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On conformal equivalence of Riemann-Finsler metrics

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Abstract. In this article a novel approach to the study of conformal change of Riemann-Finsler metrics is presented. Of particular interest is a result describing the change of the canonical spray of a Finsler manifold. Some classical results are also interpreted and proved from this new viewpoint, using coordinate free methods.

Introduction

The aim of this note is to present a new framework for the conformal theory of Riemann-Finsler metrics based on Grifone's theory of nonlinear connection [3] and the calculus of vector-valued differential forms established by A. Frölicher and A. Nijenhuis [2]. The novelty of our approach first of all lies in its purely intrinsic character (calculations with local coordinates are completely eliminated). Thus the main points are in focus and the fundamental mechanism behind the classical tensor calculus used e.g. in [7] and [10] becomes transparent. Briefly, in popular parlance, "one understands, why a theorem is true". Nevertheless, we have to emphasize that the classical source Rund [10], Ch. VI. §2 and Hashiguchi's comprehensive study [7] were indispensable and very stimulating for our work.

The plan of the paper is as follows. In the next section we offer a quite detailed exposition of the conceptual and calculational background in order to make the work self-contained as far as possible. In Section 2 some basic facts on the gradient of a function $\varphi \in C^\infty(TM)$ (M is a Finsler manifold)

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are collected, with special regard to the vertically lifted functions. Conformal equivalence of Riemann-Finsler metrics is introduced in Section 3. We give a very simple proof, based only on the homogeneity properties of the energy function, of Knebelman's famous observation which points out that the scale function must be a vertical lift. We also derive some important conformal invariants. In Section 4 we establish a key formula, describing the change of the canonical spray of a Finsler manifold under a conformal change of the Riemann-Finsler metric. As a consequence, we get immediately, how the Barthel endomorphism is changing. Having these results, one can also describe the change of the curvature of the Barthel endomorphism, the change of the Berwald and Cartan (and other) connections, etc. We shall discuss these questions and other aspects of the theory in a forthcoming paper. The concluding section is devoted to a nice application, which illustrates how the mechanism is working. We present an intrinsic proof of the classical theorem which (roughly speaking) states that in case of a simultaneous conformal and projective change the scale function is constant, i.e. the conformal change must be homothetic.

1. Preliminaries

1.1. Notations. We employ Grifone's terminology and conventions [3], [4] as far as feasible. For generalities, our base reference work is [8].

(i) M is an $n(>1)$-dimensional, C^∞, connected, paracompact manifold, $C^\infty(M)$ is the ring of real-valued smooth functions on M.

(ii) $\pi : TM \to M$ is the tangent bundle of M, $\pi_0 : \mathcal{T}M \to M$ is the bundle of nonzero tangent vectors.

(iii) $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M.

(iv) $\Omega^k(M)$ $(k \in \mathbb{N}^+)$ is the module of (scalar) k-forms on M, $\Omega^0(M) := C^\infty(M)$, $\Omega(M) := \oplus_{k=0}^n \Omega^k(M)$. $\Omega(M)$ is a graded algebra over $C^\infty(M)$, with multiplication given by the wedge product $\wedge$. $\otimes$ stands for the tensor product.

(v) $\Psi^k(M)$ $(k \in \mathbb{N}^+)$ is the $C^\infty(M)$-module of vector k-forms on M. It can be regarded as the space of k-linear (over $C^\infty(M)$) skew-symmetric maps $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. $\Psi^0(M) := \mathfrak{X}(M)$, $\Psi(M) := \oplus_{k=0}^n \Psi^k(M)$.

(vi) $i_X, \mathcal{L}_X$ $(X \in \mathfrak{X}(M))$ and d are the *insertion operator*, the *Lie derivative* (with respect to X) and the *exterior derivative*, respectively.

1.2. Frölicher-Nijenhuis theory. Our main technical tool is the Frölicher-Nijenhuis calculus of vector-valued forms and derivations, for which we refer to the wonderful original paper [2]; see also [8], [9], [12]. To make this work more readable, we assemble here some basic facts which will be applied in the sequel.

To any vector k-form $K \in \Psi^k(M)$ two derivations of $\Omega(M)$ are associated:

(i) *the derivation* i_K of degree $k-1$, completely determined by its action on $\Omega^0(M) = C^\infty(M)$ and $\Omega^1(M)$ as follows:
- $\forall f \in C^\infty(M) : i_K f = 0$
- $\forall \omega \in \Omega^1(M) : i_K \omega = \omega \circ K$;

(ii) *the derivation* d_K of degree k defined by

$$d_K := [i_K, d] := i_K \circ d - (-1)^{k-1} d \circ i_K.$$

Then, in particular,

$$\forall f \in C^\infty(M) : d_K f = i_K df \overset{(i)}{=} df \circ K$$

and d_K is uniquely determined by this formula. The graded commutator of d_K and d_L is

$$[d_K, d_L] = d_K \circ d_L - (-1)^{kl} d_L \circ d_K \quad (K \in \Psi^k(M),\ L \in \Psi^l(M)).$$

There exists a unique vector $(k+l)$-form $[K, L]$ such that

$$[d_K, d_L] = d_{[K,L]};$$

$[K, L]$ is called the *Frölicher-Nijenhuis bracket* of K and L.

We note that in case of vector 0-forms (i.e. vector fields) i_K and d_K become the usual insertion operator and the Lie derivative, while the Frölicher-Nijenhuis bracket reduces to the Lie bracket of vector fields. If $K \in \Psi^1(M)$, $Y \in \Psi^0(M) = \mathfrak{X}(M)$ then the vector 1-form $[K, Y]$ acts by the formula

$$(1) \qquad [K, Y](X) = [K(X), Y] - K[X, Y] \qquad (X \in \mathfrak{X}(M)).$$

We shall also need the identity

$$(2) \qquad [fX, K] = f[X, K] + df \wedge i_X K - d_K f \otimes X$$
$$(f \in C^\infty(M),\ X \in \mathfrak{X}(M),\ K \in \Psi^k(M)).$$

Remark 1. In the sequel we shall consider forms over TM or $\mathcal{T}M$ and apply derivations of $\Omega(TM)$ or $\Omega(\mathcal{T}M)$. *Differentiability of vector k-forms ($k \in \mathbb{N}^+$) will be required only over $\mathcal{T}M$*, unless otherwise stated.

1.3. Vertical apparatus. Homogeneity.

(a) $X^v(TM)$ and $X^v(\mathcal{T}M)$ denote the $C^\infty(TM)$-module of vertical vector fields on TM and $\mathcal{T}M$, respectively. On TM live two canonical objects which play an important role in Finslerian (and Lagrangian) theory: the *Liouville vector field* $C \in \mathfrak{X}^v(TM)$ and the *vertical endomorphism* $J \in \Psi^1(TM)$ (for the definitions see e.g. [9]). We have:

$$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(TM); \tag{3}$$

$$J^2 = 0, \qquad d_J^2 = 0; \tag{4}$$

$$[J, C] = J. \tag{5}$$

The *vertical lift* ([9], [11]) of a function $f \in C^\infty(M)$ and a vector field $X \in \mathfrak{X}(M)$ is denoted by f^v and X^v, respectively; $f^v := f \circ \pi$.

The following simple observation will be useful.

Lemma 1. *A smooth function φ on TM or $\mathcal{T}M$ is a vertical lift iff $d_J\varphi = 0$.*

Note that by 1.2 (ii) and (3)

$$d_J\varphi = 0 \iff \forall X \in \mathfrak{X}^v(TM) : X\varphi = 0.$$

We also recall [11] that the vertical lift of a 1-form $df \in \Omega^1(M)$ ($f \in C^\infty(M)$) is defined by

$$(df)^v := d(f^v). \tag{6}$$

(b) Let $k \in \mathbb{Z}$. We say that a (smooth) function φ, or a vector field X, or a scalar form ω and a vector form L, each of them given on TM or $\mathcal{T}M$, is *homogeneous of degree k* if

(i) $C\varphi = k\varphi$,

(ii) $[C, X] = (k-1)X$,

(iii) $\mathcal{L}_C\omega = k\omega$,

(iv) $\mathcal{L}_C L = (k-1)L$.

For example $\forall X \in \mathfrak{X}(M) : [C, X^v] = -X^v$, i.e. X^v is homogeneous of degree 0.

1.4. Semispray, spray. Complete lift.

Definition. A mapping $S : TM \to TTM$ is said to be a *semispray on M* if it satisfies the following conditions:

(SPR 1) S is a vector field of class C^1 on TM.

(SPR 2) S is smooth on $\mathcal{T}M$.

(SPR 3) $JS = C$.

A semispray S is called a *spray* if it is homogeneous of degree 2, i.e.

$$[C, S] = S \tag{SPR 4}$$

also holds.

Consider a vertical lift $\varphi = f \circ \pi$, $f \in C^\infty(M)$. If S is a semispray on M then the function $S\varphi$ does not depend on S because if $\bar{S}$ is another semispray then $\bar{S} - S$ is vertical (SPR 3) and

$$0 \overset{\text{Lemma 1}}{=} (\bar{S} - S)\varphi = \bar{S}\varphi - S\varphi \Rightarrow \bar{S}\varphi = S\varphi.$$

Thus the function

$$f^c := S\varphi = S(f \circ \pi) \tag{7}$$

is well-defined; it is said to be the *complete lift* of f. (For another definition see e.g. [11].) Now the *complete lift* X^c of a vector field $X \in \mathfrak{X}(M)$ can be introduced as in [11]:

$$\forall f \in C^\infty(M) : X^c f^c := (Xf)^c. \tag{8}$$

The following familiar formulas can be easily deduced:

$$\forall X, Y \in \mathfrak{X}(M),\ f \in C^\infty(M) :$$

$$X^v f^c = X^c f^v = (Xf)^v \tag{9}$$

$$[X^c, Y^c] = [X, Y]^c, \quad [X^v, Y^c] = [X, Y]^v, \tag{10}$$

$$[C, X^c] = 0 \quad \text{(i.e. } X^c \text{ is homogeneous of degree 1)}, \tag{11}$$

$$JX^c = X^v, \quad [J, X^c] = 0. \tag{12}$$

For the subsequent applications a useful observation will be the next

Local basis property. *If* $(X_1, \dots, X_n)$ *is a local basis of* $\mathfrak{X}(M)$ *then* $(X_1^v, \dots, X_n^v, X_1^c, \dots, X_n^c)$ *is a local basis for* $\mathfrak{X}(TM)$*.*

Lemma 2. $\forall f \in C^\infty(M)$*:*

(i) f^c *is homogeneous of degree 1.*

(ii) $d_J f^c = d(f^v) \overset{(6)}{=} (df)^v$.

Proof. (i) By the definition of f^c, we can write

$$f^c = S\varphi, \qquad \varphi = f \circ \pi,$$

where S is a *spray*. Thus

$$Cf^c = C(S\varphi) = [C,S]\varphi + S(C\varphi) \overset{\text{(SPR 4), Lemma 1}}{=} S\varphi = f^c,$$

which means the 1-homogeneity of f^c.

(ii) The vector 1-form $d_J f^c$ clearly kills the vertical vector fields so it is completely determined by its action on the complete lifts.

$$\forall X \in \mathfrak{X}(M):$$

$$(d_J f^c)(X^c) = df^c(JX^c) \overset{(12)}{=} df^c(X^v) = X^v f^c \overset{(9)}{=} X^c f^v = (df^v)(X^c),$$

hence $d_J f^c = d(f^v)$. $\square$

1.5. Horizontal endomorphisms. Now we recall some basic concepts of a version of Grifone's theory of nonlinear connections [3], whose role will be played by the "horizontal endomorphisms" in our approach.

Definition. A vector 1-form $h \in \Psi^1(TM)$ is said to be a *horizontal endomorphism* on M if it satisfies the following conditions:

(HE 1) h is smooth over $\mathcal{T}M$.

(HE 2) h is a projector, i.e. $h^2 = h$.

(HE 3) $\operatorname{Ker} h = \mathfrak{X}^v(TM)$.

The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ (with respect to h) is $X^h := hX^c$.

- $H := [h, C]$ is the *tension* of h,
- $t := [J, h]$ is the *weak torsion* of h,

- $T := i_S t + H$ (S is an arbitrary semispray on M) is the *strong torsion* of h.

It follows immediately from the definitions that

$$h \circ J = 0, \quad J \circ h = J; \tag{13}$$

$$\mathfrak{X}(TM) = \mathfrak{X}^v(TM) \oplus \mathfrak{X}^h(TM), \quad \mathfrak{X}^h(TM) := \operatorname{Im} h. \tag{14}$$

Owing to (HE 1), the vector forms $H, T \in \Psi^1(TM)$, $t \in \Psi^2(TM)$ are smooth over $\mathcal{T}M$; cf. also Remark 1.

Any horizontal endomorphism h naturally determines an *almost complex structure* $F \in \Psi^1(TM)$ ($F^2 = -1$, F is smooth on $\mathcal{T}M$) such that

$$F \circ h = -J, \qquad F \circ J = h; \tag{15}$$

it is called the almost complex structure associated with h.

1.6. Semibasic trace ([5], [12]). A scalar form $\omega \in \Omega^k(TM)$ ($k \in \mathbb{N}^+$) is said to be *semibasic* if

$$\forall X \in \mathfrak{X}(TM) : i_{JX}\omega = 0.$$

A vector form $L \in \Psi^l(TM)$ ($l \in \mathbb{N}^+$) is called semibasic if

$$J \circ L = 0 \quad \text{and} \quad \forall X \in \mathfrak{X}(TM) : i_{JX} L = 0.$$

Suppose that $L \in \Psi^l(TM)$ is semibasic. The *semibasic trace* $\tilde{L}$ of L is the semibasic scalar $(l-1)$-form defined by recurrence as follows:

(i) if $l = 1$, then $\tilde{L} := \operatorname{trace}(F \circ L)$, where F is the associated almost complex structure of an arbitrarily chosen horizontal endomorphism

(ii) if $l > 1$, then $i_X \tilde{L} := \widetilde{i_X L}$ for all $X \in \mathfrak{X}(TM)$.

Lemma 3. $\tilde{J} = n$.

Proof. $\tilde{J} := \operatorname{trace}(F \circ J) \overset{(15)}{=} \operatorname{trace}(h)$. Choose a local basis $(X_1, \dots, X_n)$ for $\mathfrak{X}(M)$. (14) guarantees that $(X_1^v, \dots, X_n^v, X_1^h, \dots, X_n^h)$ is a local basis of $\mathfrak{X}(TM)$. Since

$$h(X_i^v) \overset{\text{(HE 3)}}{=} 0,$$

$$h(X_i^h) := h(hX_i^c) = h^2(X_i^c) \overset{\text{(HE 2)}}{=} h(X_i^c) =: X_i^h \quad (1 \le i \le n),$$

we obtain immediately that $\operatorname{trace}(h) = n$. □

We shall need the following formula [12]:

$$\text{for each } \omega \in \Omega^k(TM) \text{ semibasic form: } \widetilde{\omega \otimes C} = (-1)^{k+1} i_S \omega \tag{16}$$

(S is an arbitrary semispray).

1.7. Finsler manifolds. Let a function $E : TM \to \mathbb{R}$, called *energy*, be given. The pair (M, E), or simply M, is said to be a *Finsler manifold* if the energy function satisfies the following conditions:

(F0) $\forall v \in \mathcal{T}M : \quad E(v) > 0, \quad E(0) = 0.$

(F1) E is of class C^1 on TM smooth on $\mathcal{T}M$.

(F2) $CE = 2E$, i.e. E is homogeneous of degree 2.

(F3) The form $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$, called the *fundamental form*, is symplectic.

The mapping

$$\begin{gathered} g : \mathfrak{X}^v(\mathcal{T}M) \times \mathfrak{X}^v(\mathcal{T}M) \to C^\infty(\mathcal{T}M) \\ (JX, JY) \mapsto g(JX, JY) := \omega(JX, Y) \end{gathered} \tag{17}$$

is a well-defined, nondegenerate symmetric bilinear form (over $C^\infty(TM)$) which we call the *Riemann-Finsler metric* of the Finsler manifold (M, E). (Note that if E is of class C^2, we get the category of Riemannian manifolds!)

We have the following important identities:

$$i_J \omega = 0, \quad i_C \omega = d_J E; \tag{18}$$

$$\mathcal{L}_C \omega = \omega; \tag{19}$$

$$E = \frac{1}{2} g(C, C). \tag{20}$$

On any Finsler manifold there is a spray $S : TM \to TTM$, which is uniquely determined on $\mathcal{T}M$ by the formula

$$i_S \omega = -dE. \tag{21}$$

This spray is called the *canonical spray* of the Finsler manifold.

The following result is the miracle of Finsler geometry:

The fundamental lemma of Finsler geometry [3]. *On a Finsler manifold (M,E) there is a unique horizontal endomorphism $h\in\Psi^1(TM)$ such that*

(B1) $d_h E = 0$ *("h is conservative"),*

(B2) *the strong torsion of h vanishes.*

h is called the Barthel endomorphism of M, and is given by the formula

$$h = \frac{1}{2}(1 + [J,S]),$$

where S is the canonical spray.

Suppose that (M,E) is a Finsler manifold with Riemann-Finsler metric g. There exists a unique tensor $\mathcal{C} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$, satisfying the following.

(CAR 1) $J \circ \mathcal{C} = 0$,

(CAR 2) $\forall X,Y,Z \in \mathfrak{X}(\mathcal{T}M) : g(\mathcal{C}(X,Y), JZ) = \frac{1}{2}(\mathcal{L}_{JX}J^*g)(Y,Z)$.

(J^* is the adjoint operator of J, see [9]). $\mathcal{C}$ is called the *Cartan tensor* of the Finsler manifold, its lowered tensor $\mathcal{C}_b$ is defined by

$$\mathcal{C}_b(X,Y,Z) := g(\mathcal{C}(X,Y), JZ).$$

Definition. [5] Let (M,E) be a Finsler manifold and consider the volume form

$$w := \frac{(-1)^{n(n+1)/2}}{n!}\omega^n$$

on $\mathcal{T}M$. The *divergence* of a vector field $X \in \mathfrak{X}(TM)$ is the function δX given by

$$(\delta X)w = \mathcal{L}_X w.$$

Lemma 4. [5] $\forall X \in \mathfrak{X}(TM) : \delta(JX) = \widetilde{[J, JX]} + 2\tilde{\mathcal{C}}(X)$.

Corollary 1. $\delta C = n$.

Proof. Applying (19) we obtain by an easy induction that

$$(\delta C)w := \mathcal{L}_C w = \frac{(-1)^{n(n+1)/2}}{n!} n\omega^n = nw. \qquad \square$$

2. Gradient operator on the tangent bundle of a Finsler manifold

Let (M, E) be a Finsler manifold with fundamental form ω. Consider a smooth function $\varphi : TM \to \mathbb{R}$. Nondegeneracy of ω guarantees the existence and unicity of the vector field $\operatorname{grad}\varphi \in \mathfrak{X}(\mathcal{T}M)$ characterized by the condition

$$d\varphi = i_{\operatorname{grad}\varphi}\omega;$$

this vector field is called the *gradient* of φ.

Lemma 5. *If M is a Finsler manifold then*

$$\forall \varphi \in C^\infty(TM) : \delta \operatorname{grad}\varphi = 0.$$

Proof. Using H. Cartan's magic formula, we have

$$\begin{aligned}\mathcal{L}_{\operatorname{grad}\varphi}\omega = i_{\operatorname{grad}\varphi} d\omega + d i_{\operatorname{grad}\varphi}\omega &\overset{(F3)}{=} \\ = d i_{\operatorname{grad}\varphi}\omega = dd\varphi = 0,&\end{aligned}$$

thus the result follows. □

Proposition 1. *Let (M, E) be a Finsler manifold and suppose that $\varphi \in C^\infty(TM)$ is a vertical lift: $\varphi = f \circ \pi$, $f \in C^\infty(M)$. Then the gradient vector field of φ has the following properties:*

(i) $\operatorname{grad}\varphi \in \mathfrak{X}^v(\mathcal{T}M)$.

(ii) $[C, \operatorname{grad}\varphi] = -\operatorname{grad}\varphi$, *i.e.* $\operatorname{grad}\varphi$ *is homogeneous of degree 0.*

(iii) $\operatorname{grad}\varphi(E) = f^c$.

Proof. (i) Since φ is a vertical lift, we get from Lemma 1 that

$$\forall X \in \mathfrak{X}(TM) : d_J\varphi(X) = d\varphi(JX) = 0.$$

Thus

$$\begin{aligned}g(JX, J\operatorname{grad}\varphi) &\overset{(17)}{=} \omega(JX, \operatorname{grad}\varphi) = -(i_{\operatorname{grad}\varphi}\omega)(JX) \\ &= -d\varphi(JX) = 0.\end{aligned}$$

We know that g is nondegenerate and (by (3)) that $JX \in \mathfrak{X}^v(TM)$ is arbitrary, so it follows that $J\,\mathrm{grad}\,\varphi = 0$. Since $\mathrm{Ker}\,J = \mathfrak{X}^v(TM)$ (also by (3)), we get the desired relation $\mathrm{grad}\,\varphi \in \mathfrak{X}^v(\mathcal{T}M)$.

(ii)
$$\begin{aligned} i_{[C,\mathrm{grad}\,\varphi]}\omega &= \mathcal{L}_C i_{\mathrm{grad}\,\varphi}\omega - i_{\mathrm{grad}\,\varphi}\mathcal{L}_C\omega \overset{(19)}{=} \\ &= \mathcal{L}_C d\varphi - i_{\mathrm{grad}\,\varphi}\omega. \end{aligned}$$

Evaluating the Lie derivative, we obtain:

$$\begin{aligned} &\forall X \in \mathfrak{X}(TM) : (\mathcal{L}_C d\varphi)(X) = C d\varphi(X) - d\varphi[C,X] \\ &= C(X\varphi) - [C,X]\varphi = X(C\varphi) \overset{\text{Lemma 1}}{=} X(0) = 0. \end{aligned}$$

Hence

$$i_{[C,\mathrm{grad}\,\varphi]}\omega = i_{-\,\mathrm{grad}\,\varphi}\omega,$$

and, therefore, $[C, \mathrm{grad}\,\varphi] = -\,\mathrm{grad}\,\varphi$.

(iii)
$$\begin{aligned} \mathrm{grad}\,\varphi(E) &= dE(\mathrm{grad}\,\varphi) \overset{(21)}{=} -i_S\omega(\mathrm{grad}\,\varphi) \\ &= \omega(\mathrm{grad}\,\varphi, S) = (i_{\mathrm{grad}\,\varphi}\omega)(S) \\ &= d\varphi(S) = S\varphi \overset{(7)}{=} f^c. \end{aligned}$$ □

3. Conformally equivalent Riemann-Finsler metrics

Definition. Consider the Finsler manifolds (M, E) and $(M, \bar{E})$ and let us denote by g and $\bar{g}$ their Riemann-Finsler metrics. g and $\bar{g}$ are said to be *conformally equivalent* if there exists a positive smooth function $\varphi : TM \to \mathbb{R}$ such that $\bar{g} = \varphi g$. In this case we also speak of a *conformal change* of the metric. The function φ is called the *scale function* or the *proportionality function*. If the scale function is constant, then we say that the conformal change is *homothetic*.

Remark 2. If $\bar{g} = \varphi g$, then

$$\bar{E} \overset{(20)}{=} \frac{1}{2}\bar{g}(C,C) = \frac{1}{2}\varphi g(C,C) \overset{(20)}{=} \varphi E.$$

Lemma 6 (Knebelman's observation). *The proportionality function between conformally equivalent Riemann-Finsler metrics is a vertical lift.*

Proof. Consider the Finsler manifolds (M, E) and $(M, \bar{E})$. In case of conformal equivalence we have just seen that $\bar{E} = \varphi E$. Using the homogeneity property (F2) of $\bar{E}$ and E it follows that

$$2\bar{E} = C\bar{E} = C(\varphi E) = (C\varphi)E + \varphi(CE) = (C\varphi)E + 2\varphi E = (C\varphi)E + 2\bar{E},$$

therefore $C\varphi = 0$; i.e. φ is homogeneous of degree 0. Since φ is smooth on the *whole* tangent manifold TM, this implies by a routine reasoning that φ is fiberwise constant which obviously means that φ is a vertical lift. $\square$

Proposition 2. *Let (M, E) and $(M, \bar{E})$ be Finsler manifolds with Riemann-Finsler metrics g and $\bar{g}$, respectively. g and $\bar{g}$ are conformally equivalent if and only if*

$$\frac{d_J E}{E} = \frac{d_J \bar{E}}{\bar{E}} \qquad (\textit{over } \mathcal{T}M).$$

In particular, $\frac{d_J E}{E}$ is invariant under the conformal changes of the Riemann-Finsler metric.

Proof. The necessity is clear: if $\bar{g} = \varphi g$, then $\bar{E} = \varphi E$, where $d_J \varphi = 0$ by Lemma 6, and so

$$\frac{d_J \bar{E}}{\bar{E}} = \frac{d_J(\varphi E)}{\varphi E} = \frac{(d_J \varphi)E + \varphi d_J E}{\varphi E} = \frac{d_J E}{E}.$$

Suppose, conversely, that over $\mathcal{T}M$

$$\frac{d_J E}{E} = \frac{d_J \bar{E}}{\bar{E}}.$$

Then we get immediately the relation

$$d_J(\ln \circ E) = d_J(\ln \circ \bar{E}),$$

or, equivalently,

$$d_J\left(\ln \circ \frac{\bar{E}}{E}\right) = 0.$$

This means by Lemma 1 that the function $\ln \circ \frac{\bar{E}}{E}$, and therefore $\frac{\bar{E}}{E}$ is a vertical lift. So there is a positive function $f \in C^\infty(M)$ such that $\frac{\bar{E}}{E} = f \circ \pi_0$. If $\varphi := f \circ \pi$, then we have

$$\begin{aligned}
&\forall X, Y \in \mathfrak{X}(\mathcal{T}M) : \bar{g}(JX, JY) := \bar{\omega}(JX, Y) := dd_J \bar{E}(JX, Y) \\
&= dd_J(\varphi E)(JX, Y) = d(E d_J \varphi + \varphi d_J E)(JX, Y) \overset{\text{Lemma 1}}{=} \\
&= d(\varphi d_J E)(JX, Y) = d\varphi \wedge d_J E(JX, Y) + \varphi dd_J E(JX, Y) \\
&= d\varphi(JX) d_J E(Y) - d\varphi(Y) d_J E(JX) + \varphi \omega(JX, Y) \\
&= d_J \varphi(X) d_J E(Y) - d\varphi(Y) dE(J^2 X) + \varphi g(JX, JY) \overset{\text{Lemma 1, (4)}}{=} \\
&= \varphi g(JX, JY);
\end{aligned}$$

hence

$$\bar{g} = \varphi g,$$

as was to be shown. $\square$

Proposition 3. *The Cartan tensor is invariant under a conformal change of a Riemann-Finsler metric, while the lowered Cartan tensor changes by the formula $\bar{\mathcal{C}}_b = \varphi \mathcal{C}_b$, where φ is the scale function.*

Proof. Consider the conformal change $\bar{g} = \varphi g$. Then $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$:

$$\begin{aligned}
&2\bar{g}(\bar{\mathcal{C}}(X, Y), JZ) \overset{\text{(CAR 2)}}{=} \mathcal{L}_{JX}(J^* \bar{g})(Y, Z) = \mathcal{L}_{JX}(J^*(\varphi g))(Y, Z) \\
&= JX(\varphi g(JY, JZ)) - \varphi g(J[JX, Y], JZ) - \varphi g(JY, J[JX, Z]) \\
&= JX(\varphi) g(JY, JZ) + 2\varphi g(\mathcal{C}(X, Y), JZ) \overset{\text{Lemma 6}}{=} 2\bar{g}(\mathcal{C}(X, Y), JZ),
\end{aligned}$$

therefore $\bar{\mathcal{C}} = \mathcal{C}$. This relation implies immediately that $\bar{\mathcal{C}}_b = \varphi \mathcal{C}_b$. $\square$

4. The fundamental relation between the canonical sprays

Theorem 1. *Suppose that g and $\bar{g}$ are conformally equivalent Riemann-Finsler metrics on M, namely*

$$\bar{g} = \varphi g; \quad \varphi = \exp \circ \alpha \circ \pi, \quad \alpha \in C^\infty(M).$$

Then the corresponding canonical sprays satisfy the relation

$$\bar{S} = S - \alpha^c C + E \operatorname{grad} \alpha^v. \tag{22}$$

Proof. On the one hand, since $\bar{E} = \varphi E$, we have

$$-d\bar{E} = -d(\varphi E) = -\varphi dE - E d\varphi \overset{(21)}{=} i_{\varphi S}\omega - E d\varphi.$$

On the other hand,

$$\begin{aligned}
-d\bar{E} &\overset{(21)}{=} i_{\bar{S}}\bar{\omega} = i_{\bar{S}} dd_J(\varphi E) = i_{\bar{S}} d(E d_J\varphi + \varphi d_J E) \\
&= i_{\bar{S}}(d\varphi \wedge d_J E + \varphi dd_J E) = (i_{\bar{S}} d\varphi) d_J E \\
&\quad - (i_{\bar{S}} d_J E) d\varphi + \varphi i_{\bar{S}}\omega = \bar{S}(\varphi) d_J E - dE(J\bar{S}) d\varphi \\
&\quad + \varphi i_{\bar{S}}\omega \overset{\text{Prop. 1, (7), (F2), (18)}}{=} i_{\operatorname{grad}\varphi(E)C}\omega - 2E d\varphi + i_{\varphi\bar{S}}\omega.
\end{aligned}$$

Comparing the right sides of these relations, we obtain that

$$i_{\varphi\bar{S}}\omega = i_{\varphi S}\omega - i_{\operatorname{grad}\varphi(E)C}\omega + E d\varphi = i_{\varphi S - \operatorname{grad}\varphi(E)C + E\operatorname{grad}\varphi}\omega.$$

Hence

$$\bar{S} = S - \frac{1}{\varphi}\operatorname{grad}\varphi(E)C + E\frac{1}{\varphi}\operatorname{grad}\varphi.$$

Since

$$\operatorname{grad}\varphi = \operatorname{grad}(\exp\circ\alpha^v) = (\exp'\circ\alpha^v)\operatorname{grad}\alpha^v = \varphi\operatorname{grad}\alpha^v$$

and therefore

$$\frac{1}{\varphi}\operatorname{grad}\varphi(E) = \operatorname{grad}\alpha^v(E) \overset{\text{Prop. 1/(iii)}}{=} \alpha^c,$$

(22) is proved. □

Corollary 2. *Under the conditions of Theorem 1, the Barthel endomorphisms are related as follows:*

$$\bar{h} = h - \frac{1}{2}(\alpha^c J + (d\alpha)^v \otimes C) + \frac{1}{2}E[J, \operatorname{grad}\alpha^v] + \frac{1}{2}d_J E \otimes \operatorname{grad}\alpha^v.$$

PROOF. $\bar{h} := \frac{1}{2}(1 + [J, \bar{S}]) \overset{(22)}{=} h - \frac{1}{2}[J, \alpha^c C] + \frac{1}{2}[J, E \operatorname{grad} \alpha^v]$. Here

- $$-[J, \alpha^c C] = [\alpha^c C, J] \overset{(2)}{=} \alpha^c [C, J] + d\alpha^c \wedge i_C J - d_J \alpha^c \otimes C \overset{(5),\ \text{Lemma 2}}{=} -\alpha^c J - (d\alpha)^v \otimes C$$

- $$[J, E \operatorname{grad} \alpha^v] \overset{(2)}{=} E[J, \operatorname{grad} \alpha^v] - dE \wedge i_{\operatorname{grad} \alpha^v} J + d_J E \otimes \operatorname{grad} \alpha^v \overset{\text{Prop. 1}}{=} E[J, \operatorname{grad} \alpha^v] + d_J E \otimes \operatorname{grad} \alpha^v,$$

thus we obtain the desired relation. □

5. An application

A well-known, classical result states (in H. Weyl's terminology) that "*the projective and conformal properties of a Finsler space determine its metric properties uniquely*" [10], p. 226. Now we are going to formulate this nice and important theorem in our framework and prove it *purely intrinsically*.

Definition (cf. [1]). *(a)* Two sprays S and $\bar{S}$ given on the manifold M are said to be *projectively equivalent* if there is a function $\lambda : TM \to \mathbb{R}$ satisfying the following conditions:

(Proj 1) λ is smooth on $\mathcal{T}M$, C^1 on TM.

(Proj 2) $\bar{S} = S + \lambda C$.

(b) We say that the Finsler manifolds (M, E) and $(M, \bar{E})$ are projectively equivalent if their canonical sprays have the property described in (a).

Remark 3. (Proj 1) and (Proj 2) easily imply that λ is homogeneous of degree 1.

Theorem 2. *Suppose that (M, E) and $(M, \bar{E})$ are projectively equivalent Finsler manifolds of dimension $n > 1$. If their Riemann-Finsler metrics are conformally equivalent as well then the conformal change is homothetic.*

PROOF. Keeping the previous notation, now we have by (Proj 2) and (22) the relations

$$\bar{S} = S + \lambda C \quad \text{and} \quad \bar{S} = S - \alpha^c C + E \operatorname{grad} \alpha^v$$

simultaneously. From these it follows that

$$\lambda C = -\alpha^c C + E \operatorname{grad} \alpha^v. \tag{23}$$

Applying both sides to the energy function E, we obtain by (F2) and Proposition 1/(iii) that

$$2\lambda E = -2\alpha^c E + \alpha^c E,$$

i.e.

$$\lambda = -\frac{1}{2}\alpha^c.$$

Substituting this into (23), we get the relation

$$\operatorname{grad} \alpha^v = \frac{\alpha^c}{2E} C.$$

Let $\mu := \frac{\alpha^c}{2E}$. In view of Lemma 2 and (F2), μ is homogeneous of degree -1. Thus we have:

$$\operatorname{grad} \alpha^v = \mu C, \tag{24}$$

$$C\mu = -\mu. \tag{25}$$

Let S_0 be an arbitrary semispray on M. First we note that

$$\tilde{\mathcal{C}}(S_0) = 0. \tag{26}$$

Indeed,

$$\begin{aligned} n &\overset{\text{Cor. 1}}{=} \delta C = \delta(JS_0) \overset{\text{Lemma 4}}{=} \widetilde{[J, JS_0]} + 2\tilde{\mathcal{C}}(S_0) \\ &= \widetilde{[J, C]} + 2\tilde{\mathcal{C}}(S_0) \overset{(5)}{=} \tilde{J} + 2\tilde{\mathcal{C}}(S_0) \overset{\text{Lemma 3}}{=} \\ &= n + 2\tilde{\mathcal{C}}(S_0), \end{aligned}$$

so (26) is true. Secondly, we claim that

$$\widetilde{[J, \operatorname{grad} \alpha^v]} = 0. \tag{27}$$

To see this, consider the vector field $X := \mu S_0$. Then

$$JX = \mu J S_0 = \mu C \overset{(24)}{=} \operatorname{grad} \alpha^v,$$

therefore

$$
\begin{aligned}
0 &\overset{\text{Lemma 5}}{=} \delta \operatorname{grad} \alpha^v = \delta(JX) \overset{\text{Lemma 4}}{=} \widetilde{[J, JX]} + 2\tilde{\mathcal{C}}(X) \\
&= [J, \widetilde{\operatorname{grad} \alpha^v}] + 2\mu\tilde{\mathcal{C}}(S_0) \overset{(26)}{=} [J, \widetilde{\operatorname{grad} \alpha^v}],
\end{aligned}
$$

so (27) is also true. Now we come back to (24). Taking Frölicher-Nijenhuis bracket with J we get the relation

$$
[J, \operatorname{grad} \alpha^v] = [J, \mu C]. \tag{28}
$$

We calculate the semibasic trace of both sides.

$$
\begin{aligned}
0 &\overset{(27)}{=} [J, \widetilde{\operatorname{grad} \alpha^v}] \overset{(28)}{=} \widetilde{[J, \mu C]} \overset{(2)}{=} \widetilde{\mu J} \\
&\quad - d\widetilde{\mu \wedge i_C} J + d_J \widetilde{\mu \otimes C} = \mu \tilde{J} + d_J \widetilde{\mu \otimes C} \\
&\overset{\text{Lemma 3, (16)}}{=} \mu n + i_{S_0} d_J \mu = \mu n + i_{S_0} i_J d\mu \\
&= \mu n + d\mu(JS_0) = \mu n + C\mu \overset{(25)}{=} \mu n - \mu = (n-1)\mu,
\end{aligned}
$$

so $\mu = 0$. This implies in view of (24) that $\operatorname{grad} \alpha^v = 0$. Thus

$$
(d\alpha)^v \overset{(6)}{=} d(\alpha^v) = i_{\operatorname{grad} \alpha^v} \omega = 0,
$$

therefore $d\alpha = 0$. Because M is connected (1.1.(i)), this means that the function α is constant which was to be proved. □

Remark 4. Observe that under the hypothesis of Theorem 2 the projective equivalence is trivial: the function λ vanishes.

Corollary 3. *Suppose that the Riemann-Finsler metrics g and $\bar{g}$ on M are conformally equivalent:*

$$
\bar{g} = \varphi g; \quad \varphi = \exp \circ \alpha \circ \pi, \quad \alpha \in C^\infty(M).
$$

Then the following assertions are equivalent:

(i) The conformal change is homothetic.

(ii) The canonical sprays S and $\bar{S}$ coincide.

(iii) The Barthel endomorphisms h and $\bar{h}$ coincide.

Proof. (i) ⇒ (ii) Indeed, if (i) holds, then α^c and $\operatorname{grad} \alpha^v$ vanish so (22) implies that $S = \bar{S}$.

(ii) $\Rightarrow$ (i) This follows immediately from Theorem 2.

(ii) $\Rightarrow$ (iii) This is evident.

(iii) $\Rightarrow$ (ii) From the hypothesis that (iii) holds we obtain the relation

$$[J, \bar{S}] = [J, S].$$

Applying both sides to the spray S, we have by (1) that

$$[C, \bar{S}] - J[S, \bar{S}] = [C, S] - J[S, S].$$

Using (SPR 4), it follows that

$$\bar{S} - J[S, \bar{S}] = S.$$

Let $Z := S - \bar{S}$. Then $Z \in \mathfrak{X}^v(\mathcal{T}M)$ and

$$J[S, \bar{S}] = J[Z, \bar{S}].$$

Here, by Proposition 4.3.7 of [9],

$$J[Z, \bar{S}] = Z = S - \bar{S}.$$

Therefore

$$\bar{S} - S + \bar{S} = S$$

hence $\bar{S} = S$, proving the implication (iii) $\Rightarrow$ (ii). $\square$

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