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AUTOMORPHISMS OF EHRESMANN CONNECTIONS

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Abstract. We prove that a diffeomorphism of a manifold with an Ehresmann connection is an automorphism of the Ehresmann connection, if and only if, it is a totally geodesic map (i.e., sends the geodesics, considered as parametrized curves, to geodesics) and preserves the strong torsion of the Ehresmann connection. This result generalizes and to some extent strengthens the classical theorem on the automorphisms of a D-manifold (manifold with covariant derivative).

1. Introduction

It is well-known (and almost trivial) that two covariant derivative operators on a manifold are equal, if and only if, they have the same geodesics and equal torsion tensors. Heuristically, this implies that a diffeomorphism of a manifold with a covariant derivative is an automorphism of the covariant derivative, if and only if, it is a totally geodesic map (i.e. sends geodesics, considered as parametrized curves, to geodesics) and "preserves the torsion".

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A formal proof of this conceptually important result may be found e.g. in a celebrated paper of J. Vilms [14] for the torsion-free case.

Here we shall investigate similar questions for manifolds endowed with an *Ehresmann connection.* Briefly, by an Ehresmann connection over a manifold M we mean a fibre preserving, fibrewise linear map from $TM \times_M TM$ into TTM, which is a right inverse of the canonical surjection $TTM \to TM \times_M TM$ and smooth over $\mathring{T}M \times_M TM$, where $\mathring{T}M$ is the bundle of the nonzero tangent vectors to M (confer C1–C4 in Section 6). In his influential paper [7], J. Grifone introduced the concept of *strong torsion* of an Ehresmann connection and proved that an Ehresmann connection is uniquely determined by its strong torsion and geodesics. This result makes it plausible that a totally geodesic map of a manifold with an Ehresmann connection which "preserves the strong torsion" is an automorphism of the Ehresmann connection (and conversely). After quite detailed, but not too difficult preparations we show that this is indeed true. Surprisingly or not, a large part of our reasoning is based on simple "commutation relations" concerning the push-forward operations by a diffeomorphism and the fundamental canonical objects introduced in Section 3 on the one hand (Section 5), and some "non-canonical" commutation relations implied by an automorphism of an Ehresmann connection (Lemmas 7.1–7.3) on the other hand.

In the concluding section we show that if the Ehresmann connection is homogeneous and of class C^1 on its whole domain, and hence it canonically leads to a covariant derivative on the base manifold, then our theorem reduces to the classical theorem on totally geodesic maps. More precisely, it turns out that the torsion condition of the classical theorem may be weakened to some extent – a phenomenon, which is hidden for the traditional approach.

2. Preliminaries

We follow the notation and conventions of [11] (see also [10] and [12]) as far as feasible. However, for the readers' convenience, in this section we fix some terminology and recall some basic facts.

"Manifold" will always mean a connected smooth manifold of dimension $n \in \mathbb{N}^*$ which is Hausdorff and has a countable basis of open sets. If M is a manifold, $C^\infty(M)$ will denote the ring of smooth functions on M and $\mathrm{Diff}\,(M)$ the group of diffeomorphisms from M onto itself. $\tau : TM \to M$ (simply, τ or TM) is the tangent bundle of M. τ_{TM} denotes the canonical projection, the "foot map", of TTM onto TM, as well as the tangent bundle of TM. If $\varphi : M \to N$ is a smooth map, then φ_* will denote the smooth map of TM into TN induced by φ, the tangent map or derivative of φ.

The vertical lift of $f \in C^\infty(M)$ is $f^{\mathsf{v}} := f \circ \tau$, the complete lift $f^{\mathsf{c}} \in C^\infty(TM)$ of f is defined by $f^{\mathsf{c}}(v) := v(f)$, $v \in TM$. It follows at once that if $\varphi : M \to M$ is a smooth map and $f \in C^\infty(M)$, then

$$(f \circ \varphi)^{\mathsf{c}} = f^{\mathsf{c}} \circ \varphi_* . \tag{1}$$

Throughout the paper, $I \subset \mathbb{R}$ will be an open interval. The velocity field of a smooth curve $\gamma : I \to M$ is

$$\dot{\gamma} := \gamma_* \circ \frac{d}{du} : I \to TM, \tag{2}$$

where $\frac{d}{du}$ is the canonical vector field on the real line. The acceleration field of γ is

$$\ddot{\gamma} := \dot{\dot{\gamma}} \overset{(2)}{=} \left(\gamma_* \circ \frac{d}{du}\right)_* \circ \frac{d}{du} = \gamma_{**} \circ \left(\frac{d}{du}\right)_* \circ \frac{d}{du}. \tag{3}$$

If $\gamma : I \to M$ is a smooth curve and $\varphi \in \operatorname{Diff}(M)$, then $\varphi \circ \gamma$ is also a smooth curve, and we have

$$\dot{\overline{\varphi \circ \gamma}} = \varphi_* \circ \dot{\gamma}, \qquad \ddot{\overline{\varphi \circ \gamma}} = \varphi_{**} \circ \ddot{\gamma}. \tag{4}$$

$\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of smooth vector fields on M. Any vector field X on M determines two vector fields on TM, the vertical lift X^{v} of X and the complete lift X^{c} of X, characterized by

$$X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}}, \quad X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}; \qquad f \in C^\infty(M). \tag{5}$$

3. Canonical constructions

Let $\tau^*TM := TM \times_M TM := \left\{ (u,v) \in TM \times TM \mid \tau(u) = \tau(v) \right\}$, and let $\pi(u,v) := u$ for $(u,v) \in \tau^*TM$. Then π is a vector bundle with total space τ^*TM and base space TM, the *pull-back* of $\tau : TM \to M$ over τ. The $C^\infty(TM)$-module of sections of π will be denoted by $\operatorname{Sec}(\pi)$. Any vector field X on M determines a section

$$\widehat{X} : v \in TM \longmapsto \big(v, X \circ \tau(v)\big) \in TM \times_M TM,$$

called the *basic section* associated to X, or the lift of X into $\operatorname{Sec}(\pi)$. The module $\operatorname{Sec}(\pi)$ is generated by the basic sections. We have a *canonical section*

$$\delta : v \in TM \longmapsto (v,v) \in TM \times_M TM.$$

Generic sections of $\mathrm{Sec}\,(\pi)$ will be denoted by $\widetilde{X}, \widetilde{Y}, \dots$. Starting from the slit tangent bundle $\mathring{\tau} : \mathring{T}M \to M$, the pull-back bundle $\mathring{\pi} : \mathring{T}M \times_M TM \to TM$ is constructed in the same way. Omitting the routine details, we remark that $\mathrm{Sec}\,(\pi)$ may naturally be embedded into the $C^\infty(\mathring{T}M)$-module $\mathrm{Sec}\,(\mathring{\pi})$.

There exists a canonical injective bundle map $\mathbf{i} : TM \times_M TM \to TTM$ given by

$$\mathbf{i}(u,v) := \dot{c}(0), \qquad \text{if} \quad c(t) := u + tv \quad (t \in \mathbb{R}),$$

and a canonical surjective bundle map

$$\mathbf{j} : TTM \to TM \times_M TM,\ w \in T_vTM$$

$$\longmapsto \mathbf{j}(w) := \big(v, \tau_*(w)\big) \in \{v\} \times T_{\tau(v)}M.$$

Then $\mathbf{j} \circ \mathbf{i} = 0$, while $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is a further important canonical object, the *vertical endomorphism* of TTM. $\mathbf{i}$ and $\mathbf{j}$ induce the tensorial maps

$$\widetilde{X} \in \mathrm{Sec}\,(\pi) \longmapsto \mathbf{i}\widetilde{X} := \mathbf{i} \circ \widetilde{X} \in \mathfrak{X}(TM),$$

and

$$\xi \in \mathfrak{X}(TM) \longmapsto \mathbf{j}\xi := \mathbf{j} \circ \xi \in \mathrm{Sec}\,(\pi),$$

so $\mathbf{J}$ may also be interpreted as a $C^\infty(TM)$-linear endomorphism of $\mathfrak{X}(TM)$. $\mathfrak{X}^{\mathsf{v}}(TM) := \mathbf{i}\,\mathrm{Sec}\,(\pi)$ is the module of *vertical vector fields* on TM. The vertical vector fields form a subalgebra of the Lie algebra $\mathfrak{X}(TM)$ at the same time. For any vector field X on M we have

$$\mathbf{i}\widehat{X} = X^{\mathsf{v}}, \qquad \mathbf{j}X^{\mathsf{c}} = \widehat{X}, \tag{6}$$

and hence

$$\mathbf{J}X^{\mathsf{v}} = 0, \quad \mathbf{J}X^{\mathsf{c}} = X^{\mathsf{v}}. \tag{7}$$

It follows that $\mathbf{J}^2 = 0$, $\mathrm{Ker}\,(\mathbf{J}) = \mathrm{Im}\,(\mathbf{J}) = \mathfrak{X}^{\mathsf{v}}(TM)$.

$C := \mathbf{i}\delta$ is a canonical vertical vector field, called the *Liouville vector field.* If $X \in \mathfrak{X}(M)$ then

$$[X^{\mathsf{v}}, C] = X^{\mathsf{v}}. \tag{8}$$

4. Semisprays

By a *semispray* over M we mean a map $S : TM \to TTM$ satisfying the following conditions:

SPR1. $\tau_{TM} \circ S = 1_{TM}$.

SPR2. $\mathbf{j}S = \delta$ (or, equivalently, $\mathbf{J}S = C$).

SPR3. S is smooth on $\mathring{T}M$.

SPR4. S sends the zero vectors of TM to the zero vectors of TTM.

A map $S : TM \to TTM$ is said to be a *spray* if it has the properties SPR1–SPR3 and satisfies the next two conditions:

SPR5. $[C, S] = S$, i.e., S is positive-homogeneous of degree 2.

SPR6. S is of class C^1 on TM.

If S is a spray then the homogeneity condition implies that SPR4 is true automatically. Notice that this concept of a spray was introduced by P. Dazord in his Thèse [4]. J. Grifone defined a semispray only by conditions SPR1–SPR3, see [7]. A spray is said to be an *affine spray* if it is of class C^2 (and hence smooth) on TM. Affine sprays were introduced by Ambrose, Palais and Singer [1] under the name "spray".

If S is a semispray over M, then for any vector field ξ on TM we have

$$\mathbf{J}[\mathbf{J}\xi, S] = \mathbf{J}\xi. \tag{9}$$

This useful relation will be cited as *Grifone's identity*; see [7], and for a simple direct proof [13]. In particular, if X is a vector field on M, then

$$\mathbf{j}[X^{\mathsf{v}}, S] = \widehat{X} = \mathbf{j}X^{\mathsf{c}}, \tag{10}$$

therefore $[X^{\mathsf{v}}, S]$ and X^{c} differ only in a vertical vector field. Thus it follows that

$$[X^{\mathsf{v}}, S] = X^{\mathsf{c}} + \eta, \qquad \eta \in \mathfrak{X}^{\mathsf{v}}(TM). \tag{11}$$

By a *geodesic* of a semispray S we mean a regular smooth curve $\gamma : I \to M$ that satisfies the relation

$$\ddot{\gamma} = S \circ \dot{\gamma}. \tag{12}$$

(Regularity excludes the constant geodesics.) A diffeomorphism $\varphi \in \mathrm{Diff}\,(M)$ is said to be an *affinity* (affine transformation, affine collineation) of a semispray S if it "preserves the geodesics as parametrized curves". This means that if γ is a geodesic of S, then $\varphi \circ \gamma$ remains a geodesic, i.e., $\ddot{\overline{\varphi \circ \gamma}} = S \circ \dot{\overline{\varphi \circ \gamma}}$. Using (3) and (12), this relation can be written in the form

$$(\varphi_{**} \circ S - S \circ \varphi_*) \circ \dot{\gamma} = 0. \tag{13}$$

The affinities of a semispray S form a group $\mathrm{Aff}\,(S)$. By an important result of O. Loos [9], $\mathrm{Aff}\,(S)$ carries a natural Lie group structure, and the Lie group $\mathrm{Aff}\,(S)$ is at most $(n^2 + n)$-dimensional.

5. Push-forward of sections. Commutation formulae

Recall that the push-forward of a vector field $X \in \mathfrak{X}(M)$ by a diffeomorphism $\varphi \in \operatorname{Diff}(M)$ is the vector field

$$\varphi_{\#}X := \varphi_* \circ X \circ \varphi^{-1}.$$

If ξ is a vector field on TM then by its push-forward by φ we mean the vector field

$$(\varphi_*)_{\#}\xi := \varphi_{**} \circ \xi \circ (\varphi_*)^{-1}.$$

In particular, the push-forward $(\varphi_*)_{\#}S$ of a semispray or spray is also a semispray or spray. The *automorphism group* of S is

$$\operatorname{Aut}(S) := \left\{ \varphi \in \operatorname{Diff}(M) \mid (\varphi_*)_{\#}S = S \right\}.$$

LEMMA 5.1. *The automorphism group of a semispray coincides with the group of affinities of the semispray.*

PROOF. Let S be a semispray and $\gamma : I \to M$ a geodesic of S. If $\varphi \in \operatorname{Aut}(S)$, then

$$S \circ \overset{\cdot}{\overbrace{\varphi \circ \gamma}} \overset{(4)}{=} S \circ \varphi_* \circ \dot{\gamma} = \varphi_{**} \circ S \circ \dot{\gamma} \overset{(12)}{=} \varphi_{**} \circ \ddot{\gamma} \overset{(4)}{=} \overset{\cdot\cdot}{\overbrace{\varphi \circ \gamma}},$$

implying that $\varphi \in \operatorname{Aff}(S)$.

The converse statement is immediate from (13) and SPR4. $\square$

The *push-forward of a section* $\widetilde{X} \in \operatorname{Sec}(\pi)$ by $\varphi \in \operatorname{Diff}(M)$ is, by definition,

$$\varphi_{\#}\widetilde{X} = (\varphi_* \times \varphi_*) \circ \widetilde{X} \circ \varphi_*^{-1}.$$

It follows at once that

$$\varphi_{\#}\delta = \delta, \tag{14}$$

and, for any vector field X on M,

$$\varphi_{\#}\widehat{X} = \widehat{\varphi_{\#}X}. \tag{15}$$

LEMMA 5.2. *For any vector field X on M and diffeomorphism $\varphi \in \operatorname{Diff}(M)$,*

$$(\varphi_*)_{\#}X^{\mathsf{c}} = (\varphi_{\#}X)^{\mathsf{c}}. \tag{16}$$

PROOF. We show that X^{c} and $(\varphi_{\#}X)^{\mathsf{c}}$ are φ-related. Since X^{c} is φ-related to $(\varphi_*)_{\#}X^{\mathsf{c}}$, this implies the claim. X and $\varphi_{\#}X$ are φ-related, so by the well-known characterization of φ-relatedness (see e.g. [6], p. 109, Lemma 5), for any function $f \in C^\infty(M)$ we have

$$\big((\varphi_{\#}X)f\big) \circ \varphi = X(f \circ \varphi).$$

Taking this into account we obtain

$$X^{\mathsf{c}}(f^{\mathsf{c}} \circ \varphi_*) \overset{(1)}{=} X^{\mathsf{c}}(f \circ \varphi)^{\mathsf{c}} \overset{(5)}{=} \big(X(f \circ \varphi)\big)^{\mathsf{c}}$$

$$= \big((\varphi_{\#}X)f \circ \varphi\big)^{\mathsf{c}} \overset{(1)}{=} \big((\varphi_{\#}X)f\big)^{\mathsf{c}} \circ \varphi_* \overset{(5)}{=} \big((\varphi_{\#}X)^{\mathsf{c}} f^{\mathsf{c}}\big) \circ \varphi_*,$$

which implies that X^{c} and $(\varphi_{\#}X)^{\mathsf{c}}$ are indeed φ_*-related. □

LEMMA 5.3. *Let* $\varphi \in \mathrm{Diff}\,(M)$. *Then*

$$(\varphi_*)_{\#} \circ \mathbf{i} = \mathbf{i} \circ \varphi_{\#}, \tag{17}$$

$$\varphi_{\#} \circ \mathbf{j} = \mathbf{j} \circ (\varphi_*)_{\#}, \tag{18}$$

$$(\varphi_*)_{\#} \circ \mathbf{J} = \mathbf{J} \circ (\varphi_*)_{\#}. \tag{19}$$

These relations may also be written in the form

$$\varphi_{**} \circ \mathbf{i} = \mathbf{i} \circ (\varphi_* \times \varphi_*), \tag{20}$$

$$(\varphi_* \times \varphi_*) \circ \mathbf{j} = \mathbf{j} \circ \varphi_{**}, \tag{21}$$

$$\varphi_{**} \circ \mathbf{J} = \mathbf{J} \circ \varphi_{**}. \tag{22}$$

The proof of (17) and (18) is a routine check, which we omit. (19) is an immediate consequence of (17) and (18).

By (17) and (14), we have at once:

$$(\varphi_*)_{\#} C = C. \tag{23}$$

Since $X^{\mathsf{v}} = \mathbf{i}\widehat{X}$, (20) and (15) imply

$$(\varphi_*)_{\#} X^{\mathsf{v}} = (\varphi_{\#}X)^{\mathsf{v}}, \qquad X \in \mathfrak{X}(M). \tag{24}$$

6. Ehresmann connections

By an *Ehresmann connection* over M we mean a map $\mathcal{H}$ from $TM \times_M TM$ into TTM satisfying the following conditions:

C1. $\mathcal{H}$ is fibre preserving and fibrewise linear, i.e., for every $v \in TM$, $\mathcal{H}_v := \mathcal{H} \upharpoonright \{v\} \times T_{\tau(v)}M$ is a linear map from $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$ into T_vTM.

C2. $\mathbf{j} \circ \mathcal{H} = 1_{TM \times_M TM}$, i.e., "$\mathcal{H}$ splits".

C3. $\mathcal{H}$ is smooth over $\overset{\circ}{T}M \times_M TM$.

C4. If $o : M \to TM$ is the zero vector field, then $\mathcal{H} = \big(o(p), v\big) = (o_*)_p(v)$, for all $p \in M$ and $v \in T_pM$.

If $\mathcal{H}$ is an Ehresmann connection over M, then there exists a unique fibre preserving, fibrewise linear map $\mathcal{V} : TTM \to TM \times_M TM$, smooth over $T\overset{\circ}{T}M$, such that

$$\mathcal{V} \circ \mathbf{i} = 1_{TM \times_M TM}, \qquad \mathrm{Ker}\,(\mathcal{V}) = \mathrm{Im}\,(\mathcal{H}).$$

$\mathcal{V}$ is called the *vertical map* associated to $\mathcal{H}$. $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ and $\mathbf{v} := \mathbf{i} \circ \mathcal{V} = 1_{TTM} - \mathbf{h}$ are projection operators on TTM such that $\mathbf{h} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{h} = 0$ and $\mathrm{Im}\,(\mathbf{h}) \oplus \mathrm{Im}\,(\mathbf{v}) = TTM$. An Ehresmann connection $\mathcal{H}$ and its associated maps $\mathcal{V}$, $\mathbf{h}$, $\mathbf{v}$ include tensorial maps at the level of sections, denoted by the same symbols. $\mathfrak{X}^{\mathsf{h}}(TM) := \mathcal{H}\big(\mathfrak{X}(TM)\big)$ is the module of *horizontal vector fields* on TM, and

$$X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) \overset{(6)}{=} \mathcal{H}(\mathbf{j}X^{\mathsf{c}}) = \mathbf{h}X^{\mathsf{c}}$$

is the *horizontal lift* of the vector field $X \in \mathfrak{X}(M)$ with respect to $\mathcal{H}$. $S_{\mathcal{H}} := \mathcal{H} \circ \delta$ is a semispray, called the *associated semispray* to $\mathcal{H}$. We have

$$\mathbf{h}[C, S_{\mathcal{H}}] = S_{\mathcal{H}}. \tag{25}$$

Indeed, $\mathbf{J}\big([C, S_{\mathcal{H}}] - S_{\mathcal{H}}\big) \overset{(9)}{=} C - \mathbf{J}S_{\mathcal{H}} = 0$, so $[C, S_{\mathcal{H}}] - S_{\mathcal{H}}$ is vertical, therefore

$$0 = \mathbf{h}\big([C, S_{\mathcal{H}}] - S_{\mathcal{H}}\big) = \mathbf{h}[C, S_{\mathcal{H}}] - \mathcal{H} \circ \mathbf{j} \circ \mathcal{H} \circ \delta$$
$$= \mathbf{h}[C, S_{\mathcal{H}}] - \mathcal{H} \circ \delta = \mathbf{h}[C, S_{\mathcal{H}}] - S_{\mathcal{H}}$$

whence (25).

A regular smooth curve $\gamma : I \to M$ is a *geodesic* of the Ehresmann connection $\mathcal{H}$ if $\mathcal{V} \circ \ddot{\gamma} = 0$, i.e., if the acceleration vector field of γ is horizontal with respect to $\mathcal{H}$. It may easily be shown (see [11], 3.3, Proposition 1) that *the geodesics of an Ehresmann connection coincide with the geodesics of its associated semispray.* If M is a manifold with an Ehresmann connection $\mathcal{H}$, then a diffeomorphism of M is said to be an *affinity* (affine collineation, or, by J. Vilms's terminology [14], a *totally geodesic map*) if it preserves the geodesics considered as parametrized curves (confer the end of Section 4). We denote by $\mathrm{Aff}\,(\mathcal{H})$ the group of these transformations. The result cited above may now be reformulated as follows:

LEMMA 6.1. *Let M be a manifold with Ehresmann connection $\mathcal{H}$, and let $S := \mathcal{H} \circ \delta$. Then* $\mathrm{Aff}\,(\mathcal{H}) = \mathrm{Aff}\,(S)$.

Any semispray S over M induces an Ehresmann connection $\mathcal{H}_S$ over M such that for all $X \in \mathfrak{X}(M)$,

$$\mathcal{H}_S(\widehat{X}) = \frac{1}{2}\big(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]\big) \tag{26}$$

(*Crampin's formula*, appeared first in [2], p. 179). The semispray $\bar{S} := \mathcal{H}_S \circ \delta$ differs from S in general; $\bar{S} = S$ if and only if $[C, S] = S$, i.e., if S is homogeneous of degree 2.

An Ehresmann connection $\mathcal{H}$ determines a covariant derivative operator ∇ in the pull-back bundle π by the rule

$$\nabla_\xi \widetilde{Y} := \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] + \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}]; \qquad \xi \in \mathfrak{X}(TM), \widetilde{Y} \in \mathrm{Sec}\,(\pi).$$

∇ is said to be the *Berwald derivative* induced by $\mathcal{H}$. Its *v-part* ∇^{v} and *h-part* ∇^{h} are defined by

$$\nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}] \tag{27}$$

and

$$\nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] \tag{28}$$

$\big(\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}\,(\pi)\big)$. Actually, the v-Berwald derivative ∇^{v} does not depend on the choice of the Ehresmann connection since the difference of two Ehresmann connections is a vertical valued tensor, the Lie bracket of two vertical vector fields is vertical, and $\mathbf{j}$ kills the vertical vector fields. If X and Y are vector fields on M, then (27) and (28) reduce to

$$\nabla^{\mathsf{v}}_{\widehat{X}} \widehat{Y} = \nabla_{X^{\mathsf{v}}} \widehat{Y} = 0 \tag{29}$$

and

$$\mathbf{i}\nabla^{\mathsf{h}}_{\widehat{X}} \widehat{Y} = \mathbf{i}\nabla_{X^{\mathsf{h}}} \widehat{Y} = [X^{\mathsf{h}}, Y^{\mathsf{v}}]. \tag{30}$$

The importance of the Berwald derivative lies, among others, in the fact that the basic geometric data (torsions, curvatures, etc.) of an Ehresmann connection $\mathcal{H}$ may conveniently be defined in terms of the Berwald derivative induced by $\mathcal{H}$. In this paper we need only the following:

$$\mathbf{t} := \nabla^{\mathsf{h}}\delta \quad - \quad \text{the } \textit{tension} \text{ of } \mathcal{H}, \tag{31}$$

$$\mathbf{T}(\widetilde{X},\widetilde{Y}) := \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} - \nabla^{\mathsf{h}}_{\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}] \quad - \quad \text{the } \textit{torsion} \text{ of } \mathcal{H}, \tag{32}$$

$$\mathbf{T}^{\mathsf{s}} := \mathbf{t} + i_\delta\mathbf{T} \quad - \quad \text{the } \textit{strong torsion} \text{ of } \mathcal{H} \tag{33}$$

($\nabla^{\mathsf{h}}\delta(\widetilde{X}) := \nabla^{\mathsf{h}}_{\widetilde{X}}\delta$, $i_\delta\mathbf{T}(\widetilde{X}) := \mathbf{T}(\delta,\widetilde{X})$; $\widetilde{X},\widetilde{Y} \in \operatorname{Sec}(\pi)$).

We may rewrite these definitions in terms of the values on basic sections $\widehat{X}$, $\widehat{Y}$ as follows:

$$\mathbf{t}(\widehat{X}) = \mathcal{V}[X^{\mathsf{h}},C] \quad \text{or} \quad \mathbf{it}(\widehat{X}) = [X^{\mathsf{h}},C], \tag{34}$$

$$\mathbf{iT}(\widehat{X},\widehat{Y}) = [X^{\mathsf{h}},Y^{\mathsf{v}}] - [Y^{\mathsf{h}},X^{\mathsf{v}}] - [X,Y]^{\mathsf{v}}, \tag{35}$$

$$\mathbf{T}^{\mathsf{s}}(\widehat{X}) = \mathcal{V}[S,X^{\mathsf{v}}] - \mathbf{j}[S,X^{\mathsf{h}}], \quad S := \mathcal{H}\circ\delta. \tag{36}$$

An Ehresmann connection is said to be *homogeneous* if its tension vanishes. Since

$$\mathbf{it}(\delta) = \mathbf{i}\nabla_{\mathcal{H}\delta}\delta \overset{(28)}{=} \mathbf{v}[\mathcal{H}\delta,C] = \mathbf{v}[S,C] = [S,C] - \mathbf{h}[S,C] \overset{(25)}{=} S - [C,S],$$

the semispray $S := \mathcal{H}\circ\delta$ associated to $\mathcal{H}$ is homogeneous of degree 2, *if and only if,* $\mathbf{t}(\delta) = 0$.

Consider the Ehresmann connection $\mathcal{H}_S$ induced by $S = \mathcal{H}\circ\delta$ according to (26). It may be shown (see [10], Proposition 4) that

$$\mathcal{H} - \mathcal{H}_S = \mathbf{i}\circ\frac{1}{2}\mathbf{T}^{\mathsf{s}}.$$

This relation clarifies the meaning of the strong torsion as well as implies that *an Ehresmann connection is uniquely determined by its associated semispray and strong torsion* (confer [7], Théorème I.55 and [5], Proposition 4.8.1).

7. Automorphisms of Ehresmann connections

Throughout this section $\mathcal{H}$ is an Ehresmann connection over M.

We begin with a simple remark. If $\varphi \in \mathrm{Diff}\,(M)$ and

$$\varphi^{\#}\mathcal{H} := \varphi_{**}^{-1} \circ \mathcal{H} \circ (\varphi_* \times \varphi_*),$$

then $\varphi^{\#}\mathcal{H}$ is also an Ehresmann connection over M. Indeed,

$$\mathbf{j} \circ \varphi^{\#}\mathcal{H} = \mathbf{j} \circ \varphi_{**}^{-1} \circ \mathcal{H} \circ (\varphi_* \times \varphi_*) \overset{(21)}{=} (\varphi_*^{-1} \times \varphi_*^{-1}) \circ \mathbf{j} \circ \mathcal{H} \circ (\varphi_* \times \varphi_*) \overset{C2}{=}$$
$$= (\varphi_*^{-1} \times \varphi_*^{-1}) \circ (\varphi_* \times \varphi_*) = 1_{TM \times_M TM},$$

so $\varphi^{\#}\mathcal{H}$ satisfies **C2**, while the remaining axioms hold immediately. $\varphi^{\#}\mathcal{H}$ is said to be the *pull-back of* $\mathcal{H}$ *by* φ. If $\varphi^{\#}\mathcal{H} = \mathcal{H}$, i.e.,

$$\varphi_{**} \circ \mathcal{H} = \mathcal{H} \circ (\varphi_* \times \varphi_*), \tag{37}$$

then φ is called an *automorphism* of $\mathcal{H}$.

$$\mathrm{Aut}\,(\mathcal{H}) := \left\{ \varphi \in \mathrm{Diff}\,(M) \mid \varphi^{\#}\mathcal{H} = \mathcal{H} \right\}$$

is the *automorphism group* of $\mathcal{H}$.

LEMMA 7.1. *For a diffeomorphism* $\varphi \in \mathrm{Diff}\,(M)$ *the following are equivalent:*

$$\text{(i) } \varphi \in \mathrm{Aut}\,(\mathcal{H}), \quad \text{(ii) } \varphi_{**} \circ \mathbf{h} = \mathbf{h} \circ \varphi_{**}, \quad \text{(iii) } \varphi_{**} \circ \mathbf{v} = \mathbf{v} \circ \varphi_{**},$$

$$\text{(iv) } (\varphi_* \times \varphi_*) \circ \mathcal{V} = \mathcal{V} \circ \varphi_{**}, \quad \text{(v) } (\varphi_*)_{\#} X^{\mathsf{h}} = (\varphi_{\#} X)^{\mathsf{h}} \textit{ for all } X \in \mathfrak{X}(M).$$

PROOF. Since $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$, the equivalence of (i) and (ii) is a consequence of (21), while the equivalence of (ii) and (iii) is obvious. The equivalence (iii) $\iff$ (iv) follows at once from (20). Finally,

$$(\varphi_*)_{\#} X^{\mathsf{h}} = (\varphi_{\#} X)^{\mathsf{h}} \overset{(15)}{\iff} \varphi_{**} \circ \mathcal{H} \circ \widehat{X} \circ \varphi_*^{-1} = \mathcal{H} \circ (\varphi_* \times \varphi_*) \circ \widehat{X} \circ \varphi_*^{-1}$$

$$\iff \varphi_{**} \circ \mathcal{H} \circ \widehat{X} = \mathcal{H} \circ (\varphi_* \times \varphi_*) \circ \widehat{X} \iff \varphi_{**} \circ \mathcal{H} = \mathcal{H} \circ (\varphi_* \times \varphi_*),$$

so (i) and (v) are also equivalent. $\square$

We recall that if $D : \mathfrak{X}(TM) \times \mathrm{Sec}\,(\pi) \to \mathrm{Sec}\,(\pi)$ is a covariant derivative operator in π, then a diffeomorphism $\varphi \in \mathrm{Diff}\,(M)$ is said to be an *automorphism of* D if

$$\varphi_{\#} D_{\xi} \widetilde{Y} = D_{(\varphi_*)_{\#}\xi} \varphi_{\#} \widetilde{Y}; \qquad \xi \in \mathfrak{X}(TM),\ \widetilde{Y} \in \mathrm{Sec}\,(\pi).$$

The group of automorphisms of D is denoted by $\mathrm{Aut}\,(D)$.

LEMMA 7.2. *If* ∇ *is the Berwald derivative induced by* $\mathcal{H}$*, then*

$$\mathrm{Aut}\,(\mathcal{H}) \subset \mathrm{Aut}\,(\nabla).$$

PROOF. Let $\varphi \in \mathrm{Aut}\,(\mathcal{H})$. It is enough to check that for any vector fields X, Y on M,

$$\nabla_{(\varphi_*)_\# X^{\mathsf{v}}} \varphi_\# \widehat{Y} = 0 \quad \text{and} \quad \nabla_{(\varphi_*)_\# X^{\mathsf{h}}} \varphi_\# \widehat{Y} = \varphi_\# \nabla_{X^{\mathsf{h}}} \widehat{Y}.$$

The first relation is an immediate consequence of (15), (24) and (29). The second needs an easy verification:

$$\varphi_\# \nabla_{X^{\mathsf{h}}} \widehat{Y} \overset{(28)}{=} (\varphi_* \times \varphi_*) \circ \mathcal{V}[X^{\mathsf{h}}, Y^{\mathsf{v}}] \circ \varphi_*^{-1} \overset{\text{Lemma 7.1}}{=}$$

$$= \mathcal{V} \circ \varphi_{**} \circ [X^{\mathsf{h}}, Y^{\mathsf{v}}] \circ \varphi_*^{-1} = \mathcal{V} \circ (\varphi_*)_\# [X^{\mathsf{h}}, Y^{\mathsf{v}}] =$$

$$= \mathcal{V}\big[(\varphi_*)_\# X^{\mathsf{h}}, (\varphi_*)_\# Y^{\mathsf{v}}\big] \overset{(24),\ \text{Lemma 7.1}}{=} \mathcal{V}\big[(\varphi_\# X)^{\mathsf{h}}, (\varphi_\# Y)^{\mathsf{v}}\big] \overset{(28)}{=}$$

$$= \nabla_{(\varphi_\# X)^{\mathsf{h}}} \widehat{\varphi_\# Y} \overset{(15),\ \text{Lemma 7.1}}{=} \nabla_{(\varphi_*)_\# X^{\mathsf{h}}} \varphi_\# \widehat{Y},$$

as was to be shown. □

LEMMA 7.3. *If* $\varphi \in \mathrm{Aut}\,(\mathcal{H})$*, then*

$$\varphi_\# \circ \mathbf{t} = \mathbf{t} \circ \varphi_\#, \qquad \varphi_\# \circ \mathbf{T} = \mathbf{T} \circ (\varphi_\# \times \varphi_\#), \qquad \varphi_\# \circ \mathbf{T}^{\mathsf{s}} = \mathbf{T}^{\mathsf{s}} \circ \varphi_\#.$$

PROOF. We verify the first relation. The second may be obtained similarly, while the third is a consequence of the first two.

Let X be a vector field on M. Then

$$\varphi_\# \mathbf{t}(\widehat{X}) = \varphi_\#(\nabla_{X^{\mathsf{h}}} \delta) \overset{\text{Lemma 7.2}}{=} \nabla_{(\varphi_*)_\# X^{\mathsf{h}}} \varphi_\# \delta \overset{(14),\ \text{Lemma 7.1}}{=}$$

$$= \nabla_{(\varphi_\# X)^{\mathsf{h}}} \delta = \mathbf{t}(\varphi_\# \widehat{X}),$$

hence $\varphi_\# \circ \mathbf{t} = \mathbf{t} \circ \varphi_\#$. □

LEMMA 7.4. *If* $\varphi \in \mathrm{Aut}\,(\mathcal{H})$*, then* $\varphi \in \mathrm{Aut}\,(S_{\mathcal{H}})$*, where* $S_{\mathcal{H}} := \mathcal{H} \circ \delta$ *is the semispray associated to* $\mathcal{H}$*. Conversely, if* S *is a semispray over* M *and* $\mathcal{H}_S$ *is the Ehresmann connection induced by* S*, then* $\mathrm{Aut}\,(S) \subset \mathrm{Aut}\,(\mathcal{H}_S)$.

PROOF. If $\varphi \in \mathrm{Aut}\,(\mathcal{H})$, then

$$\varphi_{**} \circ S_{\mathcal{H}} \circ \varphi_*^{-1} = \varphi_{**} \circ \mathcal{H} \circ \delta \circ \varphi_*^{-1} \overset{(37)}{=}$$

$$= \mathcal{H} \circ (\varphi_* \times \varphi_*) \circ \delta \circ \varphi_*^{-1} \overset{(14)}{=} \mathcal{H} \circ \delta = S_{\mathcal{H}},$$

so $\varphi \in \mathrm{Aut}\,(S_{\mathcal{H}})$.

Now let S be a semispray over M and $\varphi \in \mathrm{Aut}\,(S)$. For any vector field X on M,

$$\varphi_{**} \circ \mathcal{H}_S(\widehat{X}) \overset{(26)}{=} \frac{1}{2}\big(\varphi_{**} \circ X^{\mathsf{c}} + \varphi_{**} \circ [X^{\mathsf{v}}, S]\big) =$$

$$= \frac{1}{2}\big(\varphi_{**} \circ X^{\mathsf{c}} \circ \varphi_*^{-1} + \varphi_{**} \circ [X^{\mathsf{v}}, S] \circ \varphi_*^{-1}\big) \circ \varphi_* =$$

$$= \frac{1}{2}\big((\varphi_*)_{\#} X^{\mathsf{c}} + (\varphi_*)_{\#}[X^{\mathsf{v}}, S]\big) \circ \varphi_* \overset{(16),\,(24)}{=}$$

$$= \frac{1}{2}\big((\varphi_{\#} X)^{\mathsf{c}} + \big[(\varphi_{\#} X)^{\mathsf{v}}, (\varphi_*)_{\#} S\big]\big) \circ \varphi_* \overset{\text{cond.}}{=}$$

$$= \frac{1}{2}\big((\varphi_{\#} X)^{\mathsf{c}} + \big[(\varphi_{\#} X)^{\mathsf{v}}, S\big]\big) \circ \varphi_* = \mathcal{H}_S(\widehat{\varphi_{\#} X}) \circ \varphi_* \overset{(15)}{=}$$

$$= \mathcal{H}_S \circ \varphi_{\#} \circ \widehat{X} \circ \varphi_* = \mathcal{H}_S \circ (\varphi_* \times \varphi_*) \circ \widehat{X},$$

which proves the second claim. □

Now we obtain the results we were striving for.

THEOREM 7.5. *A diffeomorphism φ of M is an automorphism of the Ehresmann connection $\mathcal{H}$, if and only if, it is an automorphism of the associated semispray $S := \mathcal{H} \circ \delta$ and $\varphi_{\#} \circ \mathbf{T}^{\mathsf{s}} = \mathbf{T}^{\mathsf{s}} \circ \varphi_{\#}$.*

PROOF. By Lemma 7.3 and the first part of Lemma 7.4, the conditions are necessary. To prove the sufficiency, let $\varphi \in \mathrm{Aut}\,(S)$ and suppose that $\varphi_{\#} \circ \mathbf{T}^{\mathsf{s}} = \mathbf{T}^{\mathsf{s}} \circ \varphi_{\#}$. Then, for any vector field X on M,

$$\varphi_{\#} \mathbf{T}^{\mathsf{s}}(\widehat{X}) \overset{(36)}{=} \varphi_{\#}\big(\mathcal{V}[S, X^{\mathsf{v}}] - \mathbf{j}[S, X^{\mathsf{h}}]\big) \overset{(18)}{=}$$

$$= \varphi_{\#}\mathcal{V}[S, X^{\mathsf{v}}] - \mathbf{j} \circ (\varphi_*)_{\#}[S, X^{\mathsf{h}}] \overset{\text{cond.}}{=} \varphi_{\#}\mathcal{V}[S, X^{\mathsf{v}}] - \mathbf{j}\big[S, (\varphi_*)_{\#} \circ \mathcal{H} \circ \widehat{X}\big].$$

On the other hand,

$$\mathbf{T}^{\mathsf{s}}\big(\varphi_{\#}\widehat{X}\big) \overset{(15)}{=} \mathbf{T}^{\mathsf{s}}\big(\widehat{\varphi_{\#} X}\big) \overset{(36)}{=} \mathcal{V}\big[S, (\varphi_{\#} X)^{\mathsf{v}}\big] - \mathbf{j}\big[S, \mathcal{H} \circ \widehat{\varphi_{\#} X}\big] \overset{(15),(24)}{=}$$

$$= \mathcal{V}\big[S, (\varphi_*)_{\#} X^{\mathsf{v}}\big] - \mathbf{j}\big[S, \mathcal{H} \circ \varphi_{\#} \circ \widehat{X}\big] \overset{\text{cond.}}{=}$$

$$= \mathcal{V} \circ (\varphi_*)_{\#}[S, X^{\mathsf{v}}] - \mathbf{j}\big[S, \mathcal{H} \circ \varphi_{\#} \circ \widehat{X}\big].$$

So our second condition $\varphi_\# \circ \mathbf{T}^\mathsf{s} = \mathbf{T}^\mathsf{s} \circ \varphi_\#$ implies that

$$\big(\varphi_\# \circ \mathcal{V} - \mathcal{V} \circ (\varphi_*)_\#\big)\,[S, X^\mathsf{v}] = \mathbf{j}\big[S, \big((\varphi_*)_\# \circ \mathcal{H} - \mathcal{H} \circ \varphi_\#\big)\,\widehat{X}\big],$$

or, equivalently, that

$$\text{(38)} \qquad \mathbf{i} \circ \big(\varphi_\# \circ \mathcal{V} - \mathcal{V} \circ (\varphi_*)_\#\big)\,[S, X^\mathsf{v}] = \mathbf{J}\big[S, \big((\varphi_*)_\# \circ \mathcal{H} - \mathcal{H} \circ \varphi_\#\big)\,\widehat{X}\big].$$

On the left-hand side of (38)

$$\mathbf{i} \circ \big(\varphi_\# \circ \mathcal{V} - \mathcal{V} \circ (\varphi_*)_\#\big) = (\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#.$$

On the right-hand side,

$$\big((\varphi_*)_\# \circ \mathcal{H} - \mathcal{H} \circ \varphi_\#\big)\,\widehat{X} \overset{(10)}{=} \big((\varphi_*)_\# \circ \mathcal{H} \circ \mathbf{j} - \mathcal{H} \circ \varphi_\# \circ \mathbf{j}\big)\,X^\mathsf{c} \overset{(18)}{=}$$
$$= \big((\varphi_*)_\# \circ \mathbf{h} - \mathbf{h} \circ (\varphi_*)_\#\big)\,X^\mathsf{c} = \big(\mathbf{v} \circ (\varphi_*)_\# - (\varphi_*)_\# \circ \mathbf{v}\big)\,X^\mathsf{c},$$

so (38) takes the form

$$\text{(39)} \qquad \big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,[S, \mathbf{J}X^\mathsf{c}] = -\mathbf{J}\big[S, \big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,X^\mathsf{c}\big].$$

Since

$$\mathbf{J}\big((\varphi_*)_\# \mathbf{v} X^\mathsf{c}\big) = \mathbf{J} \circ \varphi_{**} \circ \mathbf{v}X^\mathsf{c} \circ \varphi_*^{-1} \overset{(22)}{=}$$
$$= \varphi_{**} \circ \mathbf{J}(\mathbf{v}X^\mathsf{c}) \circ \varphi_*^{-1} = 0,$$

it follows that $(\varphi_*)_\# \mathbf{v} X^\mathsf{c}$ is vertical, and hence $\big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,X^\mathsf{c}$ is also vertical. Thus, applying Grifone's identity (9), we find that (39) is equivalent to

$$\text{(40)} \qquad \big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,[S, \mathbf{J}X^\mathsf{c}] = \big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,X^\mathsf{c}.$$

Here, by relation (11),

$$[S, \mathbf{J}X^\mathsf{c}] = -[X^\mathsf{v}, S] = -X^\mathsf{c} + \eta, \qquad \eta \in \mathfrak{X}^\mathsf{v}(TM).$$

However, $\big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,\eta = 0$, since for any vector field Y on M,

$$\big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,Y^\mathsf{v} = (\varphi_*)_\# Y^\mathsf{v} - \mathbf{v}\big((\varphi_*)_\# Y^\mathsf{v}\big) \overset{(24)}{=}$$
$$= (\varphi_\# Y)^\mathsf{v} - \mathbf{v}(\varphi_\# Y)^\mathsf{v} = 0.$$

So relation (40) takes the form

$$-\big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,X^\mathsf{c} = \big((\varphi_*)_\# \circ \mathbf{v} - \mathbf{v} \circ (\varphi_*)_\#\big)\,X^\mathsf{c},$$

from which it follows that $(\varphi_*)_\# \circ \mathbf{v} \circ X^\mathsf{c} = \mathbf{v} \circ (\varphi_*)_\# X^\mathsf{c}$. Hence $\varphi_{**} \circ \mathbf{v} = \mathbf{v} \circ \varphi_{**}$, and we may now apply Lemma 7.1 to conclude the proof. $\square$

COROLLARY 7.6. *If M is a manifold with an Ehresmann connection $\mathcal{H}$, then a diffeomorphism φ of M is an automorphism of $\mathcal{H}$, if and only if, φ is an affinity, i.e., preserves the geodesics of $\mathcal{H}$, and $\varphi_\# \circ \mathbf{T}^{\mathsf{s}} = \mathbf{T}^{\mathsf{s}} \circ \varphi_\#$, where $\mathbf{T}^{\mathsf{s}}$ is the strong torsion of $\mathcal{H}$.*

PROOF. Let $S := \mathcal{H} \circ \delta$. Since $\operatorname{Aff}(S) = \operatorname{Aff}(\mathcal{H})$ by Lemma 6.1, the Corollary is just a reformulation of the Theorem. □

8. Discussion

Applying S. Lang's terminology [8], let M be a *D-manifold*, i.e., a manifold, endowed with a covariant derivative operator D. A diffeomorphism $\varphi \in \operatorname{Diff}(M)$ is said to be a *D-automorphism* if $\varphi_\#(D_X Y) = D_{\varphi_\# X} \varphi_\# Y$ for all vector fields X, Y on M. We denote by $\operatorname{Aut}(D)$ the group of the D-automorphisms of a D-manifold. In this concluding section we are going to clarify, how our Theorem 7.5 generalizes the classical theorem:

A diffeomorphism φ of M is a D-automorphism, if and only if, it is a totally geodesic map and $\varphi_\# \circ T(D) = T(D) \circ (\varphi_\# \times \varphi_\#)$, where $T(D)$ is the torsion of D.

Suppose that $\mathcal{H}$ is a *homogeneous* Ehresmann connection over M *of class C^1 on $TM \times_M TM$*. Then there is a unique covariant derivative operator D on M such that

$$(D_X Y)^{\mathsf{v}} = [X^{\mathsf{h}}, Y^{\mathsf{v}}] = \mathbf{i}\nabla_{X^{\mathsf{h}}} \widehat{Y}; \qquad X, Y \in \mathfrak{X}(M), \tag{41}$$

and the geodesics of D coincide with the geodesics of $\mathcal{H}$. (This fact is more or less a folklore. It is mentioned in [12], for some more details we refer to [3].) Our discussion is based on the following observation:

$$\operatorname{Aut}(D) = \operatorname{Aut}(\mathcal{H}). \tag{42}$$

Indeed, if $\varphi \in \operatorname{Aut}(\mathcal{H})$, then

$$(\varphi_\# D_X Y)^{\mathsf{v}} \overset{(24)}{=} (\varphi_*)_\# (D_X Y)^{\mathsf{v}} \overset{(41)}{=} (\varphi_*)_\# \circ \mathbf{i}\nabla_{X^{\mathsf{h}}} \widehat{Y} \overset{(17)}{=}$$

$$= \mathbf{i} \circ \varphi_\#(\nabla_{X^{\mathsf{h}}} \widehat{Y}) \overset{\text{Lemma 7.2}}{=} \mathbf{i}\nabla_{(\varphi_*)_\# X^{\mathsf{h}}} \varphi_\# \widehat{Y} \overset{(15),\ \text{Lemma 7.2}}{=}$$

$$= \mathbf{i}\nabla_{(\varphi_\# X)^{\mathsf{h}}} \widehat{\varphi_\# Y} \overset{(41)}{=} (D_{\varphi_\# X} \varphi_\# Y)^{\mathsf{v}},$$

therefore $\varphi \in \operatorname{Aut}(D)$. Thus we have $\operatorname{Aut}(\mathcal{H}) \subset \operatorname{Aut}(D)$. To prove the reverse relation, let $\varphi \in \operatorname{Aut}(D)$. Then, for any vector fields X, Y on M, $(\varphi_*)_\# (D_X Y)^{\mathsf{v}} = (D_{\varphi_\# X} \varphi_\# Y)^{\mathsf{v}}$, which is equivalent to

$$\left[(\varphi_\# X)^{\mathsf{h}} - (\varphi_*)_\# X^{\mathsf{h}}, \varphi_\# Y^{\mathsf{v}} \right] = 0. \tag{43}$$

Since

$$\mathbf{J}((\varphi_{\#}X)^{\mathsf{h}} - (\varphi_*)_{\#}X^{\mathsf{h}}) \overset{(7)}{=} (\varphi_{\#}X)^{\mathsf{v}} - \mathbf{J} \circ (\varphi_*)_{\#}X^{\mathsf{h}} \overset{(19)}{=}$$

$$= (\varphi_{\#}X)^{\mathsf{v}} - (\varphi_*)_{\#}\mathbf{J}X^{\mathsf{h}} \overset{(7)}{=} (\varphi_{\#}X)^{\mathsf{v}} - (\varphi_*)_{\#}X^{\mathsf{v}} \overset{(24)}{=} 0,$$

the vector field $(\varphi_{\#}X)^{\mathsf{h}} - (\varphi_*)_{\#}X^{\mathsf{h}}$ is vertical. Taking this into account, by 2.4(6) in [11] relation (43) implies that $(\varphi_{\#}X)^{\mathsf{h}} - (\varphi_*)_{\#}X^{\mathsf{h}}$ is a vertical lift:

$$(\varphi_{\#}X)^{\mathsf{h}} - (\varphi_*)_{\#}X^{\mathsf{h}} = Z^{\mathsf{v}}, \qquad Z \in \mathfrak{X}(M).$$

Now, using the homogeneity of $\mathcal{H}$ and relations (8) and (23), we have

$$Z^{\mathsf{v}} = [Z^{\mathsf{v}}, C] = \big[(\varphi_{\#}X)^{\mathsf{h}} - (\varphi_*)_{\#}X^{\mathsf{h}}, C\big] =$$

$$= -\big[(\varphi_*)_{\#}X^{\mathsf{h}}, (\varphi_*)_{\#}C\big] = -(\varphi_*)_{\#}[X^{\mathsf{h}}, C] = 0,$$

whence $(\varphi_*)_{\#}X^{\mathsf{h}} = (\varphi_{\#}X)^{\mathsf{h}}$. By Lemma 7.1 this implies that $\varphi \in \mathrm{Aut}\,(\mathcal{H})$ and concludes the proof of (42).

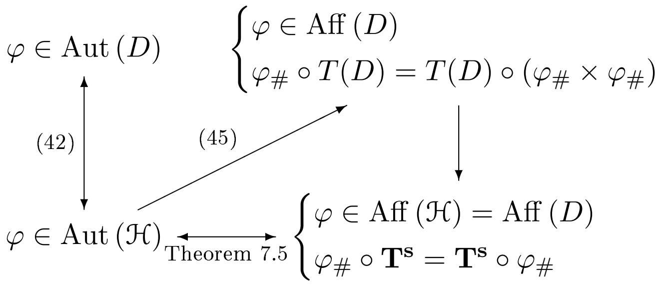
We recall that the torsion of D is related to the torsion of $\mathcal{H}$ by

$$(44) \qquad \big(T(D)(X,Y)\big)^{\mathsf{v}} = \mathbf{i}\mathbf{T}(\widehat{X}, \widehat{Y}); \qquad X, Y \in \mathfrak{X}(M).$$

Hence, taking into account Lemma 7.3,

$$(45) \qquad \varphi \in \mathrm{Aut}\,(\mathcal{H}) \quad \Longrightarrow \quad \varphi_{\#} \circ T(D) = T(D) \circ (\varphi_{\#} \times \varphi_{\#}).$$

Summarizing, if M is endowed with a homogeneous Ehresmann connection, of class C^1 on $TM \times_M TM$, D is the covariant derivative operator defined by (41) and $\varphi \in \mathrm{Diff}\,(M)$, then we have the following implications:



From the diagram we conclude the classical characterization of the D-automorphisms. As a by-product, it follows that the classical theorem may be strengthened: *a diffeomorphism φ of M is a D-automorphism, if and only if, it is a totally geodesic map and has the property* $\varphi_{\#} \circ \mathbf{T}^{\mathsf{s}} = \mathbf{T}^{\mathsf{s}} \circ \varphi_{\#}$, where $\mathbf{T}^{\mathsf{s}} = i_{\delta}\mathbf{T}$, and $\mathbf{T}$ is related to $T(D)$ by (44).

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