A NEW APPROACH TO GENERALIZED BERWALD MANIFOLDS I

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Abstract. A large class of special Finsler manifolds can be endowed with Finsler connections whose "h-part" does not depend on the directions. We call these Finsler connections *h-basic* and present a systematic treatment of them, using (in a simplified form) the Frölicher-Nijenhuis calculus. We provide an axiomatic description of a distinguished class of h-basic Finsler connections, the class of *Ichijyō connections*. With the help of an Ichijyō connection we present new characterizations of generalized Berwald manifolds, as well as – in particular – of Berwald manifolds and locally Minkowski manifolds.

Introduction

"Through the author's several experiences the author became convinced that there should exist the *best* Finsler connection for every theory of Finsler spaces" – wrote Makoto Matsumoto in 1987 ([9]). We believe that the present work will also be a manifestation of this remarkable and stimulating principle.

There is a large and very important class of Finsler manifolds whose Finsler structure, the energy – or the fundamental – function, is linked to a linear connection of the carrying manifold in a natural manner: the parallel translations with respect to the linear connection preserve the Finslerian length of the tangent vectors. This is the class of *generalized Berwald manifolds* (for an equivalent definition see **4.1**). Berwald manifolds and Wagner manifolds belong to this class, whose importance lies (among others) in the fact that generalized Berwald manifolds *may* have a rich isometry group (see [10]). We found that to any generalized Berwald manifold a *whole class* of "best" Finsler connections can be attached in general. We call the members of this class *Ichijyō connections*. One of our results is a purely intrinsic characterization of the Ichijyō connections by means of simple axioms.

Any Ichijyō connection is determined by a linear connection on the carrying manifold. Finsler connections arising from a "base linear connection" were baptized "linear Finsler connections" in [5]. This terminology would be ambiguous in our theoretical framework, so we tentatively introduce the term "*h-basic connection*" (h as "horizontally") instead. Other choices for an expressive (or a more expressive) term are also possible, of course. Finsler manifolds whose structure is connected with a base linear connection were called "point Finsler spaces" by L. Tamássy. As for his instructive geometric approach, we refer to [13].

The paper is organised as follows. In *Section 1* mainly background material is presented about the basic tools, concentrating on horizontal endomorphisms and

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general Finsler connections. The simple observation on the coincidence of two horizontal endomorphisms in **1.5** will be repeatedly applied in our investigations. *Section 2* is devoted to a concise but systematic study of the h-basic Finsler connections. The main result of this part characterizes the h-metrical h-basic Finsler connections on a non-Riemannian Finsler manifold. In *Section 3* we establish the existence and unicity of the Ichijyō connection on a Finsler manifold endowed with a "basic" linear connection. A list of essential curvature and torsion identities concerning the Ichijyō connection is also presented here. The concluding *Section 4* provides applications to generalized Berwald manifolds. Using an Ichijyō connection, we obtain a simple characterization of them, as well as of Berwald and locally Minkowski manifolds.

1. A review on horizontal endomorphisms and Finsler connections

1.1. The foundations of our present study were laid down by J. Grifone in his pioneering works [3] and [4]. A systematic approach in this spirit to Finsler manifolds, Finsler connections, and so on, was elaborated in detail in the recent surveys [11], [12]. In our subsequent considerations we almost completely adopt the conceptual and notational conventions of these papers. With occasional but characteristic exceptions, we will stay entirely within the category of C^∞ manifolds and mappings. So M always stands for a smooth manifold which is supposed to be paracompact and of finite dimension $n \geq 1$. $\pi : TM \to M$ is the tangent bundle of M, $\pi_0 : \mathcal{T}M \to M$ is the subbundle of the nonzero tangent vectors to M. $\mathfrak{X}(M)$ denotes the module of vector fields on M. The canonical objects of the tangent bundle $TM \to TTM$, namely the *vertical subbundle*, the *Liouville vector field* and the *vertical endomorphism* (or canonical almost tangent structure) are denoted by τ^{v}_{TM}, C and J, respectively. $\mathfrak{X}^{\mathrm{v}}(TM)$ denotes the module of sections of τ^{v}_{TM}; its elements are called *vertical vector fields*. We are going to use freely (and frequently) the notion and the basic properties of the vertical lift X^{v} and the complete lift X^c of a vector field $X \in \mathfrak{X}(M)$. The most important relations concerning these liftings are concisely summarized in [11] and [12]; see also the monograph [14] (of course), and [1]. A large part of our calculations is based on the following simple observation:

if $(X_i)_{i=1}^n$ is a local basis for the module $\mathfrak{X}(M)$, then $(X_i^{\mathrm{v}}, X_i^c)$ is a local base for the module $\mathfrak{X}(TM)$.

We are also going to use systematically the basic tools of the *Frölicher-Nijenhuis calculus* (operators i_K and d_K attached to a vector-valued form K, the Frölicher-Nijenhuis bracket [,] of vector forms and so on). The best source for mastering this wonderful theory still remains the original paper [2]; see also [1] and [8]. Recall that the above operators reduce to the usual insertion operator i_X, the Lie derivative $\mathcal{L}_X$ and the Lie bracket of vector fields, in particular. The operator of the exterior derivative will be denoted by d.

1.2. Semisprays and sprays. A vector field $S : TM \to TTM$ is said to be a *semispray* on the manifold M if it is of class C^1 on TM, smooth on $\mathcal{T}M$, and satisfies the relation $JS = C$. A semispray is called a *spray* if the homogeneity condition $[C, S] = S$ holds

The following formula, due to J. GRIFONE ([3], Prop. I.7) will be useful. – Let S be a semispray on M. Then for any vertical vector field X on TM we have

$$J[X,S]=X. \tag{1.2}$$

1.3. Horizontal endomorphisms. The role of nonlinear connections is played by the *horizontal endomorphisms* in our approach. Let us consider a vector 1-form on TM, i.e., a type $(1,1)$ tensor field $h:\mathfrak{X}(TM)\to\mathfrak{X}(TM)$, whose *smoothness is required only on* $\mathcal{T}M$. h is said to be a *horizontal endomorphism* on M, if it is a projector (i.e., $h^2=h$) and $\operatorname{Ker} h=\mathfrak{X}^{\mathrm{v}}(TM)$. $v:=1_{\mathfrak{X}(TM)}-h$ is called the *vertical projector* belonging to h. If $\mathfrak{X}^h(TM):=\operatorname{Im} h$, then we have the direct decomposition

$$\mathfrak{X}(TM)=\mathfrak{X}^{\mathrm{v}}(TM)\oplus\mathfrak{X}^h(TM);$$

the elements of $\mathfrak{X}^h(TM)$ are called *horizontal vector fields*. The mapping

$$X\in\mathfrak{X}(M)\mapsto X^h:=hX^c\in\mathfrak{X}^h(TM)$$

is the *horizontal lifting* with respect to h. The following "second local basis principle" (c.f. **1.1.**) will also be used systematically:

> *if $(X_i)_{i=1}^n$ is a local basis for the module $\mathfrak{X}(M)$ and h is a horizontal endomorphism on M, then $(X_i^{\mathrm{v}},X_i^h)_{i=1}^n$ is a local basis for $\mathfrak{X}(TM)$.*

It follows easily from the definitions that

$$h\circ J=0,\qquad J\circ h=J; \tag{1.3a}$$

and for any vector fields X, Y on M,

$$JX^h=X^{\mathrm{v}},\qquad J[X^h,Y^h]=[X,Y]^{\mathrm{v}}. \tag{1.3b}$$

1.4. Let a horizontal endomorphism h be given on the manifold M. If S' is an arbitrary semispray on M, then $S:=hS'$ is also a semispray on M which does not depend on the choice of S'. S is called the *semispray associated to* h. In the spirit of GRIFONE's theory, we attach to h the following data:

(1.4a) $H:=[h,C]$ – the *tension* vector 1-form;

(1.4b) $t:=[J,h]$ – the *torsion vector 2-form* or *weak torsion*;

(1.4c) $T:=i_St+H$ – the *torsion vector 1-form* or *strong torsion* (S is an arbitrary semispray);

(1.4d) $\Omega:=-\frac{1}{2}[h,h]$ – the *curvature vector 2-form*;

(1.4.e) $F:=h[S,h]-J$ – the *almost complex structure* induced by h (S is the semispray associated to h).

A horizontal endomorphism is said to be *homogeneous*, if its tension vanishes. We recall that any linear connection ∇ on the manifold M gives rise to a homogeneous, everywhere smooth horizontal endomorphism h_∇. In this case the data (1.4a)–(1.4e) are denoted by $H_\nabla,\dots,F_\nabla$.

1.5. Lemma. *Suppose that h and $\widetilde{h}$ are homogeneous horizontal endomorphisms on M. If for any vector fields X, Y on M,*

$$[X^h, Y^{\mathrm{v}}] = \left[X^{\widetilde{h}}, Y^{\mathrm{v}}\right], \tag{1.5a}$$

then $h = \widetilde{h}$.

Proof. We shall use the following simple observation:

$$\begin{aligned}&\textit{a vector field } Z \in \mathfrak{X}(TM) \textit{ is a vertical lift}\\ &\textit{if and only if } JZ = 0 \textit{ and } [J, Z] = 0.\end{aligned} \tag{1.5b}$$

Since

$$J\left(X^h - X^{\widetilde{h}}\right) = JX^h - JX^{\widetilde{h}} \overset{(1.3b)}{=} X^{\mathrm{v}} - X^{\mathrm{v}} = 0,$$

$X^h - X^{\widetilde{h}}$ is a vertical vector field. By the condition (1.5a) this vertical vector field commutes with any vertically lifted vector field. Using (1.5b) this implies easily that $X^h - X^{\widetilde{h}}$ is also a vertical lift. Thus (again by (1.5b))

$$\left[J, X^h - X^{\widetilde{h}}\right] = 0.$$

Now take an arbitrary semispray S. Then

$$\begin{aligned}0 = \left[J, X^h - X^{\widetilde{h}}\right] S &= \left[JS, X^h - X^{\widetilde{h}}\right] - J\left[S, X^h - X^{\widetilde{h}}\right]\\ &= \left[C, X^h - X^{\widetilde{h}}\right] - J\left[S, X^h - X^{\widetilde{h}}\right].\end{aligned}$$

The first term on the right hand side vanishes by the homogeneity of h and $\widetilde{h}$, while $J\left[S, X^h - X^{\widetilde{h}}\right] = X^{\widetilde{h}} - X^h$ in view of (1.2a). Thus for each vector field X on M, we have $X^h = X^{\widetilde{h}}$. This means that the horizontal endomorphisms h and $\widetilde{h}$ are identical. □

1.6. Finsler connections. A pair (D, h) is said to be *Finsler connection* on the manifold M, if D is a linear connection on the tangent manifold TM (or on the slit manifold $\mathcal{T}M$), h is a horizontal endomorphism on M, and the following conditions are satisfied:

$$D \textit{ is reducible } (\text{i.e., } Dh = 0); \tag{1.6a}$$

$$D \textit{ is almost complex } (\text{i.e., } DF = 0) \tag{1.6b}$$

(F is the almost complex structure associated to h by (1.4e)). The covariant differential DC of the Liouville vector field is said to be the *deflection* of (D, h); $h^*(DC)$ and $v^*(DC)$ are called the *h-deflection* and the *v-deflection*, respectively.

Condition (1.6b) guarantees that

$$\begin{aligned}Y \in \mathfrak{X}^{\mathrm{v}}(TM) &\Longrightarrow \forall\, X \in \mathfrak{X}(TM) : D_X Y \in \mathfrak{X}^{\mathrm{v}}(TM),\\ Y \in \mathfrak{X}^h(TM) &\Longrightarrow \forall\, X \in \mathfrak{X}(TM) : D_X Y \in \mathfrak{X}^h(TM).\end{aligned}$$

To any Finsler connection (D, h) two "partial covariant differential operators" D_h and D_v can naturally be associated as follows.

If A is a type $(r, s) \neq (0,0)$ tensor field on $\mathcal{T}M$, then we define the $(r, s+1)$ tensor fields $D_h A$ and $D_v A$ by the rules

$$i_X D_h A := D_{hX} A \quad \text{and} \quad i_X D_v A := D_{vX} A \qquad (X \in \mathfrak{X}(\mathcal{T}M)).$$

In particular, for any vector field Y on $\mathcal{T}M$,

$$(D_h Y)(X) = D_{hX} Y, \qquad (D_v Y)(X) = D_{vX} Y.$$

1.7. Curvatures and torsions of a Finsler connection. Suppose that (D, h) is a Finsler connection on the manifold M, and let us denote by $\mathbb{K}$ and $\mathbb{T}$ the classical curvature and torsion tensor of D, respectively. $\mathbb{K}$ and $\mathbb{T}$ can be determined by three "partial curvatures" and five "partial torsions", which also have importance (but not the same importance) on their own right. These data are summarized in the following table:

Curvature	
horizontal (h)	$\mathbb{R}(X,Y)Z := \mathbb{K}(hX, hY)JZ$
mixed (hv)	$\mathbb{P}(X,Y)Z := \mathbb{K}(hX, JY)JZ$
vertical (v)	$\mathbb{Q}(X,Y)Z := \mathbb{K}(JX, JY)JZ$

Torsion	
$h-$ horizontal $((h)h)$	$\mathbb{A}(X,Y) := h\mathbb{T}(hX, hY)$
$h-$ mixed $((h)hv)$	$\mathbb{B}(X,Y) := h\mathbb{T}(hX, JY)$
$v-$ horizontal $((v)h)$	$\mathbb{R}^1(X,Y) := v\mathbb{T}(hX, hY)$
$v-$ mixed $((v)hv)$	$\mathbb{P}^1(X,Y) := v\mathbb{T}(hX, JY)$
$v-$ vertical $((v)v)$	$\mathbb{S}^1(X,Y) := v\mathbb{T}(JX, JY)$

1.8. The operator D^i_J. Let $\Psi^1(TM)$ be the $C^\infty(TM)$-module of the vector 1-forms, i.e., of the type $(1,1)$ tensor fields on TM. First we consider the canonical mapping

$$D^i_J : \mathfrak{X}^{\mathrm{v}}(TM) \to \Psi^1(TM), \quad JY \mapsto D^i_J JY := [J, JY].$$

Using the property $[J, J] = 0$ it can be easily seen that for any vector field X on TM we have

$$D^i_{JX} JY := \left(D^i_J JY\right)(X) = J[JX, Y]. \tag{1.8a}$$

Now we suppose that h is a horizontal endomorphism on M, v is the complementary projection to h, and F is the almost complex structure belonging to h. Since $v = J \circ F$, we can also consider the vector field

$$D^i_{vX} JY = D^i_{JFX} JY = J[vX, Y]. \tag{1.8b}$$

With the help of h and keeping in mind the "Finslerian property" (1.6b), we prolong the operator D^i_J to $\mathfrak{X}^h(TM)$ so that for any vector field Y on TM,

$$D^i_J hY = D^i_J FJY := FD^i_J JY.$$

Then

$$D^i_{JX} hY := \left(D^i_J hY\right)(X) = FD^i_{JX} JY = F \circ J[JX, Y] = h[JX, Y].$$

In the presence of a horizontal endomorphism, D^i_J will always denote this extended operator.

1.9. Definition and lemma. Let (D, h) be a Finsler connection on the manifold M, and let us consider the (extended) operator D^i_J. If

$$\widetilde{D} : (X, Y) \in \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \mapsto \widetilde{D}_X Y := D_{hX} Y + D^i_{vX} Y,$$

then $(\widetilde{D}, h)$ is also a Finsler connection, called the *associated Finsler connection* to (D, h). For the mixed curvature $\widetilde{\mathbb{P}}$ of $(\widetilde{D}, h)$ we have the expression

$$\widetilde{\mathbb{P}}(X^c, Y^c)Z^c = -[J, D_{X^h} Z^{\mathrm{v}}]Y^c \qquad (X, Y, Z \in \mathfrak{X}(M)). \tag{1.9}$$

Proof. It may be seen immediately that $(\widetilde{D}, h)$ is indeed a linear connection. If $\widetilde{\mathbb{K}}$ is the curvature tensor of $\widetilde{D}$, then for any vector fields X, Y, Z on M we have

$$\widetilde{\mathbb{P}}(X^c, Y^c)Z^c = \widetilde{\mathbb{K}}(X^h, Y^{\mathrm{v}})Z^{\mathrm{v}} = \widetilde{D}_{X^h} \widetilde{D}_{Y^{\mathrm{v}}} Z^{\mathrm{v}} - \widetilde{D}_{Y^{\mathrm{v}}} \widetilde{D}_{X^h} Z^{\mathrm{v}} - \widetilde{D}_{[X^h, Y^{\mathrm{v}}]} Z^{\mathrm{v}}.$$

Here

$$\widetilde{D}_{Y^{\mathrm{v}}} Z^{\mathrm{v}} = D^i_{JY^c} J Z^c \overset{(1.8a)}{=} J[Y^{\mathrm{v}}, Z^c] = J[Y, Z]^{\mathrm{v}} = 0;$$

$$\widetilde{D}_{[X^h, Y^{\mathrm{v}}]} Z^{\mathrm{v}} = \widetilde{D}_{v[X^h, Y^{\mathrm{v}}]} Z^{\mathrm{v}} = D^i_{v[X^h, Y^{\mathrm{v}}]} J Z^c$$
$$\overset{(1.8b)}{=} J\big[[X^h, Y^{\mathrm{v}}], Z^c\big] = [J[X^h, Y^{\mathrm{v}}], Z^c] - [J, Z^c][X^h, Y^{\mathrm{v}}] = 0,$$

since $[X^h, Y^{\mathrm{v}}]$ is vertical, while $[J, Z^c] = 0$ by (1.3c) of [12]. The remaining second term can be formed as follows:

$$-\widetilde{D}_{Y^{\mathrm{v}}} \widetilde{D}_{X^h} Z^{\mathrm{v}} = -\widetilde{D}_{Y^{\mathrm{v}}}(D_{X^h} Z^{\mathrm{v}}) = -D^i_{Y^{\mathrm{v}}}(D_{X^h} Z^v) = -D^i_{JY^c}(JFD_{X^h} Z^{\mathrm{v}})$$
$$= -\left[D^i_J(JFD_{X^h} Z^v)\right] Y^c = -[J, JFD_{X^h} Z^{\mathrm{v}}]Y^c = -[J, D_{X^h} Z^{\mathrm{v}}]Y^c. \qquad \square$$

1.10. Finsler manifolds. Vertical and prolonged metric. Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E) is said to be a *Finsler manifold* with *energy function* E if the following conditions are satisfied:

$$\forall\, a \in \mathcal{T}M : E(a) > 0, \quad E(0) = 0; \tag{1.10a}$$

$$E \text{ is of class } C^1 \text{ on } TM \text{ and smooth on } \mathcal{T}M; \tag{1.10b}$$

$$CE = 2E, \text{ i.e., } E \text{ is homogeneous of degree 2}; \tag{1.10c}$$

$$\text{the } \textit{fundamental form } \omega := dd_J E \text{ is symplectic.} \tag{1.10d}$$

Then there is a spray $S_0 : TM \to TTM$, uniquely determined on $\mathcal{T}M$ by the relation

$$i_{S_0} \omega = -dE \tag{1.10e}$$

and prolonged to a C^1-mapping of TM such that $S_0(0) = 0$. This spray is called the *canonical spray* of the Finsler manifold. The mapping

$$\overline{g} : \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M) \times \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M) \to C^\infty(\mathcal{T}M), (JX, JY) \mapsto \overline{g}(JX, JY) := \omega(JX, Y) \tag{1.10f}$$

is a well-defined, nondegenerate symmetric bilinear form, which is said to be the *vertical metric* of (M, E). Taking an arbitrary horizontal endomorphism h on M, $\overline{g}$ can be prolonged to $\mathfrak{X}(\mathcal{T}M)$ as follows: for any vector fields $X, Y \in \mathfrak{X}(\mathcal{T}M)$,

$$g(X, Y) := \overline{g}(JX, JY) + \overline{g}(vX, vY), \qquad v := 1_{\mathfrak{X}(\mathcal{T}M)} - h. \tag{1.10g}$$

Then g is a pseudo-Riemannian metric on $\mathcal{T}M$, called the *prolongation of* $\overline{g}$ *along* h.

1.11. The Cartan tensors. The *first Cartan tensor*

$$\mathcal{C} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M), \qquad (X,Y) \mapsto \mathcal{C}(X,Y)$$

of the Finsler manifold (M,E) is defined by the rules

(1.11a) $$J \circ \mathcal{C} := 0,$$

(1.11b) $$\overline{g}(\mathcal{C}(X,Y), JZ) := \frac{1}{2}\left(\mathcal{L}_{JX} J^*\overline{g}\right)(Y,Z) \qquad (Z \in \mathfrak{X}(\mathcal{T}M)).$$

The lowered tensor $\mathcal{C}_\flat$ of $\mathcal{C}$ is given by the formula

(1.11c) $$\mathcal{C}_\flat(X,Y,Z) := \overline{g}(\mathcal{C}(X,Y), JZ); \qquad X,Y,Z \in \mathfrak{X}(\mathcal{T}M).$$

Let us note that

(1.11d) (M,E) *is a Riemannian manifold if and only if* $\mathcal{C} = 0$.

Now we consider a horizontal endomorphism h on M, and the prolongation g of the vertical metric $\overline{g}$ along h. The *second Cartan tensor* $\mathcal{C}'$ of (M,E) (belonging to h) is given by the condition

(1.11c) $$J \circ \mathcal{C}' := 0$$

and the formula

(1.11f) $$g(\mathcal{C}'(X,Y), JZ) := \frac{1}{2}\left(\mathcal{L}_{hX} g\right)(JY, JZ).$$

For the basic properties of $\mathcal{C}'$ we refer to [12].

1.12. The Barthel endomorphism. If (M,E) is a Finsler manifold then there exists a unique horizontal endomorphism h_0 on M such that

(1.12a) h_0 *is conservative, i.e.,* $d_{h_0}E = 0$;

(1.12b) h_0 *is homogeneous*;

(1.12c) *the weak torsion of* h_0 *vanishes.*

This fundamental discovery is due to J. Grifone [3]. The horizontal endomorphism characterized by (1.12a)–(1.12c) will be called the *Barthel endomorphism* of the Finsler manifold. It can be explicitly given by the formula

(1.12d) $$h_0 = \frac{1}{2}\left(1_{\mathfrak{X}(\mathcal{T}M)} + [J, S_0]\right),$$

where S_0 is the canonical spray of (M,E). Note that conditions (1.12b)–(1.12c) can be replaced by the single condition of the vanishing of the strong torsion.

1.13. The Hashiguchi connection. We have an abundance of nice Finsler connections on any Finsler manifold (but see MATSUMOTO's principle from the Introduction!); for recent surveys we again refer to [11] and [12]. In our forthcoming considerations we need only one of them, the *Hashiguchi connection* $\left(\overset{H}{D}, h\right)$ characterized by the following axioms:

(1.13a) *the v-mixed torsion of $\overset{H}{D}$ vanishes*;

(1.13b) $\overset{H}{D}$ *is v-metrical*, i.e., $\overset{H}{D}_v g = 0$;

(1.13c) *the v-vertical torsion of $\overset{H}{D}$ vanishes*

(g is the prolongation of the vertical metric along h).

The covariant derivatives with respect to $\overset{H}{D}$ can be calculated by the following formulas:

(1.13d,e) $\overset{H}{D}_{JX} JY = J[JX, Y] + \mathcal{C}(X, Y);$ $\qquad \overset{H}{D}_{hX} JY = v[hX, JY];$

(1.13f,g) $\overset{H}{D}_{JX} hY = h[JX, Y] + F\mathcal{C}(X, Y);$ $\qquad \overset{H}{D}_{hX} hY = hF[hX, JY]$

($X, Y \in \mathfrak{X}(\mathcal{T}M)$, F is the almost complex structure induced by h). If, in addition,

(1.13h) *h is conservative*;

(1.13i) *the h-horizontal torsion of $\overset{H}{D}$ vanishes*;

(1.13j) the *h-deflection of $\overset{H}{D}$ vanishes*,

then h becomes the Barthel endomorphism. In this case $\left(\overset{H}{D}, h\right)$ is said to be the *standard Hashiguchi connection* of (M, E).

The method of an intrinsic proof can be found in [11].

2. h-basic Finsler connections

2.1. Definition. A Finsler connection (D, h) is said to be an *h-basic Finsler connection* if there exists a linear connection ∇ on the manifold M such that for any vector fields X, Y on M, we have

$$D_{X^h} Y^{\mathrm{v}} = (\nabla_X Y)^{\mathrm{v}} .$$

Then ∇ is called the *base connection* belonging to (D, h).

2.2. Remark. The base connection of an h-basic connection is clearly unique.

2.3. Lemma (c.f. [5], Proposition 1.1)**.** *A Finsler connection (D,h) is h-basic if and only if the mixed curvature of the associated Finsler connection $(\widetilde{D},h)$ vanishes.*

Proof. – Suppose that the mixed curvature $\widetilde{\mathbb{P}}$ of the associated Finsler connection vanishes. Then, taking into account (1.9), for any vector fields X, Y, Z on M we have

$$0=[J,D_{X^h}Z^{\mathrm{v}}]\,Y^c=[Y^{\mathrm{v}},D_{X^h}Z^{\mathrm{v}}]-J\,[Y^c,D_{X^h}Z^{\mathrm{v}}]=[Y^{\mathrm{v}},D_{X^h}Z^{\mathrm{v}}]\,.$$

This means that the vertical vector field $D_{X^h}Y^{\mathrm{v}}$ commutes with any vertically lifted vector field. Hence, by the same argument as in **1.5**, $D_{X^h}Y^{\mathrm{v}}$ is also a vertical lift. Using this fact we can see easily that the mapping

$$\nabla:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M);\quad \forall\,(X,Y)\in\mathfrak{X}(M)\times\mathfrak{X}(M):(\nabla_XY)^{\mathrm{v}}:=D_{X^h}Y^{\mathrm{v}}$$

is a well-defined linear connection on M, and so (D,h) is an h-basic Finsler connection.

Conversely, if (D,h) is an h-basic Finsler connection with the base connection ∇, then for any vector fields X, Y, Z on M we have

$$\begin{aligned}\widetilde{\mathbb{P}}(X^c,Y^c)Z^c&=\widetilde{\mathbb{K}}(X^h,Y^{\mathrm{v}})Z^{\mathrm{v}}=D_{X^h}D^i_{Y^{\mathrm{v}}}Z^{\mathrm{v}}-D^i_{Y^{\mathrm{v}}}D_{X^h}Z^{\mathrm{v}}-D^i_{[X^h,Y^{\mathrm{v}}]}Z^{\mathrm{v}}\\&=-D^i_{Y^{\mathrm{v}}}D_{X^h}Z^{\mathrm{v}}=-D^i_{Y^{\mathrm{v}}}\left(\nabla_XZ\right)^{\mathrm{v}}=0,\end{aligned}$$

hence the mixed curvature of the associated Finsler connection vanishes. □

2.4. Lemma. *Suppose that (D,h) is an h-basic Finsler connection with the base connection ∇, and let h_∇ be the horizontal endomorphism induced by h. Then*

$$D_{X^h}C=X^h-X^{h_\nabla}\qquad(X\in\mathfrak{X}(M)),\tag{2.4a}$$

therefore h_∇ coincides with h if and only if the h-deflection of (D,h) vanishes.

Proof. Let $(U,(u^i)_{i=1}^n)$ be a chart on M. Then over $\pi^{-1}(U)$ the Liouville vector field can be represented in the form

$$C\restriction\pi^{-1}(U)=(u^i)^c\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}.$$

So in the neighborhood $\pi^{-1}(U)$ we have:

$$\begin{aligned}D_{X^h}C&=D_{X^h}(u^i)^c\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}=\left(X^h(u^i)^c\right)\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}+(u^i)^cD_{X^h}\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}\\&=\left(X^h(u^i)^c\right)\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}+(u^i)^c\left(\nabla_X\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}\overset{(1)}{=}\left(X^h(u^i)^c\right)\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}\\&\quad+(u^i)^c\left[X^{h_\nabla},\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}\right]=\left(X^h(u^i)^c\right)\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}+\left[X^{h_\nabla},(u^i)^c\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}\right]\\&\quad-\left(X^{h_\nabla}(u^i)^c\right)\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}=(X^h-X^{h_\nabla})(u^i)^c\left(\frac{\partial}{\partial u^i}\right)^{\mathrm{v}}+[X^{h_\nabla},C]\\&\overset{(2)}{=}X^h-X^{h_\nabla},\end{aligned}$$

at the steps (1) and (2) using the fact that h_∇ arises from a linear connection on M. □

2.5 Remark. Consider a Finsler connection (D, h) with *vanishing h-deflection*. By the lemma just proved, in order that (D, h) be h-basic it is necessary that h be *smooth on the whole tangent manifold* and should satisfy the *homogeneity condition* $H := [h, C] = 0$.

2.6. Corollary. *If (D, h) is an h-basic Finsler connection with vanishing h-deflection, then for any vector fields X, Y on $\mathcal{T}M$ we have the rules for calculation*

$$D_{hX}JY = v[hX, JY], \tag{2.6a}$$

$$D_{hX}hY = hF[hX, JY]. \tag{2.6b}$$

□

2.7. Proposition. *Let (D, h) be an h-basic Finsler connection and suppose that the horizontal endomorphism h is homogeneous. Then the h-deflection of (D, h) vanishes if and only if the v-mixed torsion of D vanishes, i.e.,*

$$\textit{under the homogeneity condition, } h^*DC \Leftrightarrow \mathbb{P}^1 = 0. \tag{2.7a}$$

Proof. For any vector fields X, Y on M we have

$$\begin{aligned}\mathbb{P}^1(X^h, Y^h) &= v\mathbb{T}(X^h, Y^{\mathrm{v}}) = v\left(D_{X^h}Y^{\mathrm{v}} - D_{Y^{\mathrm{v}}}X^h - [X^h, Y^{\mathrm{v}}]\right) \\ &= D_{X^h}Y^{\mathrm{v}} - [X^h, Y^{\mathrm{v}}].\end{aligned}$$

If ∇ is the base connection of (D, h), then

$$D_{X^h}Y^{\mathrm{v}} = (\nabla_X Y)^{\mathrm{v}} = \left[X^{h_\nabla}, Y^{\mathrm{v}}\right]$$

by the conditions. So it follows that

$$\mathbb{P}^1(X^h, Y^h) = 0 \;\Leftrightarrow\; \left[X^{h_\nabla}, Y^{\mathrm{v}}\right] = [X^h, Y^{\mathrm{v}}].$$

In view of Lemma 1.5 the last relation holds if and only if $h = h_\nabla$, which (by Lemma 2.4) is equivalent to the vanishing of the h-deflection of (D, h). □

2.8. Proposition. *Let us consider an h-basic Finsler connection (D, h) with the base connection ∇. Suppose that the horizontal endomorphism h is smooth on the whole tangent manifold. Then the h-deflection of (D, h) coincides with the tension of h if and only if the v-mixed torsion of D vanishes, i.e.,*

$$h^*(DC) = H \Leftrightarrow \mathbb{P}^1 = 0, \textit{ if } h \textit{ is smooth everywhere.} \tag{2.8a}$$

Proof of $\mathbb{P}^1 = 0 \implies h^*DC = H$. – We have just seen that under the condition $\mathbb{P}^1 = 0$, for any vector fields X, Y on M we can write

$$D_{X^h}Y^{\mathrm{v}} = [X^h, Y^{\mathrm{v}}].$$

From this the general rule for calculation

$$D_{hX}JY = v[hX, JY]; \qquad X, Y \in \mathfrak{X}(TM) \tag{2.8b}$$

can be deduced easily. Taking an arbitrary semispray S on M, for each vector field X on M we obtain

$$D_{X^h}C = D_{X^h}JS \overset{(2.8b)}{=} v[X^h, JS] = v[X^h, C] = H(X^c).$$

This means that $h^*DC = H$.

Proof of $h^*(DC) = H \implies \mathbb{P}^1 = 0$. – Let $X \in \mathfrak{X}(M)$ be arbitrary. On the one hand, $D_{X^h}C = [X^h, C]$. On the other hand, in view of **2.4**, $D_{X^h}C = X^h - X^{h_\nabla}$. Thus, taking into account the homogeneity of h_∇, it follows that

$$\left[C, X^h - X^{h_\nabla}\right] = [C, X^h] = -\left(X^h - X^{h_\nabla}\right).$$

This relation implies in a well-known manner that the vertical vector field $X^h - X^{h_\nabla}$ is homogeneous of degree 0. Since h is smooth on the whole tangent manifold, we can conclude that $X^h - X^{h_\nabla}$ is a vertical lift. Hence for any vector field Y on M we have

$$0 = \left[X^h - X^{h_\nabla}, Y^{\mathrm{v}}\right] = [X^h, Y^{\mathrm{v}}] - \left[X^{h_\nabla}, Y^{\mathrm{v}}\right];$$

therefore

$$\mathbb{P}^1(X^h, Y^h) = (\nabla_X Y)^{\mathrm{v}} - [X^h, Y^{\mathrm{v}}] = \left[X^{h_\nabla}, Y^{\mathrm{v}}\right] - [X^h, Y^{\mathrm{v}}] = 0,$$

and the implication is verified. □

2.9. Theorem. *Let (D, h) be an h-basic Finsler connection on the non-Riemannian Finsler manifold (M, E). (D, h) is h-metrical if and only if h is conservative and the h-deflection of (D, h) vanishes. That is,*

$$D_h g = 0 \Leftrightarrow d_h E = 0 \wedge h^*DC = 0 \tag{2.9a}$$

(g is the prolongation of the vertical metric along h).

Proof of $D_h g = 0 \implies d_h E = 0 \wedge h^*DC = 0$. – We do this in several steps.

First step. Let ∇ be the base connection of (D, h). We show that the horizontal endomorphism h_∇ is conservative, i.e., $d_{h_\nabla}E = 0$. – Taking an arbitrary semispray S and a vector field X on M, we have

$$\begin{aligned} 2X^hE &= X^h(2E) = X^h[g(C, C)] = 2g(C, D_{X^h}C) = 2g(C, JD_{X^h}S) = 2\omega(C, D_{X^h}S) \\ &= 2i_C\omega(D_{X^h}S) = 2(d_JE)(D_{X^h}S) = 2(dE)(D_{X^h}C) = 2(D_{X^h}C)E, \end{aligned}$$

by the condition $D_h g = 0$ and using some well-known relations concerning the fundamental form ω. So we conclude that

$$(D_{X^h}C)\,E = X^hE.$$

On the other hand,

$$(D_{X^h}C)\,E \overset{(2.4a)}{=} \left(X^h - X^{h_\nabla}\right)E = X^hE - X^{h_\nabla}E,$$

and the last two relations imply that for any vector field X on M, $X^{h_\nabla}E = 0$. This means that $d_{h_\nabla}E = 0$, as we claimed.

Second step. Let X, Y, Z be arbitrary vector fields on M. Using (3.4b) of [12], we obtain

$$
\begin{aligned}
X^{h_\nabla} g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) - g\big((\nabla_X Y)^{\mathrm{v}}, Z^{\mathrm{v}}\big) - g\big(Y^{\mathrm{v}}, (\nabla_X Z)^{\mathrm{v}}\big) &= X^{h_\nabla}[Y^{\mathrm{v}}(Z^{\mathrm{v}}E)] \\
- (\nabla_X Y)^{\mathrm{v}}(Z^{\mathrm{v}}E) - Y^{\mathrm{v}}\big((\nabla_X Z)^{\mathrm{v}}E\big) &= X^{h_\nabla}[Y^{\mathrm{v}}(Z^{\mathrm{v}}E)] - \left[X^{h_\nabla}, Y^{\mathrm{v}}\right](Z^{\mathrm{v}}E) \\
- Y^{\mathrm{v}}\left(\left[X^{h_\nabla}, Z^{\mathrm{v}}\right]E\right) &= Y^{\mathrm{v}}\left[Z^{\mathrm{v}}\left(X^{h_\nabla}E\right)\right] = 0,
\end{aligned}
$$

since h_∇ is conservative, as we have just seen. Thus we obtain the relation

$$
\text{(2.9b)} \qquad X^{h_\nabla} g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) = g\big((\nabla_X Y)^{\mathrm{v}}, Z^{\mathrm{v}}\big) + g\big(Y^{\mathrm{v}}, (\nabla_X Z)^{\mathrm{v}}\big); \quad X, Y, Z \in \mathfrak{X}(M).
$$

Third step. Let $X, Y, Z \in \mathfrak{X}(M)$ be arbitrary again. By the condition $D_h g = 0$ we get

$$
\begin{aligned}
0 = (D_{X^h} g)(Y^{\mathrm{v}}, Z^{\mathrm{v}}) &= X^h\big[g(Y^{\mathrm{v}}, Z^{\mathrm{v}})\big] - g\left(D_{X^h} Y^{\mathrm{v}}, Z^{\mathrm{v}}\right) - g(Y^{\mathrm{v}}, D_{X^h} Z^{\mathrm{v}}) \\
&= X^h\big[g(Y^{\mathrm{v}}, Z^{\mathrm{v}})\big] - g\big((\nabla_X Y)^{\mathrm{v}}, Z^{\mathrm{v}}\big) - g\big(Y^{\mathrm{v}}, (\nabla_X Z)^{\mathrm{v}}\big) \\
&\overset{(2.9b)}{=} X^h\big[g(Y^{\mathrm{v}}, Z^{\mathrm{v}})\big] - X^{h_\nabla}\big[g(Y^{\mathrm{v}}, Z^{\mathrm{v}})\big] = \big(X^h - X^{h_\nabla}\big) g(Y^{\mathrm{v}}, Z^{\mathrm{v}}).
\end{aligned}
$$

On the other hand, using the well-known symmetries of the first Cartan tensor, we can write

$$
\begin{aligned}
2g\left(\mathcal{C}(Y^c, Z^c), X^h - X^{h_\nabla}\right) &= 2g\left(\mathcal{C}\big(F\big(X^h - X^{h_\nabla}\big), Y^c\big), Z^{\mathrm{v}}\right) \\
&\overset{(1.11b)}{=} \Big(\mathcal{L}_{(X^h - X^{h_\nabla})} J^* g\Big)(Y^c, Z^c) = \big(X^h - X^{h_\nabla}\big) g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) \\
&- g\left(J[X^h - X^{h_\nabla}, Y^c], Z^{\mathrm{v}}\right) - g\left(Y^{\mathrm{v}}, J[X^h - X^{h_\nabla}, Z^c]\right) \\
&= (X^h - X^{h_\nabla}) g(Y^{\mathrm{v}}, Z^{\mathrm{v}}).
\end{aligned}
$$

Comparing the last two results it follows that for any vector fields X, Y, Z on M

$$
g\left(\mathcal{C}(Y^c, Z^c), X^h - X^{h_\nabla}\right) = 0.
$$

This implies that $X^h - X^{h_\nabla} = 0$ and hence $h = h_\nabla$, since g is nondegenerate and $\mathcal{C}$ does not vanish identically by the condition and (1.11d). Thus h is also conservative and, in view of **2.4**, $h^* DC = 0$.

Proof of $(d_h E = 0 \wedge h^* DC = 0) \implies D_h g = 0$. – The condition $h^* DC = 0$ implies by **2.4** the coincidence of h and h_∇. Then h_∇ is automatically conservative. Using this fact we obtain by the calculation of the previous third step that

$$
(D_{X^h} g)(Y^{\mathrm{v}}, Z^{\mathrm{v}}) = (X^h - X^{h_\nabla}) g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) = 0 \quad (X, Y, Z \in \mathfrak{X}(M));
$$

thus the desired relation $D_h g = 0$ is true. $\square$

3. The Ichijyō connection

3.1. Theorem. *Suppose that (M,E) is a Finsler manifold and ∇ is a linear connection on M. Let h_∇ be the horizontal endomorphism induced by ∇, and let us consider the prolongation g of the vertical metric along h_∇. There exists a unique Finsler connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ on M such that*

(3.1a) $\overset{\nabla}{D}$ *is v-metrical, i.e.,* $\overset{\nabla}{D}_v g = 0$;

(3.1b) *the v-vertical torsion* $\overset{\nabla}{\mathbb{S}}{}^1$ *of* $\overset{\nabla}{D}$ *vanishes*;

(3.1c) *the mixed curvature of the associated Finsler connection* $\left(\widetilde{D^\nabla}, h_\nabla\right)$ *vanishes*;

(3.1d) *the h-deflection of* $\left(\overset{\nabla}{D}, h_\nabla\right)$ *vanishes.*

The covariant derivatives with respect to $\overset{\nabla}{D}$ can be calculated explicitly by the following formulas:

$$\overset{\nabla}{D}_{JX} JY = J[JX,Y] + \mathcal{C}(X,Y); \tag{3.1e}$$

$$\overset{\nabla}{D}_{h_\nabla X} JY = v_\nabla [h_\nabla X, JY]; \tag{3.1f}$$

$$\overset{\nabla}{D}_{JX} h_\nabla Y = h_\nabla [JX,Y] + F_\nabla \mathcal{C}(X,Y); \tag{3.1g}$$

$$\overset{\nabla}{D}_{h_\nabla X} h_\nabla Y = h_\nabla F_\nabla [h_\nabla X, JY] \tag{3.1h}$$

$(X, Y \in \mathfrak{X}(\mathcal{T}M))$.

Proof of the unicity. – We show that axioms (3.1a)–(3.1d) force the rules for calculation (3.1e) and (3.1f), from these (3.1g) and (3.1h) immediately follow by (1.6b). We do this in two steps.

First step. Since $\overset{\nabla}{D}$ is v-metrical, the relations

$$X^{\mathrm{v}} g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) = g\left(\overset{\nabla}{D}_{X^{\mathrm{v}}} Y^{\mathrm{v}}, Z^{\mathrm{v}}\right) + g\left(Y^{\mathrm{v}}, \overset{\nabla}{D}_{X^{\mathrm{v}}} Z^{\mathrm{v}}\right),$$

$$Y^{\mathrm{v}} g(Z^{\mathrm{v}}, X^{\mathrm{v}}) = g\left(\overset{\nabla}{D}_{Y^{\mathrm{v}}} Z^{\mathrm{v}}, X^{\mathrm{v}}\right) + g\left(Z^{\mathrm{v}}, \overset{\nabla}{D}_{Y^{\mathrm{v}}} X^{\mathrm{v}}\right),$$

$$-Z^{\mathrm{v}} g(X^{\mathrm{v}}, Y^{\mathrm{v}}) = -g\left(\overset{\nabla}{D}_{Z^{\mathrm{v}}} X^{\mathrm{v}}, Y^{\mathrm{v}}\right) - g\left(X^{\mathrm{v}}, \overset{\nabla}{D}_{Z^{\mathrm{v}}} Y^{\mathrm{v}}\right)$$

hold for any vector fields X, Y, Z on M. Adding the corresponding sides of these three equations and using the vanishing of $\overset{\nabla}{\mathbb{S}}{}^1$,we obtain

$$g\left(2\overset{\nabla}{D}_{X^{\mathrm{v}}} Y^{\mathrm{v}}, Z^{\mathrm{v}}\right) = X^{\mathrm{v}} g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) + Y^{\mathrm{v}} g(Z^{\mathrm{v}}, X^{\mathrm{v}}) - Z^{\mathrm{v}} g(X^{\mathrm{v}}, Y^{\mathrm{v}}).$$

Taking into account (3.7a) of [12], here the right hand side is just

$$2\mathcal{C}_\flat(X^c, Y^c, Z^c) = 2g(\mathcal{C}(X^c, Y^c), Z^{\mathrm{v}}).$$

Consequently

$$\overset{\nabla}{D}_{X^{\mathrm{v}}} Y^{\mathrm{v}} = \mathcal{C}(X^c, Y^c) = \mathcal{C}\big(X^{h_\nabla}, Y^{h_\nabla}\big),$$

so rule (3.1e) is verified for vertically lifted vector fields. Having obtained this result, we can immediately deduce the general form (3.1e).

Second step. Now we conclude (3.1f) from (3.1c) and (3.1d). In view of Lemma 2.3, the latter condition implies that $\left(\overset{\nabla}{D}, h_\nabla\right)$ is an h-basic Finsler connection. So there exists a unique linear connection $\widetilde{\nabla}$ on M such that for any vector fields X, Y on M,

$$\left(\widetilde{\nabla}_X Y\right)^{\mathrm{v}} = \overset{\nabla}{D}_{X^{h_\nabla}} Y^{h_\nabla}.$$

Using condition (3.1d) and Lemma 2.4, we infer immediately that $\widetilde{\nabla}$ coincides with the given linear connection ∇. Thus

$$\overset{\nabla}{D}_{X^{h_\nabla}} Y^{\mathrm{v}} = (\nabla_X Y)^{\mathrm{v}} = [X^{h_\nabla}, Y^{\mathrm{v}}],$$

from which the formula (3.1f) can be derived easily.

Proof of the existence. – Define the mapping

$$\overset{\nabla}{D} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$$

by the rule

$$(X, Y) \mapsto \overset{\nabla}{D}_X Y := \overset{\nabla}{D}_{v_\nabla X}(v_\nabla Y) + \overset{\nabla}{D}_{v_\nabla X}(h_\nabla Y) + \overset{\nabla}{D}_{h_\nabla X}(v_\nabla Y) + \overset{\nabla}{D}_{h_\nabla X}(h_\nabla Y),$$

where the terms of the right hand side are determined by (3.1e)–(3.1h). Then it can be checked by a straightforward calculation that $\overset{\nabla}{D}$ is a linear connection on $\mathcal{T}M$, $\left(\overset{\nabla}{D}, h_\nabla\right)$ is a Finsler connection on M, and axioms (3.1a)–(3.1d) are satisfied. □

3.2. Remarks.

(i) We propose to call the Finsler connection described in **3.1** the *Ichijyō connection* induced by ∇ in honour of Y. Ichijyō, who used its coordinate version effectively in his excellent papers [6], [7].

(ii) Rules (3.1e)–(3.1h) take the following more convenient form for the vertically and horizontally lifted vector fields:

(3.2a,b) $\quad \overset{\nabla}{D}_{X^{\mathrm{v}}} Y^{\mathrm{v}} = \mathcal{C}(X^{h_\nabla}, Y^{h_\nabla}); \qquad \overset{\nabla}{D}_{X^{h_\nabla}} Y^{\mathrm{v}} = (\nabla_X Y)^{\mathrm{v}};$

(3.2c,d) $\quad \overset{\nabla}{D}_{X^{\mathrm{v}}} Y^{h_\nabla} = F_\nabla \mathcal{C}(X^{h_\nabla}, Y^{h_\nabla}); \qquad \overset{\nabla}{D}_{X^{h_\nabla}} Y^{h_\nabla} = (\nabla_X Y)^{h_\nabla}$

$(X, Y \in \mathfrak{X}(M))$.

3.3. Proposition. *Let (M, E) be a Finsler manifold, ∇ a linear connection on M, and consider the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ induced by ∇. Then*

$$\left(\overset{\nabla}{D}_{JX}\mathcal{C}\right)(Y, Z) = \left(\overset{\nabla}{D}_{JY}\mathcal{C}\right)(X, Z), \tag{3.3a}$$

where X, Y, Z are any vector fields in $\mathcal{T}M$.

The proof parallels that of (A.16) in [4] and is omitted.

3.4. Lemma. *Let $\left(\overset{\nabla}{D}, h_\nabla\right)$ be an Ichijyō connection with the base connection ∇. The torsion $\overset{\nabla}{\mathbb{T}}$ of $\overset{\nabla}{D}$ satisfies the identities*

$$\overset{\nabla}{\mathbb{T}}(X^{h_\nabla}, Y^{h_\nabla}) = \big(\mathbb{T}_\nabla(X, Y)\big)^{h_\nabla} + \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla}); \tag{3.4a}$$

$$\overset{\nabla}{\mathbb{T}}(X^{h_\nabla}, Y^{\mathrm{v}}) = -F_\nabla\mathcal{C}(X^{h_\nabla}, Y^{h_\nabla}); \qquad \overset{\nabla}{\mathbb{T}}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0, \tag{3.4b,c}$$

where $\mathbb{T}_\nabla$ denotes the torsion tensor of ∇; Ω_∇ and F_∇ are the curvature and the associated almost complex structure of h_∇, respectively; while X and Y are arbitrary vector fields on M.

The proof is very straightforward so we omit it.

3.5. Corollary. *For the partial curvatures and torsions of an Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ we have the following representation:*

Curvature	$(X, Y, Z \in \mathfrak{X}(\mathcal{T}M))$
horizontal	$\overset{\nabla}{\mathbb{R}}(X, Y)Z = [J, \Omega_\nabla(X, Y)]h_\nabla Z + \mathcal{C}\big(F\Omega_\nabla(X, Y), Z\big)$
mixed	$\overset{\nabla}{\mathbb{P}}(X, Y)Z = \left(\overset{\nabla}{D}_{h_\nabla X}\mathcal{C}\right)(h_\nabla Y, h_\nabla Z)$
vertical	$\overset{\nabla}{\mathbb{Q}}(X, Y)Z = \mathcal{C}(F\mathcal{C}(X, Z), Y) - \mathcal{C}\big(X, F\mathcal{C}(Y, Z)\big)$

Torsion	$(X, Y \in \mathfrak{X}(M))$
$h-$ *horizontal*	$\overset{\nabla}{\mathbb{A}}(X^{h_\nabla}, Y^{h_\nabla}) = (\mathbb{T}_\nabla(X, Y))^{h_\nabla}$
$h-$ *mixed*	$\overset{\nabla}{\mathbb{B}}(X^{h_\nabla}, Y^{\mathrm{v}}) = -F_\nabla\mathcal{C}(X^{h_\nabla}, Y^{h_\nabla})$
$v-$ *horizontal*	$\overset{\nabla}{\mathbb{R}}{}^1(X^{h_\nabla}, Y^{h_\nabla}) = \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla})$
$v-$ *mixed*	$\overset{\nabla}{\mathbb{P}}{}^1 = 0$
$v-$ *vertical*	$\overset{\nabla}{\mathbb{S}}{}^1 = 0$

(F is an arbitrary almost complex structure on TM).

Applying our previous results including (3.3a), these formulas can be obtained by a routine but lengthy calculation that we will not present here.

3.6. Corollary. *The horizontal curvature of an Ichijyō connection vanishes if and only if the curvature of the base connection ∇, or – what is essentially the same – the curvature of h_∇ – vanishes.*

Proof. It is clearly enough to show that the vanishing of $\overset{\nabla}{\mathbb{R}}$ is equivalent to the vanishing of Ω_∇. The implication $\Omega_\nabla = 0 \implies \overset{\nabla}{\mathbb{R}} = 0$ is evident from **3.5**. Conversely, suppose that $\overset{\nabla}{\mathbb{R}} = 0$, and let S_∇ be the semispray associated to h_∇. For any vector fields X, Y, on M, we have

$$0 = \overset{\nabla}{\mathbb{R}}(X^{h_\nabla}, Y^{h_\nabla})S_\nabla \overset{\mathbf{3.5}}{=} \Big[J, \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla})\Big]S_\nabla + \mathcal{C}\Big(F\Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla}), S_\nabla\Big)$$
$$= \Big[J, \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla})\Big]S_\nabla = \Big[C, \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla})\Big] - J\Big[S_\nabla, \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla})\Big].$$

Using the graded Jacobi identity and taking into consideration the homogeneity of h_∇, we readily obtain that the first term of the right hand side vanishes, while the second term is just $-\Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla})$ by (1.2a).

Thus Ω_∇ vanishes, which ends the proof. □

3.7. Corollary. *The mixed curvature of an Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ vanishes if and only if the h-covariant derivative of the first Cartan tensor with respect to $\overset{\nabla}{D}$ vanishes, i.e.,*

$$\overset{\nabla}{\mathbb{P}} = 0 \iff \overset{\nabla}{D}_{h_\nabla}\mathcal{C} = 0. \qquad \square$$

3.8. Corollary. *The h-horizontal torsion of an Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ and the torsion tensor of ∇ (or the weak torsion of h_∇) vanish at the same time.*

Proof. The assertion is clear from the relations

$$\overset{\nabla}{\mathbb{A}}(X^{h_\nabla}, Y^{h_\nabla}) = (\mathbb{T}_\nabla(X,Y))^{h_\nabla} = (F_\nabla \circ t_\nabla)(X^{h_\nabla}, Y^{h_\nabla}) \qquad (X, Y \in \mathfrak{X}(M)),$$

where the latter equality can be obtained in the same way as Corollary 2/(ii) in [11]. □

4. Generalized Berwald manifolds

4.1. Definition. Suppose that (M, E) is a Finsler manifold and let ∇ be a linear connection on M. The triplet (M, E, ∇) is said to be a *generalized Berwald manifold* if the horizontal endomorphism h_∇ is conservative, i.e., $d_{h_\nabla}E = 0$. A generalized Berwald manifold (M, E, ∇) is called a *Berwald manifold* if ∇ is a torsion-free linear connection. If, in addition, ∇ is flat, then we speak of a *locally Minkowski* Finsler manifold.

4.2. Remark. Generalized Berwald manifolds were introduced by V. V. WAGNER in 1943. Their systematic investigation, within the framework of MATSUMOTO's theory, was initiated by M. HASHIGUCHI and Y. ICHIJYŌ in the middle of the seventies. Our definition was inspired by SZABÓ's paper [10]. – It can easily be seen that in the particular case of Berwald manifolds the horizontal endomorphism h_∇ coincides with the Barthel endomorphism, hence the linear connection ∇ is unique. Then we speak of *the* linear connection of the Berwald manifold and write (M, E) rather than (M, E, ∇).

4.3. Proposition. *Let (M, E) be a Finsler manifold and suppose that ∇ is a linear connection on M. The following conditions are equivalent:*

(a) *(M, E, ∇) is a generalized Berwald manifold;*

(b) *the second Cartan tensor $\mathcal{C}'_\nabla$ belonging to h_∇ vanishes;*

(c) *the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ is h_∇-metrical, i.e., $\overset{\nabla}{D}_{h_\nabla} g = 0$.*

Proof of (a) $\Longrightarrow$ (b). – Starting from the definition of $\mathcal{C}'_\nabla$ and using (3.4b) of [12], we obtain for any vector fields X, Y, Z on M that

$$
\begin{aligned}
2(\mathcal{C}'_\nabla)_\flat(X^c, Y^c, Z^c) &:= 2g(\mathcal{C}'_\nabla(X^c, Y^c)JZ^c) = \left(\mathcal{L}_{h_\nabla X^c} g\right)(JY^c, JZ^c) \\
&= X^{h_\nabla} g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) - g\big([X^{h_\nabla}, Y^{\mathrm{v}}], Z^{\mathrm{v}}\big) - g(Y^{\mathrm{v}}, [X^{h_\nabla}, Z^{\mathrm{v}}]) \\
&= X^{h_\nabla}[Y^{\mathrm{v}}(Z^{\mathrm{v}}E)] - [X^{h_\nabla}, Y^{\mathrm{v}}](Z^{\mathrm{v}}E) - Y^{\mathrm{v}}([X^{h_\nabla}, Z^{\mathrm{v}}]E) \\
&= Y^{\mathrm{v}}[Z^{\mathrm{v}}(X^{h_\nabla}E)] = 0,
\end{aligned}
$$

since h_∇ is conservative. Thus $\mathcal{C}'_\nabla = 0$.

Proof of (b) $\Longleftrightarrow$ (c). – For any vector fields X, Y, Z on M, we have

$$
\begin{aligned}
\left(\overset{\nabla}{D}_{X^{h_\nabla}} g\right)(Y^{\mathrm{v}}, Z^{\mathrm{v}}) &= X^{h_\nabla} g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) - g\left(\overset{\nabla}{D}_{X^{h_\nabla}} Y^{\mathrm{v}}, Z^{\mathrm{v}}\right) - g\left(Y^{\mathrm{v}}, \overset{\nabla}{D}_{X^{h_\nabla}} Z^{\mathrm{v}}\right) \\
&= X^{h_\nabla} g(Y^{\mathrm{v}}, Z^{\mathrm{v}}) - g([X^{h_\nabla}, Y^{\mathrm{v}}], Z^{\mathrm{v}}) - g(Y^{\mathrm{v}}[X^{h_\nabla}, Z^{\mathrm{v}}]) = 2g\left(\mathcal{C}'_\nabla(X^c, Y^c), Z^{\mathrm{v}}\right),
\end{aligned}
$$

so it is obvious that assertions (b) and (c) are equivalent.

Proof of (c) $\Longrightarrow$ (a). – Since the h-deflection of $\left(\overset{\nabla}{D}, h_\nabla\right)$ vanishes by axiom (3.1d), we obtain that

$$
\begin{aligned}
0 &\overset{\text{(c)}}{=} \left(\overset{\nabla}{D}_{X^{h_\nabla}} g\right)(C, C) = X^{h_\nabla} g(C, C) - 2g\left(\overset{\nabla}{D}_{X^{h_\nabla}} C, C\right) \\
&= 2X^{h_\nabla} E = 2d_{h_\nabla} E(X^c),
\end{aligned}
$$

for any vector field X on M. This means that $d_{h_\nabla} E = 0$. □

4.4. Proposition. *If (M, E, ∇) is a generalized Berwald manifold, then the mixed curvature of the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ vanishes.*

Proof. Taking into account **3.5**, it is enough to check the vanishing of $\overset{\nabla}{D}_{h_\nabla}\mathcal{C}$. This requires only a quite immediate (but lengthy) calculation which we omit. □

4.5. A counterexample. Suppose that (M, E, ∇) is a generalized Berwald manifold, and let σ be a non-constant smooth function on M. Then

$$\overline{h}_\nabla := h_\nabla - d\sigma^{\mathrm{v}} \otimes C \qquad (\sigma^{\mathrm{v}} := \sigma \circ \pi)$$

is an everywhere smooth, homogeneous horizontal endomorphism, so $\overline{h}_\nabla$ generates a linear connection $\overline{\nabla}$ on M. It can be checked by a direct calculation that the mixed curvature of the corresponding Ichijyō connection vanishes. However, $(M, E, \overline{\nabla})$ is *not* a generalized Berwald manifold, since $h_{\overline{\nabla}}$ is obviously non-conservative. Thus the converse of **4.4** is not true in general.

4.6. Lemma. (c.f. [12], 6.5.) *Let (M, E) be a Finsler manifold and let its Barthel endomorphism be denoted by h_0. (M, E) is a Berwald manifold if and only if there is a linear connection ∇ on M such that for any vector fields X, Y, on M,*

$$(\nabla_X Y)^{\mathrm{v}} = [X^{h_0}, Y^{\mathrm{v}}]. \tag{4.6 a}$$

Then ∇ is just the linear connection of the Berwald manifold.

Proof. In view of **4.2**, the necessity of the condition is obvious. Conversely, if a linear connection ∇ satisfies (4.6a) then we obtain that

$$[X^{h_0}, Y^{\mathrm{v}}] = [X^{h_\nabla}, Y^{\mathrm{v}}]$$

for any vector fields X, Y on M. This implies by Lemma 1.5 the coincidence of h_0 and h_∇. Then it follows at once that (M, E) is a Berwald manifold. □

4.7. Theorem. *A Finsler manifold is a Berwald manifold if and only if its Hashiguchi connection is an Ichijyō connection.*

Proof. Consider a Finsler manifold (M, E). Let the Barthel endomorphism be denoted by h_0, and let $\left(\overset{H}{D}, h_0\right)$ be the Hashiguchi connection (**1.13**) on M.

Neccessity. Suppose that (M, E) is a Berwald manifold with the linear connection ∇. Then $h_\nabla = h_0$. We show that the Hashiguchi connection $\left(\overset{H}{D}, h_0\right)$ is just the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right) = \left(\overset{\nabla}{D}, h_0\right)$. We have only to check that $\left(\overset{H}{D}, h_0\right)$ satisfies the axioms formulated in **3.1**. – (3.1a) and (3.1b) are just the axioms (1.13b) and (1.13c) of the Hashiguchi connection. The vanishing of the mixed curvature of the associated Finsler connection to $\left(\overset{H}{D}, h_0\right)$ follows at once from **4.6** and **2.3**, hence (3.1c) is satisfied. Finally, the h-deflection of $\left(\overset{H}{D}, h_0\right)$ also vanishes, as the following simple calculation shows: for any semispray S on M and any vector field X on $\mathcal{T}M$,

$$h_0^*\left(\overset{H}{D}C\right)(X) = \overset{H}{D}C(h_0X) = \overset{H}{D}_{h_0X}C = \overset{H}{D}_{h_0X}JS \overset{(1.13e)}{=} v_0[h_0X, C] = 0,$$

since h_0 is homogeneous.

Sufficiency. Suppose that there is a linear connection ∇ on M such that the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ coincides with the Hashiguchi connection $\left(\overset{H}{D}, h_0\right)$. Then $h_0 = h_\nabla$, therefore (4.6a) is satisfied and consequently (M, E) is a Berwald manifold. □

4.8. Theorem. *A Finsler manifold (M, E) is a locally Minkowski manifold if and only if there exists a torsion-free, flat linear connection ∇ on M such that the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ is "h_∇-metrical", i.e., $\overset{\nabla}{D}_{h_\nabla} g = 0$.*

Proof of the necessity. – If (M, E) is a locally Minkowski manifold, then – of course – it is a Berwald manifold at the same time. By the assumption the linear connection ∇ of this Berwald manifold is torsion-free and flat. But (M, E, ∇) is a generalized Berwald manifold as well, so the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ is h-metrical by Proposition 4.3.

Proof of the sufficiency. – If ∇ is a torsion-free, flat linear connection on M and the Ichijyō connection $\left(\overset{\nabla}{D}, h_\nabla\right)$ is h_∇-metrical then Proposition 4.3 assures that (M, E, ∇) is a generalized Berwald manifold, hence h_∇ is conservative. Since h_∇ arises from a symmetric linear connection, its tension and its weak torsion vanish. Thus, by the unicity statement of **1.12**, h_∇ is just the Barthel endomorphism and consequently (M, E) is a locally Minkowski manifold. □

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