned}
(P_\gamma^*(g_M)_q)(v, w) &= (g_M)_q(P_\gamma(v), P_\gamma(w)) \\
&:= \int_{S_q} \big(z \longmapsto (E_q)''(z)(P_\gamma(v), P_\gamma(w))\big)\omega_q \\
&\overset{\text{Lemma 14}}{=} n \int_{B_q} \big(z \longmapsto (E_q)''(z)(P_\gamma(v), P_\gamma(w))\big)\Omega_q \\
&= n \int_{B_q} \varphi\, \Omega_q \overset{\text{Claim 1}}{=} \pm n \int_{B_q} \varphi (P_\gamma^{-1})^* \Omega_p \\
&\overset{\text{change of variables}}{=} n \int_{B_p} (\varphi \circ P_\gamma)\Omega_p \overset{\text{Lemma 14}}{=} \int_{S_p} (\varphi \circ P_\gamma)\omega_p \\
&\overset{\text{Claim 2}}{=} \int_{S_p} \big(u \longmapsto (E_p)''(u)(v, w)\big)\omega_p = (g_M)_p(v, w).
\end{aligned}$$

This concludes the proof. □

7.11. Let (M, F) be a *pre-Finsler manifold*; $[a, b] \subset \mathbb{R}$, where $a \neq b$, a compact interval, and choose two points, p and q, in M. Denote by $\mathcal{C}(p, q)$ the set of all C^1 curve segments $\gamma : [a, b] \to M$ such that $\gamma(a) = p$, $\gamma(b) = q$. Define the *energy functional*

$$\mathbb{E} : \mathcal{C}(p, q) \to \mathbb{R}$$

by

$$\mathbb{E}(\gamma) := \int_a^b E \circ \dot\gamma = \int_a^b E(\dot\gamma(t))dt, \quad \gamma \in \mathcal{C}(p, q).$$

The regular extremals of $\mathbb{E}$ (i.e., the critical 'points' $\gamma \in \mathcal{C}(p, q)$ with $\dot\gamma(t) \neq 0$ for all $t \in [a, b]$) are said to be the *geodesics* corresponding to (or of) F. In terms of coordinates, a regular curve γ in $\mathcal{C}(p,q)$ is a geodesic of F if, and only if, in any induced chart $\left(\tau^{-1}(\mathcal{U}), (x^i, y^i)_{i=1}^n\right)$ such that $\operatorname{Im}(\gamma) \cap \mathcal{U} \neq \emptyset$, the *Euler – Lagrange equations*

$$\frac{\partial E}{\partial x^i} \circ \dot{\gamma} - \left(\frac{\partial E}{\partial y^i} \circ \dot{\gamma}\right)' = 0, \quad i \in \{1, \dots, n\}$$

are satisfied. If, in particular, F is a Finsler function, then the concept of an F-geodesic just introduced yields the same curves as our earlier definition in 4.2(a).

Remark. One may also consider the arclength functional

$$\mathbb{F} : \mathcal{C}(p,q) \to \mathbb{R}, \quad \gamma \mapsto \mathbb{F}(\gamma) := \int_a^b F \circ \dot{\gamma} = \int_a^b F(\dot{\gamma}(t))dt.$$

It is not difficult to show (see, e.g., [11], p. 185) that the set of the geodesics corresponding to F coincides with the set of positive constant speed extremals of $\mathbb{F}$.

Proposition 18. *Let (M, F, D) be an at least two-dimensional Berwald – Matveev manifold with associated Riemannian metric g_M defined by Proposition 17. Then any geodesic of the Riemannian manifold (M, g_M) is also a geodesic of F.*

Proof. Let E_R be the energy function associated to the Riemannian metric g_M, given by

$$E_R(v) := \frac{1}{2}(g_M)_{\tau(v)}(v, v), \quad v \in TM.$$

We define the function

$$\widetilde{E} := E_R + E,$$

and let $\widetilde{F} := \sqrt{2\widetilde{E}}$. Then $\widetilde{F}$ is of class C^2 on $\mathring{T}M$ and satisfies (F2). Since at each point $p \in M$ and for any tangent vectors $u \in \mathring{T}_pM$, $v, w \in T_pM$ we have

$$\left(\widetilde{E}_p\right)''(u)(v, w) := (g_M)_p(v, w) + (E_p)''(u)(v, w),$$

and on the right-hand side of this relation $(E_p)''(u)$ is positive semidefinite, while $(g_M)_p$ is positive definite, it follows that

$$\left(\widetilde{E}_p\right)''(u) = \left(\widetilde{E} \restriction T_pM\right)''(u)$$

is positive definite. This implies that $\widetilde{F}$ satisfies condition (F4) in 4.1, therefore $\left(M, \widetilde{F}\right)$ is a Finsler manifold of class C^2. Since the parallel translations with respect to D preserve the E-norms and, by Proposition 17, also the E_R-norms of the tangent vectors to M, it follows that they also preserve the $\widetilde{E}$-norms, and hence the $\widetilde{F}$-norms. So, by (B8) in Proposition 7, $\big(M, \widetilde{F}\big)$ is a Berwald manifold with Finsler function of class C^2, and the set of geodesics of $\widetilde{F}$ coincides with the set of geodesics corresponding to $F_R := \sqrt{2E_R}$.

In an induced chart $\big(\tau^{-1}(\mathcal{U}), (x^i, y^i)_{i=1}^n\big)$, the Euler–Lagrange equations of the energy functional

$$\widetilde{\mathbb{E}} : \mathcal{C}(p,q) \to \mathbb{R}, \quad \gamma \mapsto \widetilde{\mathbb{E}}(\gamma) := \int_a^b \widetilde{E} \circ \dot{\gamma} = \int_a^b E_R \circ \dot{\gamma} + \int_a^b E \circ \dot{\gamma}$$

take the form

$$\begin{aligned} 0 &= \frac{\partial E}{\partial x^i} \circ \dot{\gamma} - \left(\frac{\partial E}{\partial y^i} \circ \dot{\gamma}\right)' + \left(\frac{\partial E_R}{\partial x^i} \circ \dot{\gamma} - \left(\frac{\partial E_R}{\partial y^i} \circ \dot{\gamma}\right)'\right) \\ &= \frac{\partial E}{\partial x^i} \circ \dot{\gamma} - \left(\frac{\partial E}{\partial y^i} \circ \dot{\gamma}\right)', \end{aligned}$$

since, as we have just seen, if γ is a geodesic of $\widetilde{F}$, then it is also a geodesic of F_R – and vice versa. Thus it follows that the geodesics of F_R, i.e., of the Riemannian manifold (M, g_M), are also geodesics of the gauge function F. □

7.12. We present a further natural and simple construction to show that if (M, F, D) is a Berwald–Matveev manifold, then D is the Levi-Civita derivative of a Riemannian metric on M.

In what follows, by an *ellipsoid* on a vector space V we mean the unit ball of a positive definite scalar product $b : V \times V \to \mathbb{R}$, i.e., a set of the form $\mathcal{E}(b) := \{v \in V | b(v,v) \leq 1\}$. Ellipsoids are preserved by linear isomorphisms, namely, if $\Phi : V \to W$ is an isomorphism, then

$$\Phi\big(\mathcal{E}(b)\big) = \mathcal{E}\big((\Phi^{-1})^* b\big).$$

By the classical Loewner–Behrend theorem (see, e.g., [5]), if V is endowed with a Lebesgue measure and K is a compact subset of V with non-empty interior, then there exists a unique least-volume ellipsoid containing K. We call this ellipsoid the *Loewner ellipsoid* determined by K.

Now, keeping the notation of Proposition 17, consider a Berwald–Matveev manifold (M, F, D). At each point $p \in M$, denote by $\mathcal{E}(b_p)$ the Loewner ellipsoid determined by the unit F_p-ball B_p. Then

$$g_L : p \in M \longmapsto (g_L)_p := b_p \in \mathrm{L}^2(T_pM, \mathbb{R})$$

is a rough Riemannian metric on M; we show that it is actually a Riemannian metric.

As in the proof of Proposition 17, let p and q be two points in M, and let γ be a curve segment connecting p with q. Then, by our previous remark,

$$P_\gamma\big(\mathcal{E}(b_p)\big) = \mathcal{E}\big((P_\gamma^{-1})^* b_p\big)$$

is an ellipsoid in T_qM. Since P_γ preserves the F-norm of tangent vectors to M, we have $B_q \subset P_\gamma\big(\mathcal{E}(b_p)\big)$. If $\mathcal{E}_q$ is another ellipsoid containing B_q, then $P_\gamma^{-1}(\mathcal{E}_q)$ is an ellipsoid in T_pM containing B_p, and

$$\begin{aligned}\int_{\mathcal{E}_q} \Omega_q &\overset{\text{Claim 1, 7.10}}{=} \pm \int_{\mathcal{E}_q} (P_\gamma^{-1})^* \Omega_p \\ &\overset{\text{change of variables}}{=} \int_{P_\gamma^{-1}(\mathcal{E}_q)} \Omega_p \geq \int_{\mathcal{E}(b_p)} \Omega_p \\ &\overset{\text{Claim 1, 7.10}}{=} \pm \int_{\mathcal{E}(b_p)} P_\gamma^* \Omega_q \overset{\text{change of variables}}{=} \int_{P_\gamma(\mathcal{E}(b_p))} \Omega_q \\ &= \int_{\mathcal{E}((P_\gamma^{-1})^* b_p)} \Omega_q.\end{aligned}$$

This implies that $P_\gamma\big(\mathcal{E}(b_p)\big)$ is the least-volume ellipsoid containing B_q, therefore $\mathcal{E}\big((P_\gamma^{-1})^* b_p\big) = \mathcal{E}(b_q)$ and hence $P_\gamma^*(g_L)_q = (g_L)_p$. Thus, by Lemma 16, g_L is smooth, so it is indeed a Riemannian metric on M. It is clear from the construction that the Levi-Civita derivative for g_L is the given covariant derivative D.

We note finally that *if the holonomy group of D is irreducible at a point of M*, then g_L is proportional to the Riemannian metric g_M constructed above. In general, if g_1 and g_2 are two Riemannian metrics on M such that $Dg_1 = Dg_2 = 0$, then – under the irreducibility of the holonomy group of D – we have $g_2 = \lambda g_1$, where λ is a positive real number.

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