On Projectively Flat Finsler Spaces

Dedicated to the memory of Shiing-Shen Chern the outstanding geometer on the centenary of his birth

Tran Quoc BINH

University of Debrecen, Institute of Mathematics,
H-4010 Debrecen, P.O. Box 12, Hungary

E-mail: binh@math.klte.hu

József SZILASI

University of Debrecen, Institute of Mathematics,
H-4010 Debrecen, P.O. Box 12, Hungary

E-mail: szilasi@math.klte.hu

Lajos TAMÁSSY

University of Debrecen, Institute of Mathematics,
H-4010 Debrecen, P.O. Box 12, Hungary

E-mail: tamassy@math.klte.hu

Abstract First we present a short overview of the long history of projectively flat Finsler spaces. We give a simple and quite elementary proof of the already known condition for the projective flatness. We show that a 1-homogeneous Lagrange function, whose level surfaces have non-vanishing Gaussian curvature, is a pseudo-Finsler function, and we give a criterion for the projective flatness of the space (Theorem 1). After this we obtain a second order PDE system, whose solvability is necessary and sufficient for a Finsler space to be projectively flat (Theorem 2). We also derive a condition in order that an infinitesimal transformation takes geodesics of a Finsler space into geodesics. This yields a Killing type vector field (Theorem 3). In the last section we present a characterization of the Finsler spaces which are projectively flat in a parameter-preserving manner (Theorem 4), and we show that these spaces over $\mathbb{R}^n$ are exactly the Minkowski spaces (Theorem 5).

Keywords Finsler functions; projectively flat; projectively Euclidean

MSC(2000) 53B40, 53C60

Abbreviated title: Projectively flat

1 Introduction and historical overview

Let $F^n = (M, \mathcal{F})$ be a Finsler space over a connected n-dimensional base manifold M with a regular, 1^+-homogeneous and strongly convex Finsler metric (fundamental function, Finsler function) $\mathcal{F}$ [4]. F^n or $\mathcal{F}$ is said to be *locally projectively flat* (*locally projectively Euclidean* or, by M. Matsumoto's terminology, a *Finsler space with rectilinear extremals*) if there exists an atlas $\{(U_\alpha, \varphi_\alpha) | \alpha \in \mathcal{A}\}$ on M satisfying the following condition: for each α and for any geodesic

$$\gamma : I \to U_\alpha \subset \mathbb{R}^n$$

of F^n there exists a strictly monotone smooth function $f : I \to \mathbb{R}$, and two vectors $a, b \in \mathbb{R}^n$, $a \neq 0$ such that

$$\varphi_\alpha(\gamma(t)) = f(t)a + b, \quad t \in I. \tag{1.a}$$

Our investigations are throughout local, so we drop the word 'locally' and we say simply *locally flat*. If, in particular, $f(t) = t$ for all $t \in I$, i.e. the image curve $\varphi_\alpha \circ \gamma$ of γ is the affine parametrization

$$t \in I \mapsto ta + b \in \varphi_\alpha(U_\alpha) \tag{1.b}$$

of a straight line segment in $\varphi_\alpha(U_\alpha) \subset R^n$, then we say that F^n is projectively flat in a *parameter-preserving manner*. Previous investigations concerned mainly with the general case (1.a), and less attention was paid on the special case (1.b).

Although the question to find Lagrangians whose geodesics are straight lines emerged already in the 19th century (raised by G. Darboux), the real starting point of the investigations of projectively flat metrics is Hilbert's fourth problem [13] raised on the International Congress of Mathematicians (Paris 1900), in which he asked about the spaces in which the shortest curves between any pair of points are segments of straight lines. The first answer was given by Hilbert's student G. Hamel. In a 34 pages long paper he found necessary and sufficient conditions in order that a space satisfying an axiom system, which is a modification of Hilbert's axioms for Euclidean geometry, removing a strong congruence axiom and including the Desargues axiom, be projectively flat ([12] esp. p. 250 (I,b); see also [20] p. 185; [6] p. 66; or [10]). Later E. Bompiani (1924) showed that among the Riemannian manifolds exactly that of the constant curvature are projectively flat.

The term 'projective metric' was introduced by Busemann and Kelly [8]. Following them, a metric, i.e. a distance function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ is called to be projective if (i) d is continuous with respect to the canonical topology of $\mathbb{R}^n$, and *(ii)* d is 'additive along the straight lines', i.e. $d(a, b) + d(b, c) = d(a, c)$, whenever a, b, c are points on a straight line in the given order. In this more general context the 'fourth problem requests, among many other things, a method of construction for the cone of such projective metrics, and this task lies at the heart of the problem' (quoted verbatim from R. Alexander [1].

The first 75 years of study related to the question is summarized in a survey article of Busemann [7]. Obviously the three classical geometries (Euclidean, hyperbolic, and elliptic) solve the problem over a suitable domain of E^n. Hilbert himself mentioned two solutions of non-Riemannian type: the Minkowski spaces (i.e. the finite dimensional Banach spaces), and a modification of the Cayley-Klein construction of hyperbolic geometry, now called Hilbert geometry. An interesting, non-symmetric projective metric was discovered by Funk in 1929 [11].

Differentiable distance functions supplemented with certain mild and natural additional conditions yield also Finsler metrics [26]. Concerning Finsler manifolds, it is known that an F^n with Hilbert, Funk or Klein metric is projectively flat ([20] pp. 32–33; [6] pp. 105–106).

For general Finsler spaces A. Rapcsák gave conditions for projective flatness ([19]; or [20] section 12.2; see also [10], and [25], Th.8.1). Both Hamel's and Rapcsák's results say basically that the condition for the projective flatness is the existence of a coordinate system (x) on the base manifold in which

$$(*) \qquad \frac{\partial^2 \mathcal{F}}{\partial x^k \partial y^i} y^k = \frac{\partial^2 \mathcal{F}}{\partial x^i \partial y^k} y^k.$$

Actually Rapcsák gave conditions in order that two Finsler spaces $(M, \mathcal{F})$ and $(M, \bar{\mathcal{F}})$ admit projective mapping on each other. This condition is

$$(2) \qquad \mathcal{F}_{|i} = \frac{\partial \mathcal{F}_{|v}}{\partial y^i} y^v,$$

where $|$ means the Berwald derivative determined by $\bar{\mathcal{F}}$. However if $(M, \bar{\mathcal{F}})$ is a Euclidean space in a Cartesian coordinate system, then (2) reduces to $(*)$.

For a recent account in the different formulations and coordinate free reformulation of Rapcsák's equations we refer to [25] and [3]. In two dimensions, using an integral representation of the Finsler function, A. V. Pogorelov presented an elegant solution of Hamel's equation [18]. He also showed that the smoothness of the Finsler function is not an essential condition, since any continuous projective Finsler function can be uniformly approximated by smooth projective Finsler functions solving Hilbert's problem on each compact subset. He gave a comprehensive solution of Hilbert's problem in 2 and 3 dimensions. However some important questions remained open. Among others, the following: how can the continuous Finsler functions be constructed in dimensions greater than two? The exciting questions arising from Pogorelov's investigations were discussed and exhaustively answered by Z. I. Szabó [24]. Many papers dealt with the problem from different aspects as that of M. Matsumoto [15], [16], X. Mo, Z. Shen, C. Yang [17], Alvarez Paiva [2] etc.. In the last time especially the projectively flatness of Randers and Einstein spaces came into the lime light. See T. Q. Binh and X. Cheng [5], Z. Shen, and C. Yildrim, [23], B. Li and Z. Shen, [14], Z. Shen [20], [21], [22], X. Cheng and M. Li [9].

Recently M. Crampin [10] introduced the notion of pseudo-Finsler spaces by a slight modification of the basic assumptions on the fundamental function. In a pseudo-Finsler space the positiveness of $\mathcal{F}$ on $\overset{\circ}{T}M := TM \setminus \{0_p \in T_pM | p \in M\}$ is dropped, positive homogeneity is kept, and the positive definiteness of the Hessian $\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^k}$ is replaced by the condition that the 'reduced Hessian' of $\mathcal{F}$ defines at every point (x, y) a *non-degenerate* symmetric bilinear form on the factor space $T_xM/\langle y \rangle$. Then

$$\text{rank}\, \frac{\partial^2 \mathcal{F}}{\partial y^i \partial y^k} = n - 1. \tag{3}$$

If $\mathcal{F} > 0$ (everywhere on $\overset{\circ}{T}M$), then a pseudo-Finsler function becomes a Finsler function (see Crampin [10], Lemma 2). He also realized the importance of (3) in the proof of $(*)$.

In this paper first we want to present a simple proof of condition $(*)$ for the projective flatness. This proof is very near to the known ones (Crampin, Shen, etc.), but it is quite short, and uses elementary considerations only. Starting with a smooth and 1^+-homogeneous Lagrange function L, from the powerful condition of the non-vanishing of the Gaussian curvature K of the level surfaces of L we conclude relation (3). Thus L becomes a Finsler function. Then from (3) the projective flatness follows easily (Theorem 1). After this we obtain a second order PDE system, whose solvability is necessary and sufficient for the existence of a coordinate system (x), in which $(*)$ is satisfied, that is for the projectively flat character of F^n (Theorem 2). Also in form of a PDE system we find necessary and sufficient conditions for the existence of a vector field $v(x)$ such that the infinitesimal transformations $x \to x+v(x)d\tau$ take geodesics into geodesics (Theorem 3). Thus this $v(x)$ is an analogue of a Killing vector field in case when the transformations preserve geodesics. In the last section we investigate the Finsler spaces which are projectively flat in a parameter-preserving manner. The obtained characterization of these spaces is very similar to $(*)$ (Theorem 4). Finally we show that a Finsler function $\mathcal{F}$ over $\mathbb{R}^n$ is projectively flat in a parameter-preserving manner, if and only if $(\mathbb{R}^n, \mathcal{F})$ is a Minkowski space (Theorem 5).

2 Projectively flat Finsler spaces

1. Our investigations are of local character, so we may assume that our base manifold M admits a global chart, and hence it can be identified with $\mathbb{R}^n$, or with a Euclidean space $\mathbb{E}^n$ with coordinate system (x^i).

First we give a simple proof of the following well known theorem, which is elementary also at the crucial points.

Theorem A (Hamel [12]; Rapcsak [19]; see also Z. Shen [20]; M. Crampin [10]; or J. Szilasi [25])**.** *A Finsler space $F^n = (\mathbb{E}^n, \mathcal{F})$ is projectively flat if and only if there exists a coordinate system (x) such that*

$$\frac{\partial^2 \mathcal{F}}{\partial x^k \partial y^i} y^k = \frac{\partial^2 \mathcal{F}}{\partial x^i \partial y^k} y^k. \tag{4}$$

Proof. a/ ((4) is necessary). The geodesics $x(t)$ of a Finsler space $F^n = (\mathbb{E}^n, \mathcal{F})$ satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{F}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{F}}{\partial y^i} = \frac{\partial \mathcal{F}}{\partial x^i} - \frac{\partial^2 \mathcal{F}}{\partial x^k \partial y^i} y^k - \frac{\partial^2 \mathcal{F}}{\partial y^k \partial y^i} y'^k = 0, \tag{5}$$

where $y = x'$. If in (x) the geodesics are straight lines $x(t) = f(t)a + b$ (see (1.a)), then $x'' = y'$ is parallel to $x' = y$, that is $y' = \lambda y$. By the first order homogeneity of $\mathcal{F}$ in y, the last term of (5) vanishes. Then (5) reduces to

$$\frac{\partial^2\mathcal{F}}{\partial x^k \partial y^i} y^k = \frac{\partial \mathcal{F}}{\partial x^i},$$

and by the homogeneity of $\mathcal{F}$ we obtain

$$\frac{\partial \mathcal{F}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{F}}{\partial y^k} y^k\right) = \frac{\partial^2\mathcal{F}}{\partial x^i \partial y^k} y^k,$$

which yields (4).

b/ ((4) is sufficient) In case of (4) the Euler-Lagrange equations reduce to

$$\frac{\partial^2\mathcal{F}}{\partial y^k \partial y^i} y'^k = 0. \tag{6}$$

$y' = 0$ is a solution of (6). This means that in case of (4) any

$$x(t) = ta + b, \qquad a \neq 0 \tag{7}$$

is a solution of (6), and thus (7) is a geodesic of F^n. Since a and b are arbitrary in (7), and in a Finsler space from each point and in any direction there exists exactly one geodesic with the given initial data, (7) yields all geodesics of F^n. Thus, under condition (4), the geodesics of F^n are straight lines in $\mathbb{E}^n$. □

Remark. t is a constant speed parameter of (7) in $\mathbb{E}^n$ ($t = c\,s$, where c is a constant, and s is the Euclidean arc length). However if (7) is considered as a geodesic of F^n, then t is not a constant speed parameter in general. If τ is a constant speed parameter on F^n, then $t = f(\tau)$ and (7) gets the form $x(\tau) = f(\tau)a + b$ (see (1.a)). Hence (4) does not mean that F^n, in this case, was projectively flat in a parameter-preserving manner.

2. Now we investigate 1^+-homogeneous Lagrange functions L, whose level surfaces have non-vanishing Gaussian curvature K.

Theorem 1. *Let L be a positive homogeneous Lagrange function, smooth on $\mathring{T}\mathbb{R}^n$ such that its level surfaces $L(x_0, y) = \mathrm{const}$ have nowhere vanishing Gaussian curvature. Then L is a pseudo-Finsler function, and the geodesics of L are straight lines if and only if*

$$\frac{\partial^2 L}{\partial y^k \partial x^i}(x,y) y^k = \frac{\partial^2 L}{\partial y^i \partial y^k}(x,y) y^k. \tag{8}$$

Proof. Let $L(y) := L(x_0, y),\ y \in \mathbb{R}^n \setminus \{0\}$. Since $\frac{\partial L}{\partial y^i}(y)y^i = L(y) = c \neq 0$, every $L(y)$ is a regular value of L, and thus $L^{-1}[c]$ is a hypersurface for each $c \neq 0$. Both $\{L^{-1}[c],\ c > 0\}$ and $\{L^{-1}[c],\ c < 0\}$ consist of homothetic hypersurfaces $\phi(c)$, but $L^{-1}[1]$ and $L^{-1}[-1]$ are not homothetic in general. (If L is a Finsler metric, then $\phi(1)$ is the indicatrix $\mathcal{I}(x)$). At a $y_0 \in \phi$ let e_α, $\alpha = \overline{1, n-1}$ be unit vectors in the principal directions of ϕ, and $e_n = N$ the unit normal vector of ϕ. Let u^i (u^n denoted also by z) be coordinates in $R^n(y)$ with respect to the base (e_α, N) with origin o. Then

$$u^i = A^i_k y^k + b^i \quad \text{and} \quad y^i = B^i_k u^k + c^i \tag{9}$$

denote the transformations $(y) \leftrightarrow (u)$ with an appropriate regular matrix A and its inverse B. ϕ has the representation of the form

$$(u^\alpha) \to \mathfrak{r}(u^\alpha) = u^\alpha e_\alpha + z(u^\alpha)N.$$

Then $L(y) = \bar{L}(u^1, \dots, u^{n-1}, z)$, and the Gauss formula takes the form

$$\mathfrak{r}_{u^\alpha u^\beta} = \Gamma_{\alpha}{}^{\sigma}{}_{\beta}\mathfrak{r}_\sigma + h_{\alpha\beta}\mathfrak{n}, \quad \frac{\partial \mathfrak{r}}{\partial u^\alpha} =: \mathfrak{r}_\alpha,$$

where $\mathfrak{n}$ is the unit normal vector field of ϕ. At o we obtain

$$\mathfrak{r}_{u^\alpha u^\beta} = (0, \dots, 0, z_{u^\alpha u^\beta}) = \Gamma_{\alpha}{}^{\sigma}{}_{\beta}\, e_\sigma + h_{\alpha\beta}N,$$

and hence

$$z_{u^\alpha u^\beta} = h_{\alpha\beta}. \tag{10}$$

Since the Gaussian curvature K of ϕ is nowhere vanishing according to our assumption, we conclude

$$K = \frac{\det(h_{\alpha\beta})}{\det(g_{\alpha\beta})} \neq 0 \Rightarrow \det(h_{\alpha\beta}) \neq 0 \Rightarrow \det(z_{\alpha\beta}) \neq 0. \tag{11}$$

We know that $\bar{L}(u^\alpha, z(u^\alpha)) = c$. From this we obtain

$$\frac{\partial \bar{L}}{\partial u^\alpha} + \frac{\partial \bar{L}}{\partial z}\frac{\partial z}{\partial u^\alpha} = 0,$$

and

$$\frac{\partial^2 \bar{L}}{\partial u^\beta \partial u^\alpha} + \frac{\partial^2 \bar{L}}{\partial z \partial u^\alpha}\frac{\partial z}{\partial u^\beta} + \frac{\partial^2 \bar{L}}{\partial u^\beta \partial z}\frac{\partial z}{\partial u^\alpha} + \frac{\partial^2 \bar{L}}{\partial z \partial z}\frac{\partial z}{\partial u^\alpha}\frac{\partial z}{\partial u^\beta} + \frac{\partial \bar{L}}{\partial z}\frac{\partial^2 z}{\partial u^\alpha \partial u^\beta} = 0. \tag{12}$$

At o $u^\alpha = 0$, and $\frac{\partial z}{\partial u^\alpha}(o) = 0$. Thus from (12), (10) and (11) at o we get

$$\left(\frac{\partial \bar{L}}{\partial z}\right)^{-n} \det\left(\frac{\partial^2 \bar{L}}{\partial u^\beta \partial u^\alpha}\right) = \det(h_{\alpha\beta}) \neq 0 \Rightarrow \det\left(\frac{\partial^2 \bar{L}}{\partial u^\beta \partial u^\alpha}(0)\right) \neq 0,$$

hence

$$\operatorname{rank} \frac{\partial^2 \bar{L}}{\partial u^k \partial u^i} \geq n-1.$$

According to (9), $\frac{\partial^2 \bar{L}}{\partial u^k \partial u^i} = \frac{\partial^2 L}{\partial y^r \partial y^s} A^r_k\, A^s_i$, where $\det A \neq 0$. So we also have

$$\operatorname{rank} \frac{\partial^2 L}{\partial y^r \partial y^s} \geq n-1. \tag{13}$$

On the other hand, since the functions $\frac{\partial L}{\partial y^k}$ are positive homogeneous of degree 0, we have

$$\frac{\partial^2 L}{\partial y^k \partial y^i} y^i = 0,$$

and hence $\operatorname{rank} \frac{\partial^2 L}{\partial y^k \partial y^i} < n$. With (13) this yields

$$\operatorname{rank} \frac{\partial^2 L}{\partial y^k \partial y^i} = n-1, \tag{14}$$

therefore L is a pseudo-Finsler function.

To prove that (8) is necessary and sufficient for the projective flatness, one can follow the considerations of sec. 1, replacing $\mathcal{F}$ by L. In the case of a Finsler function $\mathcal{F}$ the geodesic equation is explicit for y', and consequently for each point x and any direction y at x, there is a unique geodesic starting from x with initial velocity y. However, in the case of pseudo-Finslerian functions this is not true, that is the equation of the geodesics is not explicit for y'. Therefore it is not assured that the straight lines representing the solution $y' = 0$ of (6), yield every geodesic.

However, there are several ways to find all solutions of (6), and thus all geodesics. We proceed as follows: By the psuedo-Finslerian property of L, the quadratic equation

$$\frac{\partial^2 L}{\partial y^i \partial y^k}(x_0, y) u^i u^k = 0 \tag{15}$$

for u has the only nontrivial solution $u = \lambda y$, $\lambda \neq 0$. But every solution of

$$\frac{\partial^2 L}{\partial y^i \partial y^k}(x_0, y)u^i = 0 \tag{16}$$

is also a solution of (15). Nevertheless, with $u = y'$ (16) is the same as (6). Hence every solution of (6) (different of $y' = 0$) is given by $y' = \lambda y$. Then $x'' = \lambda x'$, and hence $x(t) = f(t)a + b$. This means that in case of (8) every geodesic is a straight line. $\square$

3. We know that a Riemannian space $V^n = (\mathbb{R}^n, g)$ is of constant curvature iff it is projectively flat. This yields

Corollary 1. *A Riemannian space $V^n = (\mathbb{R}^n, g)$ is of constant curvature if and only if*

$$\frac{\partial^2 \sqrt{g_{ij}y^i y^j}}{\partial x^k \partial y^m} y^k = \frac{\partial^2 \sqrt{g_{ij}y^i y^j}}{\partial x^m \partial y^k} y^k.$$

Also we obtain a corollary for a Randers space $\mathcal{R}^n$ with metric function $\mathcal{F}(x, y) = \sqrt{g_{ij}(x)y^i y^j} + b_i(x)y^i$, $V^n = (\mathbb{R}^n, g)$, where the Riemannian norm of b is smaller than 1, and b is a gradient vector field (more precisely, a closed 1-form). In this case

$$\begin{aligned} \frac{\partial^2 \mathcal{F}}{\partial x^s \partial y^i} y^s - \frac{\partial^2 \mathcal{F}}{\partial x^i \partial y^s} y^s = & \left[\frac{\partial^2 \sqrt{g_{kj}y^k y^j}}{\partial x^s \partial y^i} y^s - \frac{\partial^2 \sqrt{g_{kj}y^k y^j}}{\partial x^i \partial y^s} y^s \right] \\ & - \left(\frac{\partial b_s}{\partial x^i} - \frac{\partial b_i}{\partial x^s} \right) y^s. \end{aligned} \tag{17}$$

The term in the parenthesis vanishes because we supposed that b is closed. Then, since the vanishing of the expression in the bracket means the projective flatness of V^n by Theorem A, and the vanishing of the left hand side of (17) means the projective flatness of the Randers space $\mathcal{R}^n$ we obtain

Corollary 2. *If in a Randers space the deforming* 1*-form is closed, then it is projectively flat, if and only if, the corresponding Riemannian space is projectively flat.*

4. (4) is no tensor relation. So given a Finsler space $F^n = (\mathbb{R}^n, \mathcal{F})$ in a coordinate system (x), and by choosing another coordinate system $(\bar{x})$, the fact that relation

$$\frac{\partial^2 \mathcal{F}}{\partial \bar{x}^k \partial \bar{y}^i} \bar{y}^k \neq \frac{\partial^2 \mathcal{F}}{\partial \bar{x}^i \partial \bar{y}^k} \bar{y}^k \tag{4'}$$

holds, does not decide the projective flatness of F^n. We give conditions for the existence of a coordinate system (x) in which (4) is satisfied, and thus F^n is projectively flat.

Let $\bar{x}^i = \bar{x}^i(x)$ be a coordinate transformation. Then

$$\mathcal{F}(x, y) = \mathcal{F}(\bar{x}(x), \bar{y}(x, y)) \quad \text{with} \quad \bar{y}^p = \frac{\partial \bar{x}^p}{\partial x^k}(x) y^k,$$

and

$$\frac{\partial \mathcal{F}}{\partial y^i} = \frac{\partial \mathcal{F}}{\partial \bar{y}^r} \frac{\partial \bar{x}^r}{\partial x^i}.$$

After calculating $\frac{\partial^2 \mathcal{F}}{\partial x^s \partial y^i}$ we obtain

$$\frac{\partial^2 \mathcal{F}}{\partial x^s \partial y^i} y^s - \frac{\partial^2 \mathcal{F}}{\partial x^i \partial y^s} y^s = \tag{18}$$

$$a) \quad = \begin{cases} \left(\dfrac{\partial^2 \mathcal{F}}{\partial \bar{x}^m \partial \bar{y}^p} - \dfrac{\partial^2 \mathcal{F}}{\partial \bar{x}^p \partial \bar{y}^m} \right) \dfrac{\partial \bar{x}^m}{\partial x^s} \dfrac{\partial \bar{x}^p}{\partial x^i} y^s \\ + \dfrac{\partial^2 \mathcal{F}}{\partial \bar{y}^r \partial \bar{y}^p} \left[\dfrac{\partial^2 \bar{x}^p}{\partial x^s \partial x^k} \dfrac{\partial \bar{x}^r}{\partial x^i} - \dfrac{\partial^2 \bar{x}^p}{\partial x^i \partial x^k} \dfrac{\partial \bar{x}^r}{\partial x^s} \right] y^s y^k = 0 \end{cases} \tag{19}$$

$$b) \quad \bar{y}^j = \frac{\partial \bar{x}^j}{\partial x^p} y^p.$$

For a given Finsler function $\mathcal{F}$ (19) is a second order PDE system for $\bar{x}^i(x)$. If (19) is solvable, then (18) vanishes, which means that F^n is projectively flat, and the obtained coordinate system (x) is rectilinear (i.e. a geodesically linear coordinate system). Conversely, if F^n is projectively flat, then (18) vanishes, and the transformation $\bar{x}^i = \bar{x}^i(x)$ describing the transition between the starting coordinate system $(\bar{x})$ and the coordinate system (x), in which (18) vanishes, is a solution of (19). This yields

Theorem 2. *A Finsler space with Finsler metric $\mathcal{F}(\bar{x}, \bar{y})$ is projectively flat if and only if the second order PDE system (19) is solvable for $\bar{x} = \bar{x}(x)$. Then (x) is a rectilinear (geodesically linear) coordinate system.*

5. Let (x) be a geodesically linear (rectilinear) coordinate system for F^n. It is easy to see that exactly those other coordinate systems are geodesically linear, which are projectively related to (x). The projective flat character of F^n is characterized by the existence of a coordinate system (x), in which (4) is satisfied.

However, as we have already emphasized, (4) is no tensor relation.

Proposition 1. *Relation* (4) *is preserved by a coordinate transformation* $(x) \to (\bar{x})$*, if and only if, the transformation is linear.*

Proof. If $(x) \to (\bar{x})$ is linear, then the bracket in (19.a) vanishes, and the remaining part of (18) and (19) shows that for linear transformations (4) is a tensor relation.

Conversely, if (4) is a tensor relation, then the parenthesis in (19.a) vanishes. Denoting the expression in the bracket by $D^{pr}_{ski}(x)$, at (x_0, y) we obtain

$$D^{pr}_{ski}(x_0)\left[\frac{\partial^2 \mathcal{F}}{\partial \bar{y}^p \partial \bar{y}^r}(\bar{x}(x_0), \bar{y}(x_0, y_0))y^s y^k\right] = 0.$$

However the expression in the bracket depends on y, and can take many different values. Thus D^{pr}_{ski} must vanish at every x_0. Let $A^p_k := \frac{\partial \bar{x}^p}{\partial x^k}$, $(B^p_k) := (A^p_k)^{-1}$. Then relation $D^{pr}_{ski}(x) = 0$ gets the form

$$\left(\frac{\partial}{\partial x^s} A^p_k\right) A^r_i - \left(\frac{\partial}{\partial x^i} A^p_k\right) A^r_s = 0.$$

Multiplying by B^m_r, we obtain

$$\left(\frac{\partial}{\partial x^s} A^p_k\right) \delta^m_i - \left(\frac{\partial}{\partial x^i} A^p_k\right) \delta^m_s = 0.$$

A contraction for m and i yields $\frac{\partial}{\partial x^s} A^p_k = 0$, and hence $\bar{x}^i(x)$ is linear. ☐

One can see in a similar way that $\frac{\partial^2 \mathcal{F}}{\partial x^k \partial y^i}$ has also a tensorial character for linear transformations only.

6. We want to find conditions under which an infinitesimal transformation

$$\varphi : x^i(t) \to \bar{x}^i(t) = x^i(t) + v^i(x(t))d\tau \tag{20}$$

takes the geodesics of $F^n = (\mathbb{R}^n, \mathcal{F})$ again into geodesics of F^n, up to linear terms in $d\tau$. From (20) we obtain

$$\bar{y}^i = \bar{x}'^i = \left(\delta^i_k + \frac{\partial v^i}{\partial x^k} d\tau\right) y^k \tag{21}$$

and

$$\bar{y}'^k = y'^k + \left(\frac{\partial v^s}{\partial x^i} y'^i + \frac{\partial^2 v^k}{\partial x^s \partial x^i} y^s y^i\right) d\tau. \tag{22}$$

$\bar{x}(t)$ is a geodesic, if and only if

$$\frac{\partial \mathcal{F}}{\partial x^i}(\bar{x}(t), \bar{y}(t)) = \frac{\partial^2 \mathcal{F}}{\partial x^k \partial y^i}(\bar{x}(t), \bar{y}(t))\bar{y}^k + \frac{\partial^2 \mathcal{F}}{\partial y^k \partial y^i}(\bar{x}(t)), \bar{y}(t))\bar{y}'^k. \tag{23}$$

Developing the partial derivatives of $\mathcal{F}(\bar{x}, \bar{y})$ into power series around $\bar{x}(0) = x(0)$, $\bar{y}(0) = y(0)$, taking into consideration the value of $\bar{x}^s - x^s$, $\bar{y}^s - y^s$ and $\bar{y}'^k - y'^k$ obtained form (20), (21), (22), and neglecting the terms with $(d\tau)^k$, $k > 1$, a straightforward calculation gives

$$\begin{aligned}
&\left[\frac{\partial^2 \mathcal{F}}{\partial x^s \partial x^i} - \frac{\partial^3 \mathcal{F}}{\partial x^s \partial x^k \partial y^i} y^k - \frac{\partial^3 \mathcal{F}}{\partial x^s \partial y^k \partial y^i} C^k\right] v^s \\
&+ \left[\left(\frac{\partial^2 \mathcal{F}}{\partial y^s \partial x^i} - \frac{\partial^2 \mathcal{F}}{\partial y^i \partial x^s}\right) y^r - \frac{\partial^2 \mathcal{F}}{\partial y^s \partial y^i} C^r\right. \\
&\left.- \left(\frac{\partial^3 \mathcal{F}}{\partial y^s \partial x^k \partial y^i} y^k + \frac{\partial^3 \mathcal{F}}{\partial y^s \partial y^k \partial y^i} C^k\right) y^r\right] \frac{\partial v^s}{\partial x^r} \\
&- \left[\frac{\partial^2 \mathcal{F}}{\partial y^s \partial y^i} y^m y^r\right] \frac{\partial^2 v^s}{\partial x^m \partial x^r} = 0.
\end{aligned} \tag{24}$$

Here we took into consideration that $x(t)$ is a geodesic of F^n, and thus it satisfies (5). Furthermore $C^k(x, y)$ means a solution of

$$\frac{\partial^2 \mathcal{F}}{\partial y^k \partial y^i} C^k = \frac{\partial \mathcal{F}}{\partial x^i} - \frac{\partial^2 \mathcal{F}}{\partial x^k \partial y^i} y^k.$$

So $v^i(x)$ must satisfy (24) for every y in order that φ be an infinitesimal geodesic mapping. That is (24) is a necessary condition. However (24) is also sufficient. Namely (5) and (24) lead to (23). These yield

Theorem 3. *An infinitesimal transformation $\varphi : x \to x + v(x)d\tau$ takes the geodesics of F^n into geodesics (up to linear terms in $d\tau$) if and only if the second order PDE system* (24) *is solvable for $v(x)$.*

In (24) $y^1, \ldots, y^n$ can be considered as parameters. But we can complete (24) with the identities

$$\frac{\partial v^i}{\partial y^r} = 0. \tag{25}$$

Then (24) and (25) together form a PDE system for $v^i(x,y)$.

If φ is an infinitesimal isometry, then the role of (24) is played by the Killing equations. Thus (24) can be considered as a Killing type equation when infinitesimal isometries are replaced by infinitesimal geodesic mappings.

3 The parameter-preserving case

Sometimes it is more convenient to use the energy function $\mathcal{F}^2$ in place of $\mathcal{F}$. It is easy to see that applying constant speed parameter t, the Euler-Lagrange equation (5) is equivalent to

$$\frac{\partial \mathcal{F}^2}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{F}^2}{\partial y^i} = \frac{\partial \mathcal{F}^2}{\partial x^i} - \frac{\partial^2 \mathcal{F}^2}{\partial x^k \partial y^i} y^k - \frac{\partial^2 \mathcal{F}^2}{\partial y^k \partial y^i} y'^k = 0. \tag{26}$$

Indeed, since $\mathcal{F}$ is constant along any geodesic $x(t)$ with constant speed parameter t, i.e., $\mathcal{F}(x(t), x'(t)) = \text{const.} = c$ (in arc length parameter $c = 1$), we have

$$\begin{aligned} \frac{\partial \mathcal{F}^2}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{F}^2}{\partial y^i} &= 2\mathcal{F}\frac{\partial \mathcal{F}}{\partial x^i} - \frac{d}{dt}\left(2\mathcal{F}\frac{\partial \mathcal{F}^2}{\partial y^i}\right) \\ &= 2\mathcal{F}\left[\frac{\partial \mathcal{F}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{F}}{\partial y^i}\right] = 0. \end{aligned} \tag{27}$$

Then the counterpart of Theorem A is the following

Theorem 4. *A Finsler space $F^n = (\mathbb{R}^n, \mathcal{F})$ is projectively flat in a parameter-preserving manner if and only if there exists a coordinate system (x) in which*

$$\frac{\partial^2 \mathcal{F}^2}{\partial x^k \partial y^i} y^k = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial x^i \partial y^k} y^k. \tag{28}$$

Proof. a/ If F^n is projectively flat in a parameter-preserving manner, then there exists a coordinate system (x), in which the geodesics have the form $x(t) = ta + b$. Then $y' = x'' = 0$, and the Euler-Lagrange equation (26) reduces to

$$\frac{\partial \mathcal{F}^2}{\partial x^i} = \frac{\partial^2 \mathcal{F}^2}{\partial x^k \partial y^i} y^k.$$

However, by the second order homogeneity of $\mathcal{F}^2$, we have

$$\frac{\partial \mathcal{F}^2}{\partial x^i} = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial x^i \partial y^k} y^k, \tag{29}$$

and these give (28).

b/ Conversely, if one has (28), then (29) and (26) yield

$$\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^k} y'^k = 2 g_{ik} y'^k = 0. \tag{30}$$

Then $y' = x'' = 0$, and thus $x(t) = ta + b$. □

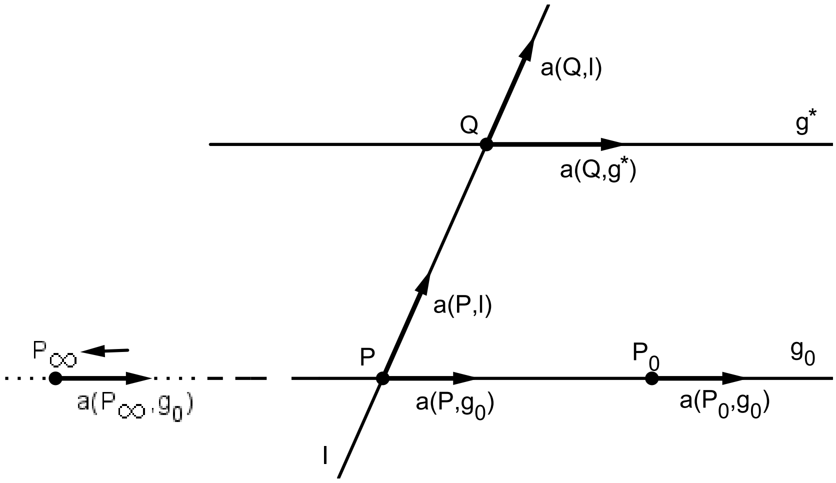
Theorem 5. *A Finsler space $F^n = (\mathbb{R}^n, \mathcal{F})$ is projectively flat in a parameter-preserving manner if and only if it is a Minkowski space.*

Proof. a) In an adapted coordinate system (x) the metric function $\mathcal{F}$ of any Minkowski space $\mathcal{M}^n$ is independent of x, and it satisfies (28). Then (26) reduces to (30), which is satisfied by any straight line $g(t) \equiv ta + b$. Thus any $\mathcal{M}^n$ is projectively flat in a parameter-preserving manner.

b) If F^n is projectively flat (in a parameter-preserving manner), then the geodesics can be represented by $g(t) = ta+b$, where a is the constant velocity vector of the geodesic: $\mathcal{F}(g(t), g'(t)) = \mathcal{F}(g(t), a) = \|a\|_F = \text{const}$ along $g(t)$.

It is no restriction of the generality if we assume that t is the Finslerian arc length parameter, and thus $c = 1$. Then a, which is the same at any point of the straight line $g(t)$, is an element of the indicatrices $\mathcal{I}(g(t))$. At a point P, and in the direction of a straight line g, a will be denoted by $a(P, g)$.

Let P_0 be an arbitrary point of $\mathbb{R}^n$, and g_0 a straight line through it. Let Q be an arbitrary point of $\mathbb{R}^n$ out of g_0, and g^* a straight line through Q parallel to g_0. Finally let ℓ be a straight line through Q intersecting g_0 in P.



Then let us turn ℓ around Q into $g^*: \ell \to g^*$. Then P tends to infinity on $g_0: P \to P_\infty$. So we have

$$\left.\begin{array}{l} a(Q,\ell) \to a(Q,g^*) \\ \text{and} \\ a(Q,\ell) = a(P,\ell) \to a(P_\infty,g_0) = a(P_0,g_0) \end{array}\right\} \Rightarrow a(Q,g^*) = a(P_0,g_0)$$

From these

$$\mathcal{I}(P_0) \ni a(P_0,g_0) \cong a(Q,g^*) \in \mathcal{I}(Q).$$

($\cong$ means: congruent by parallel translation.) But Q was an arbitrary point of $\mathbb{R}^n$, and g_0 can be any line through P_0. Then

$$\mathcal{I}(Q) \cong \mathcal{I}(P_0),$$

which means that F^n is a Minkowski space. $\square$

If $a(P_0,g_0) \cong a(P_0,g)$ for any g through P_0, that is $a(P_0,\cdot)$ is independent of the direction, then F^n is a Euclidean space $\mathbb{E}^n$.

Corollary 3. *If a Riemannian space $V^n = (\mathbb{R}^n, g)$ is projectively flat in a parameter-preserving manner, then it is a Euclidean space* (for the indicatrices are congruent ellipsoids).

This corollary can also be proved by a short calculation. The Riemannian energy function is $\mathcal{F}^2(x,y) = g_{ih}(x)y^iy^k$. By (28)

$$2\frac{\partial g_{ri}}{\partial x^k}y^ry^k = \frac{\partial g_{rk}}{\partial x^i}y^ry^k,$$

and hence

$$2\frac{\partial g_{ri}}{\partial x^k} = \frac{\partial g_{rk}}{\partial x^i}. \tag{31}$$

Dividing by 2, and changing i and k, we obtain $\frac{\partial g_{rk}}{\partial x^i} = \frac{1}{2}\frac{\partial g_{ri}}{\partial x^k}$. Replacing this into (31) gives

$$4\frac{\partial g_{ri}}{\partial x^k} = \frac{\partial g_{ri}}{\partial x^k}, \text{ therefore } \frac{\partial g_{ij}}{\partial x^k} = 0,$$

and hence g_{ij} is constant. □

Acknowledgments The second author was supported by the National Science Research Foundation OTKA No. NK-81402.

References

[1] Alexander, R., Zonoid theory and Hilbert's fourth problem, *Geometria Dedicata* **28**, 199–211 (1988)

[2] Alvarez Paiva, J. C., Symplectic geometry and Hilbert's fourth problem, *J. Diff. Geom.* **69**, 353–378 (2005)

[3] Bácsó, S., Szilasi, Z., On the projective theory of sprays, *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, **26**, 171–207 (2010)

[4] Bao, D., Chern, S.S., Shen, Z., An introduction to Riemann-Finsler geometry, Springer Verlag New York, (2000)

[5] Binh, T. Q.; Cheng, Xinyue, On a class of projectively flat (α,β)-Finsler metric, *Publ. Math. Debrecen*, **73**, 391–400 (2008)

[6] Busemann, H., The Geometry of Geodesics, Academic Press, New York, (1955)

[7] Busemann, H., Problem IV: Desarguesian Spaces, *Proc. A.M.S. Symp. Pure Math.* **28**, 131–144 (1976)

[8] Busemann, H., Kelly, P., Projective Geometry and Projective Metrics, Academic Press, New York, (1953)

[9] Cheng, Xinyue; Li, Ming, On a class of projectively flat Randers metrics, *Publ. Math. Debrecen*, **71**, 195–205 (2007)

[10] Crampin, M., Some remarks on the Finslerian version of Hilbert's fourth problem (to appear in *Houston J. of Math.*)

[11] Funk, P., Über Geometrien, bei denen die Geraden die Kürzesten sind, *Math. Ann.* **101**, 226–237 (1929)

[12] Hamel, G., Über die Geometrien, in denen die Geraden die Kürzesten sind, *Math. Ann.* **57**, 231–264 (1903)

[13] Hilbert, D., Gesammelte Abhandlugen, Band III, Seite 303–305, or *Archiv f. Math. u. Phys. 3. Reihe, Bd. I*, S. 44–65, S. 213–232, (1901)

[14] Li, Benling; Shen, Zhongmin, On a class of projectively flat Finsler metrics with constant flag cuvature, *Internat. J. Math.* **18**, 749–760 (2007)

[15] Matsumoto, M., Projective changes of Finsler metrics and projectively flat Finsler spaces, *Tensor* **34**, 303–315 (1980)

[16] Matsumoto, M., Wei, X., Projective changes of Finsler spaces of constant curvature, *Publ. Math. Debrecen* **44**, 175–181 (1994)

[17] Mo, Xiaohuan; Shen, Zhongmin; Yang, Chunhong, Some constructions of projectively flat Finsler metrics, *Sci. China Ser. A* **49**, 708–714 (2006)

[18] Pogorelov, A. V., Hilbert's fourth problem, Scripta Series in Mathematics, Winston and Sons, New York, (1979)

[19] Rapcsák, A., Über die bahntreuen Abbildungen metrischer Räume, *Publ. Math. Debrecen* **8**, 285–290 (1961)

[20] Shen, Zhongmin, Differential Geometry of Spray and Finsler Spaces, Kluwer Ac. Publ., Dordrecht (2001)

[21] Shen, Zhongmin, On projectively flat (α, β)-metrics, *Canad. Math. Bull.* **52**, 132–144 (2009)

[22] Shen, Zhongmin, Projectively flat Randers metrics with constant flag curvature, *Math. Ann.* **325**, 19–30 (2003)

[23] Shen, Zongmin; Yildrim, C., On a class with projectively flat metrics with constant flag curvature, *Canad. J. Math.* **60**, 443–456 (2008)

[24] Szabó, Z. I., Hilbert's Fourth Problem I, *Advances in Math.* **59**, 185–301 (1986)

[25] Szilasi, J., Calculus along the tangent bundle projection and projective metrizability, Diff. Geom. Appl. 539–568; World Sci. Publ., New Jersey (2009)

[26] Tamássy, L., Relation between metric spaces and Finsler spaces, *Diff. Geom. Appl.* **26**, 483–494 (2008)