

## Conformal vector fields on Finsler manifolds

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**Abstract.** Applying concepts and tools from classical tangent bundle geometry and using the apparatus of the calculus along the tangent bundle projection (‘pull-back formalism’), first we enrich the known lists of the characterizations of affine vector fields on a spray manifold and conformal vector fields on a Finsler manifold. Second, we deduce consequences on vector fields on the underlying manifold of a Finsler structure having one or two of the mentioned geometric properties.

### Introduction

The theory of ‘geometrical’ – projective, affine, conformal, isometric – vector fields on a Finsler manifold has a vast literature, mainly from the period dominated technically by the classical tensor calculus, visually, ‘the debauch of indices’. Chapter VIII of K. Yano’s book ‘The theory of Lie derivatives and its applications’ presents a survey of the main achievements from the beginning of the 20th century to 1957. A good overview of the developments of the next decades can be found in R. B. Misra’s paper [15], written in 1981, revised and updated in 1993. It is important to note that in a 2-part paper, see [13], [14], M. Matsumoto clarified and improved some results of Yano in the framework of his theory of Finsler connections.

From the (relatively) modern, but partly tensor calculus based literature the works of H. Akbar-Zadeh [2], [3], J. Grifone [9], [10] and R. L. Lovas [12] are worth mentioning. Grifone applies systematically the ‘ $\tau_{TM}: TTM \rightarrow TM$  formalism’, combining with the Frölicher-Nijenhuis calculus of vector-valued forms; Lovas formulates and proves his results in terms of the ‘pull-back formalism  $\overset{\circ}{\pi}: \overset{\circ}{TM} \times_M TM \rightarrow \overset{\circ}{TM}$ ’. Our paper is a continuation of both Grifone’s and Lovas’s works. Although we are going to develop the greater part of the theory in terms of the pull-back bundle, the concepts and techniques of the tangent bundle geometry,

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including the vertical calculus on  $\widetilde{TM}$ , also play an eminent role in our considerations. To make the paper more readable, in section 1 we summarize in a coherent way the various concepts and tools which will be indispensable in the following.

We apply two types of a Lie derivative operator: beside the classical Lie derivative operator  $\mathcal{L}_\xi$  on  $TM$  ( $\xi \in \mathfrak{X}(TM)$ ) we need a further operator, denoted by  $\widetilde{\mathcal{L}}_\xi$ , which acts on the tensor algebra of the  $C^\infty(TM)$ -module of the sections of the vector bundle  $\pi: TM \times_M TM \rightarrow TM$  (or of the bundle  $\overset{\circ}{\pi}$ ). To assure the validity of the crucial identity  $[\widetilde{\mathcal{L}}_\xi, \widetilde{\mathcal{L}}_\eta] = \widetilde{\mathcal{L}}_{[\xi, \eta]}$  in case of the ‘new’ operator, we are forced to differentiate with respect to *projectable* vector fields on  $TM$ . In section 2 some basic properties of the operator  $\widetilde{\mathcal{L}}_\xi$  are established.

The affine and projective properties of a Finsler manifold depend only on its canonical spray, so it is natural to examine affine and projective vector fields in the (virtual) generality of spray manifolds. A vector field  $X$  on a manifold  $M$  is said to be an affine vector field or a Lie symmetry for a spray  $S: TM \rightarrow TTM$  if  $S$  is invariant under the flow of the complete lift  $X^c$  of  $X$ , that is, if  $\mathcal{L}_{X^c}S = [X^c, S] = 0$ . In Lovas’s paper [12] various equivalents of this property are established. In section 3 we enrich his list with some new items, which will be technically useful in the next section.

By a conformal vector field on a Finsler manifold  $(M, F)$  we mean a vector field  $X$  on  $M$  satisfying

$$\widetilde{\mathcal{L}}_{X^c}g = \varphi g,$$

where  $g$  is the metrical tensor of the Finsler manifold (the vertical Hessian of the energy function  $E = \frac{1}{2}F^2$ ) and  $\varphi$  is a function, defined and continuous on  $TM$ , smooth on the deleted bundle  $\overset{\circ}{TM}$ . It turns out at once that  $\varphi$  has to be fibrewise constant, i.e., of the form  $\varphi = f \circ \tau$ , where  $f$  is a smooth function on  $M$  and  $\tau$  is the tangent bundle projection. Homothetic and isometric (or Killing) vector fields are the particular cases for which  $\varphi$  is a constant function, resp. identically zero. In section 4 we present further characterizations of conformal vector fields on a Finsler manifold (Proposition 2), one of them has already been proposed by Grifone in [10]. We show that if a vector field  $X \in \mathfrak{X}(M)$  is both affine and conformal on a Finsler manifold  $(M, F)$ , then  $X^c$  is a conformal vector field for the Sasaki extension of the metric tensor of  $(M, F)$  (Proposition 3).

At this stage, the following ‘expectable’, but non-trivial conclusions may be deduced fairly easily:

- (a) Homothetic vector fields on a Finsler manifold are affine vector fields (Proposition 4).
- (b) If a vector field on a Finsler manifold is both projective and conformal, then it is a homothetic vector field (Proposition 5).
- (c) If a vector field preserves the Dazord volume form of a Finsler manifold and it is also projective, then it is an affine vector field (Proposition 6, (i)).
- (d) If a vector field is both volume-preserving (in the above sense) and conformal, then it is a Killing field (Proposition 6, (ii)).

# 1 Basic setup

## 1.1 Generalities

Most of our basic notations and conventions will be the same as in [4], see also [16]. However, for the reader's convenience, we present here a short review on the most essential things.

(a) By a manifold we mean a finite dimensional smooth manifold whose underlying topological space is Hausdorff, second countable and connected. In what follows,  $M$  will be an  $n$ -dimensional manifold, where  $n \geq 2$ . Let  $k \in \mathbb{N} \cup \{\infty\}$ . We denote by  $C^k(M)$  the set of  $k$ -times continuously differentiable real-valued functions on  $M$ , with the convention that  $C^0(M)$  is the set of the continuous functions on  $M$ . In particular,  $C^\infty(M)$  is the real algebra of smooth functions on  $M$ .

(b) The tangent space of  $M$  at a point  $p \in M$  is denoted by  $T_pM$ ;  $TM := \bigcup_{p \in M} T_pM$ . The tangent bundle of  $M$  is the triplet  $(TM, \tau, M)$ , where the tangent bundle projection  $\tau$  is defined by  $\tau(v) := p$  if  $v \in T_pM$ . Instead of  $(TM, \tau, M)$  we usually write  $\tau: TM \rightarrow M$  or simply  $\tau$ . Similarly, the tangent bundle of  $TM$  is  $(TTM, \tau_{TM}, TM)$  or  $\tau_{TM}: TTM \rightarrow TM$  or  $\tau_{TM}$ . In general, we prefer to denote a bundle by the same symbol as we use for its projection.

A *vector field* on  $M$  is a smooth section of the tangent bundle  $\tau: TM \rightarrow M$ . The vector fields on  $M$  form a  $C^\infty(M)$ -module which will be denoted by  $\mathfrak{X}(M)$ . The *zero vector field*  $o$  on  $M$  is defined by

$$p \in M \mapsto o(p) := 0_p := \text{the zero vector in } T_pM.$$

The *deleted bundle* for  $\tau$  is the fibre bundle  $\overset{\circ}{\tau}: \overset{\circ}{TM} \rightarrow M$ , where  $\overset{\circ}{TM} := TM \setminus o(M)$ ,  $\overset{\circ}{\tau} := \tau \upharpoonright \overset{\circ}{TM}$ .

(c) If  $\varphi: M \rightarrow N$  is a smooth mapping between smooth manifolds, then we denote its derivative by  $\varphi_*$ , which is a fibrewise linear smooth mapping of  $TM$  into  $TN$ . Two vector fields  $X \in \mathfrak{X}(M)$  and  $Y \in \mathfrak{X}(N)$  are  $\varphi$ -related if  $\varphi_* \circ X = Y \circ \varphi$ ; then we write  $X \underset{\varphi}{\sim} Y$ . A vector field  $\xi$  on  $TM$  is said to be *projectable* if there exists a vector field  $X$  on  $M$  such that  $\xi \underset{\tau}{\sim} X$ .

(d) The classical graded derivations of the graded algebra  $\Omega(M) := \bigoplus_{k=0}^n \Omega^k(M)$  of the differential forms on  $M$  are

- the Lie derivative  $\mathcal{L}_X$  ( $X \in \mathfrak{X}(M)$ ),
- the substitution operator  $i_X$  ( $X \in \mathfrak{X}(M)$ ),
- the exterior derivative  $d$ ,

related by H. Cartan's 'magic' formula

$$\mathcal{L}_X = i_X \circ d + d \circ i_X. \tag{1}$$

## 1.2 Canonical constructions and objects

(a) By the *vertical lift* of a smooth function  $f$  on  $M$  we mean the function

$$f^\nu := f \circ \tau \in C^\infty(TM);$$

the *complete lift* of  $f$  is the function  $f^c \in C^\infty(TM)$  given by

$$f^c(v) := v(f), v \in TM.$$

(b) A vector field  $\xi$  on  $TM$  is *vertical* if  $\xi \approx_\tau o$ . The vertical vector fields form a  $C^\infty(TM)$ -module  $\mathfrak{X}^\vee(TM)$ , which is also a subalgebra of the Lie algebra  $\mathfrak{X}(TM)$ . The *Liouville vector field* on  $TM$  is the unique vertical vector field  $C \in \mathfrak{X}^\vee(TM)$  such that

$$Cf^c = f^c \text{ for all } f \in C^\infty(M). \quad (2)$$

The *vertical lift* of a vector field  $X$  on  $M$  is the unique vertical vector field  $X^\vee \in \mathfrak{X}^\vee(TM)$  satisfying

$$X^\vee f^c = (Xf)^\vee \text{ for all } f \in C^\infty(M); \quad (3)$$

the *complete lift*  $X^c \in \mathfrak{X}(TM)$  of  $X$  is characterized by

$$X^c f^c = (Xf)^c, \quad f \in C^\infty(M) \quad (4)$$

(see [19], Ch. I.3). Then we have

$$X^c f^\vee = (Xf)^\vee, \quad f \in C^\infty(M). \quad (5)$$

Both  $X^\vee$  and  $X^c$  are projectable:  $X^\vee \approx_\tau o$ ,  $X^c \approx_\tau X$ . Lie brackets involving vertical and complete lifts satisfy the rules

$$[X^\vee, Y^\vee] = 0, \quad [X^c, Y^\vee] = [X, Y]^\vee, \quad [X^c, Y^c] = [X, Y]^c, \quad (6a-c)$$

$$[C, X^\vee] = -X^\vee, \quad [C, X^c] = 0. \quad (7a-b)$$

(c) Let

$$\begin{aligned} TM \times_M TM &:= \{(u, v) \in TM \times TM \mid \tau(u) = \tau(v)\}, \\ \overset{\circ}{TM} \times_M TM &:= \{(u, v) \in \overset{\circ}{TM} \times TM \mid \overset{\circ}{\tau}(u) = \tau(v)\}. \end{aligned}$$

If

$$\pi := \text{pr}_1 \upharpoonright TM \times_M TM, \quad \overset{\circ}{\pi} := \text{pr}_1 \upharpoonright \overset{\circ}{TM} \times_M TM,$$

then both  $\pi$  and  $\overset{\circ}{\pi}$  are vector bundles over  $TM$  and  $\overset{\circ}{TM}$ , resp., with fibres

$$\{u\} \times T_{\tau(u)}M \cong T_{\tau(u)}M; \quad u \in TM, \text{ resp. } u \in \overset{\circ}{TM}.$$

We denote by  $\text{Sec}(\pi)$  and  $\text{Sec}(\overset{\circ}{\pi})$  the  $C^\infty(TM)$ -, resp.  $C^\infty(\overset{\circ}{TM})$ -module of the sections of these bundles. A typical section in  $\text{Sec}(\pi)$  is of the form

$$\tilde{X}: v \in TM \longmapsto (v, \underline{X}(v)) \in TM \times_M TM,$$

where  $\underline{X}: TM \rightarrow TM$  is a smooth mapping such that  $\tau \circ \underline{X} = \tau$ .  $\underline{X}$  is called the *principal part* of  $\tilde{X}$ . We have a *canonical section* in  $\text{Sec}(\pi)$ , denoted by  $\delta$ , whose principal part is the identity mapping of  $TM$ . Every vector field  $X$  on  $M$  yields a

section  $\widehat{X}$  in  $\text{Sec}(\pi)$ , called a *basic section*, whose principal part is  $X \circ \tau$ . Locally, the  $C^\infty(TM)$ -module  $\text{Sec}(\pi)$  is generated by the basic sections.

We denote by  $\mathcal{T}_l^k(\pi)$  the  $C^\infty(TM)$ -module of the type  $(k, l)$  tensors over the module  $\text{Sec}(\pi)$ ; the meaning of  $\mathcal{T}_l^k(\overset{\circ}{\pi})$  is analogous.

(d) We have a canonical  $C^\infty(TM)$ -linear injection  $\mathbf{i}: \text{Sec}(\pi) \rightarrow \mathfrak{X}(TM)$  given on the basic sections by

$$\mathbf{i}(\widehat{X}) := X^\vee, \quad X \in \mathfrak{X}(M), \quad (8)$$

and a canonical  $C^\infty(TM)$ -linear surjection  $\mathbf{j}: \mathfrak{X}(TM) \rightarrow \text{Sec}(\pi)$  such that

$$\mathbf{j}(X^\vee) := 0, \quad \mathbf{j}(X^c) := \widehat{X}. \quad (9)$$

Then  $\text{Im}(\mathbf{i}) = \text{Ker}(\mathbf{j}) = \mathfrak{X}^\vee(TM)$ . The mapping  $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$  is said to be the *vertical endomorphism* of  $\mathfrak{X}(TM)$ . It follows immediately that

$$\text{Im}(\mathbf{J}) = \text{Ker}(\mathbf{J}) = \mathfrak{X}^\vee(TM), \mathbf{J}^2 = 0.$$

Due to their  $C^\infty(TM)$ -linearity,  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{J}$  have a natural pointwise interpretation.

### 1.3 Some vertical calculus

(a) We define the *vertical differential*  $\nabla^\vee F$  of a function  $F \in C^\infty(TM)$  as a 1-form in  $\mathcal{T}_1^0(\pi)$  given by

$$\nabla^\vee F(\widetilde{X}) := \nabla_{\widetilde{X}}^\vee F := (\mathbf{i}\widetilde{X})F, \widetilde{X} \in \text{Sec}(\pi). \quad (10)$$

The vertical differential  $\nabla^\vee \widetilde{Y}$  of a section  $\widetilde{Y} \in \text{Sec}(\pi)$  is the type  $(1, 1)$  tensor in  $\mathcal{T}_1^1(\pi)$  defined by

$$\begin{cases} \nabla^\vee \widetilde{Y}(\widetilde{X}) := \nabla_{\widetilde{X}}^\vee \widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \eta], \\ \eta \in \mathfrak{X}(TM), \quad \mathbf{j}(\eta) = \widetilde{Y}. \end{cases} \quad (11)$$

(It is easy to check that  $\nabla_{\widetilde{X}}^\vee \widetilde{Y}$  does not depend on the choice of  $\eta$  satisfying  $\mathbf{j}(\eta) = \widetilde{Y}$ .)

By the standard technique, to make sure that Leibniz's rule holds, the operators  $\nabla_{\widetilde{X}}^\vee$  may be extended to tensor derivations of the full tensor algebra of  $\text{Sec}(\pi)$ .

(b) Next we consider the graded algebra  $\Omega(TM)$  of the differential forms on  $TM$ , and we define an operator

$$d_{\mathbf{J}}: \Omega(TM) \longrightarrow \Omega(TM)$$

by the rules

$$d_{\mathbf{J}}F := dF \circ \mathbf{J}, \quad d_{\mathbf{J}}dF := -dd_{\mathbf{J}}F; \quad F \in C^\infty(TM). \quad (12)$$

Then  $d_{\mathbf{J}}$  is a graded derivation of degree 1 of  $\Omega(TM)$ , called the *vertical differentiation on  $TM$* . We have (and we shall need) the following important relation:

$$d_{\mathbf{J}} \circ \mathcal{L}_C - \mathcal{L}_C \circ d_{\mathbf{J}} = d_{\mathbf{J}}. \quad (13)$$

For details, we refer to the book [6]. We mention that  $\nabla^\vee$  and  $d_{\mathbf{J}}$ , at the level of functions, are related by

$$d_{\mathbf{J}}F = \nabla^\vee F \circ \mathbf{j}, \quad F \in C^\infty(TM).$$

(c) Let  $K$  be a type  $(1, 1)$  tensor on  $TM$ , interpreted as an endomorphism of the  $C^\infty(TM)$ -module  $\mathfrak{X}(TM)$ . It will be convenient to denote the Lie derivative  $-\mathcal{L}_\eta K$  ( $\eta \in \mathfrak{X}(TM)$ ) by  $[K, \eta]$ . Then, for any vector field  $\xi$  on  $TM$ ,

$$[K, \eta]\xi = [K\xi, \eta] - K[\xi, \eta].$$

We have, in particular,

$$[\mathbf{J}, C] = \mathbf{J}; \quad [\mathbf{J}, X^\vee] = 0, \quad [\mathbf{J}, X^c] = 0 \quad (X \in \mathfrak{X}(M)). \quad (14a-c)$$

In what follows, for simplicity, we shall denote also by  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{J}$  the restrictions of these mappings to  $\text{Sec}(\overset{\circ}{\pi})$  and  $\mathfrak{X}(\overset{\circ}{TM})$ .

#### 1.4 Ehresmann connections

(a) By an *Ehresmann connection* in  $\overset{\circ}{TM}$  we mean a  $C^\infty(\overset{\circ}{TM})$ -linear mapping

$$\mathcal{H}: \text{Sec}(\overset{\circ}{\pi}) \longrightarrow \mathfrak{X}(\overset{\circ}{TM})$$

such that

$$\mathbf{j} \circ \mathcal{H} = 1_{\text{Sec}(\overset{\circ}{\pi})}.$$

We emphasize (cf. 1.2(d)) that the  $C^\infty(\overset{\circ}{TM})$ -linearity of  $\mathcal{H}$  makes it possible to interpret an Ehresmann connection as a strong bundle map

$$\mathbf{H}: \overset{\circ}{TM} \times_M TM \longrightarrow T\overset{\circ}{TM}$$

as follows:

For each  $(u, v) \in \overset{\circ}{TM} \times_M TM$  there exists a section  $\tilde{X} \in \text{Sec}(\overset{\circ}{\pi})$  such that  $\tilde{X}(u) = (u, v)$ . Let  $\mathbf{H}_u(v) := \mathcal{H}(\tilde{X})(u)$ . Then  $\mathbf{H}_u$  is well-defined and

$$\mathcal{H}(\tilde{X})(u) = \mathbf{H}_u(\tilde{X}(u)) \text{ for all } \tilde{X} \in \text{Sec}(\overset{\circ}{\pi}).$$

Obviously, the mappings

$$\mathbf{H}_u: \{u\} \times T_{\overset{\circ}{\tau}(u)}M \longrightarrow T_u\overset{\circ}{TM}, \quad u \in \overset{\circ}{TM}$$

are linear. Now we obtain the desired mapping  $\mathbf{H}: \overset{\circ}{TM} \times_M TM \rightarrow T\overset{\circ}{TM}$  by setting

$$\mathbf{H} \upharpoonright \{u\} \times T_{\overset{\circ}{\tau}(u)}M := \mathbf{H}_u.$$

(b) Let  $\mathcal{H}: \text{Sec}(\overset{\circ}{\pi}) \rightarrow \mathfrak{X}(\overset{\circ}{TM})$  be an Ehresmann connection in  $\overset{\circ}{TM}$ . Then  $\mathfrak{X}^h(\overset{\circ}{TM}) := \text{Im}(\mathcal{H})$  is a submodule of  $\mathfrak{X}(\overset{\circ}{TM})$ , and we have the direct decomposition  $\mathfrak{X}(\overset{\circ}{TM}) = \mathfrak{X}^v(\overset{\circ}{TM}) \oplus \mathfrak{X}^h(\overset{\circ}{TM})$ . Vector fields on  $\overset{\circ}{TM}$  belonging to  $\mathfrak{X}^h(\overset{\circ}{TM})$  are called *horizontal*. Notice that they do not form, in general, a subalgebra of the Lie algebra  $\mathfrak{X}(\overset{\circ}{TM})$ . The mappings

$$\mathbf{h} := \mathcal{H} \circ \mathbf{j}, \quad \mathbf{v} := 1_{\mathfrak{X}(\overset{\circ}{TM})} - \mathbf{h},$$

$$\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}: \mathfrak{X}(\overset{\circ}{TM}) \longrightarrow \text{Sec}(\overset{\circ}{\pi})$$

are called the *horizontal projection*, the *vertical projection* and the *vertical mapping* associated to  $\mathcal{H}$ , respectively.  $\mathbf{h}$  and  $\mathbf{v}$  are indeed projection operators in  $\mathfrak{X}(\overset{\circ}{TM})$ , while the mapping  $\mathcal{V}$  has the properties

$$\mathcal{V} \circ \mathbf{i} = 1_{\text{Sec}(\overset{\circ}{\pi})}, \quad \text{Ker}(\mathcal{V}) = \text{Im}(\mathcal{H}).$$

The *horizontal lift* of a vector field  $X$  on  $M$  (with respect to  $\mathcal{H}$ ) is

$$X^h := \mathcal{H}(\widehat{X}) = \mathbf{h}(X^c).$$

( $\widehat{X}$  and  $X^c$  are regarded here as a section in  $\text{Sec}(\overset{\circ}{\pi})$  and a vector field on  $\overset{\circ}{TM}$ , resp.; for simplicity, we make no notational distinction.)

(c) An Ehresmann connection  $\mathcal{H}$  is said to be *homogeneous* if

$$[C, X^h] = 0 \text{ for all } X \in \mathfrak{X}(M).$$

Then  $\mathcal{H}$ , as a strong bundle map of  $\overset{\circ}{TM} \times_M TM$  to  $\overset{\circ}{TTM}$ , may be extended continuously to a mapping  $TM \times_M TM \rightarrow TTM$  such that

$$\mathcal{H}(0_p, v) = (o_*)_p(v) \text{ for all } p \in M, v \in T_pM.$$

Thus, in what follows, we shall always assume that a homogeneous Ehresmann connection is defined on the entire  $TM \times_M TM$  (or on  $\text{Sec}(\pi)$ ).

(d) If  $\mathcal{H}$  is an Ehresmann connection in  $\overset{\circ}{TM}$ , then the mapping

$$\nabla: \mathfrak{X}(\overset{\circ}{TM}) \times \text{Sec}(\overset{\circ}{\pi}) \longrightarrow \text{Sec}(\overset{\circ}{\pi}), \quad (\xi, \widetilde{Y}) \longmapsto \nabla_\xi \widetilde{Y}$$

given by

$$\nabla_{\mathbf{v}\xi} \widetilde{Y} := \nabla_{\mathbf{v}\xi}^v \widetilde{Y} \stackrel{(11)}{=} \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] \tag{15a}$$

$$\nabla_{\mathbf{h}\xi} \widetilde{Y} := \nabla_{\mathbf{j}\xi}^h \widetilde{Y} := \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}] \tag{15b}$$

is a covariant derivative operator in the vector bundle  $\overset{\circ}{\pi}$ , called the *Berwald derivative* induced by  $\mathcal{H}$ .

By the *tension* of  $\mathcal{H}$  we mean the  $\nabla^h$ -differential  $\mathbf{t} := \nabla^h \delta$  of the canonical section. Then, for any section  $\tilde{X} \in \text{Sec}(\overset{\circ}{\pi})$ ,

$$\mathbf{t}(\tilde{X}) := (\nabla^h \delta)(\tilde{X}) := \nabla_{\tilde{X}}^h \delta = \mathcal{V}[\mathcal{H}\tilde{X}, C]. \quad (16)$$

In particular,

$$\mathbf{it}(\hat{X}) = [X^h, C], \quad X \in \mathfrak{X}(M);$$

therefore  $\mathcal{H}$  is homogeneous if, and only if, its tension vanishes.

With the help of the induced Berwald derivative we define the *torsion*  $\mathbf{T}$  of an Ehresmann connection  $\mathcal{H}$  by

$$\mathbf{T}(\tilde{X}, \tilde{Y}) := \nabla_{\mathcal{H}\tilde{X}} \tilde{Y} - \nabla_{\mathcal{H}\tilde{Y}} \tilde{X} - \mathbf{j}[\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}]; \quad \tilde{X}, \tilde{Y} \in \text{Sec}(\overset{\circ}{\pi}).$$

Evaluating on basic sections, we obtain the more expressive formula

$$\mathbf{iT}(\hat{X}, \hat{Y}) = [X^h, Y^\vee] - [Y^h, X^\vee] - [X, Y]^\vee; \quad X, Y \in \mathfrak{X}(M).$$

## 2 Lie derivative along the tangent bundle projection

Let  $\xi$  be a projectable vector field on  $TM$  (1.1(c)). We define a Lie derivative operator  $\tilde{\mathcal{L}}_\xi$  on the tensor algebra of the  $C^\infty(TM)$ -module  $\text{Sec}(\pi)$  by the rules

$$\tilde{\mathcal{L}}_\xi \varphi := \xi \varphi, \quad \text{if } \varphi \in C^\infty(TM); \quad (17a)$$

$$\tilde{\mathcal{L}}_\xi \tilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}\tilde{Y}], \quad \text{if } \tilde{Y} \in \text{Sec}(\pi), \quad (17b)$$

and by extending it to the whole tensor algebra in such a way that  $\tilde{\mathcal{L}}_\xi$  satisfies the product rule of tensor derivations. Since  $\xi$  is a projectable and  $\mathbf{i}\tilde{Y}$  is a vertical vector field, it follows that the vector field  $[\xi, \mathbf{i}\tilde{Y}]$  is vertical, so  $\tilde{\mathcal{L}}_\xi \tilde{Y}$  is well-defined. If  $\mathbf{v} = \mathbf{i} \circ \mathcal{V}$  is the vertical projection associated to an Ehresmann connection  $\mathcal{H}$  in  $TM$ , then  $\mathbf{i}^{-1}[\xi, \mathbf{i}\tilde{Y}] = \mathcal{V}[\xi, \mathbf{i}\tilde{Y}]$ , so we get the useful formula

$$\tilde{\mathcal{L}}_\xi \tilde{Y} = \mathcal{V}[\xi, \mathbf{i}\tilde{Y}]. \quad (18)$$

Notice, however, that the Lie derivative operator  $\tilde{\mathcal{L}}_\xi$  does not depend on any Ehresmann connection in  $TM$ .

If, in particular,  $\xi := X^c$  or  $\xi := X^h$ , where  $X$  is a vector field on  $M$ , then (18) takes the form

$$\tilde{\mathcal{L}}_{X^c} \tilde{Y} = \mathcal{V}[X^c, \mathbf{i}\tilde{Y}], \quad (19)$$

resp.

$$\tilde{\mathcal{L}}_{X^h} \tilde{Y} = \mathcal{V}[X^h, \mathbf{i}\tilde{Y}] \stackrel{(15b)}{=} \nabla_{\hat{X}}^h \tilde{Y}. \quad (20)$$

Since  $[X^c, \mathbf{id}] = [X^c, C] \stackrel{(7b)}{=} 0$ , it follows that

$$\tilde{\mathcal{L}}_{X^c} \delta = 0. \quad (21)$$



The Lie derivative of a basic section with respect to a complete lift leads essentially to the ordinary Lie derivative. Namely, for any vector fields  $X, Y$  on  $M$  we have

$$\tilde{\mathcal{L}}_{X^c} \widehat{Y} \stackrel{(19)}{=} \mathcal{V}[X^c, Y^c] \stackrel{(6b)}{=} \mathcal{V}[X, Y]^c = \mathcal{V} \circ \mathbf{i}[\widehat{X}, \widehat{Y}] = \widehat{[X, Y]} = \widehat{\mathcal{L}_X Y}.$$

This relation indicates that our Lie derivative operator  $\tilde{\mathcal{L}}_{X^c}$  is a natural extension of the classical Lie derivative  $\mathcal{L}_X$  on  $M$ .

**Lemma 1.** *For any projectable vector fields  $\xi, \eta$  on  $TM$ ,*

$$[\tilde{\mathcal{L}}_\xi, \tilde{\mathcal{L}}_\eta] = \tilde{\mathcal{L}}_{[\xi, \eta]}. \quad (22)$$

*Proof.* Obviously, both sides of (22) act in the same way on smooth functions on  $TM$ . If  $\tilde{Y}$  is a section of  $\pi$ , then, applying (18) repeatedly,

$$\begin{aligned} [\tilde{\mathcal{L}}_\xi, \tilde{\mathcal{L}}_\eta] \tilde{Y} &= \tilde{\mathcal{L}}_\xi \mathcal{V}[\eta, \mathbf{i}\tilde{Y}] - \tilde{\mathcal{L}}_\eta \mathcal{V}[\xi, \mathbf{i}\tilde{Y}] = \mathcal{V}([\xi, \mathbf{i}\mathcal{V}[\eta, \mathbf{i}\tilde{Y}]] - [\eta, \mathbf{i}\mathcal{V}[\xi, \mathbf{i}\tilde{Y}]]) \\ &= \mathcal{V}([\xi, [\eta, \mathbf{i}\tilde{Y}]] + [\eta, [\mathbf{i}\tilde{Y}, \xi]]) = -\mathcal{V}[\mathbf{i}\tilde{Y}, [\xi, \eta]] = \mathcal{V}[[\xi, \eta], \mathbf{i}\tilde{Y}] \\ &= \tilde{\mathcal{L}}_{[\xi, \eta]} \tilde{Y}. \quad \square \end{aligned}$$

**Lemma 2.** *Let  $X \in \mathfrak{X}(M)$ ,  $\eta \in \mathfrak{X}(TM)$ . Then*

$$\tilde{\mathcal{L}}_{X^c} \mathbf{j}\eta = \mathbf{j}\mathcal{L}_{X^c} \eta. \quad (23)$$

*Proof.* Since

$$0 \stackrel{(14c)}{=} [\mathbf{J}, X^c] \eta = [\mathbf{J}\eta, X^c] - \mathbf{J}[\eta, X^c],$$

we find

$$\mathbf{i}\tilde{\mathcal{L}}_{X^c} \mathbf{j}\eta = [X^c, \mathbf{J}\eta] = \mathbf{J}[X^c, \eta] = \mathbf{i}(\mathbf{j}\mathcal{L}_{X^c} \eta),$$

which implies (23).  $\square$

We end this section with the definition of the Lie derivative  $\tilde{\mathcal{L}}_\xi D$  of a covariant derivative  $D: \mathfrak{X}(TM) \times \text{Sec}(\pi) \rightarrow \text{Sec}(\pi)$ : it is given by the rule

$$(\tilde{\mathcal{L}}_\xi D)(\eta, \tilde{Z}) := \tilde{\mathcal{L}}_\xi (D_\eta \tilde{Z}) - D_\eta (\tilde{\mathcal{L}}_\xi \tilde{Z}) - D_{[\xi, \eta]} \tilde{Z},$$

where  $\eta \in \mathfrak{X}(TM)$ ,  $\tilde{Z} \in \text{Sec}(\pi)$ .

Notice finally that the theory of Lie derivatives ‘along the tangent bundle projection’ sketched here works without any change also on the bundle  $\overset{\circ}{\pi}: \overset{\circ}{TM} \times_M TM \rightarrow \overset{\circ}{TM}$ .

### 3 Affine vector fields on a spray manifold

#### 3.1

By a *spray* for  $M$  we mean a  $C^1$  mapping  $S: TM \rightarrow TTM$ , smooth on  $\overset{\circ}{TM}$ , such that

$$\tau_{TM} \circ S = 1_{TM}; \quad (24)$$

$$\mathbf{J}S = C; \quad (25)$$

$$[C, S] = S. \quad (26)$$

Condition (25) is equivalent to the requirement  $\tau_* \circ S = 1_{TM}$ , so a spray for  $M$  is a section also of the secondary vector bundle  $\tau_* : TTM \rightarrow TM$ . In view of (26), a spray is a *homogeneous* vector field (of class  $C^1$ ) of degree 2. We say that a manifold endowed with a spray is a *spray manifold*.

### 3.2

If  $\mathcal{H}$  is a homogeneous Ehresmann connection in  $TM$ , then  $S := \mathcal{H} \circ \delta$  is a spray for  $M$ , called the *spray associated to  $\mathcal{H}$* . Indeed, for any vector  $w$  in  $TM$ ,  $S(w) = \mathcal{H}(w, w) \in T_w TM$ , therefore  $\tau_{TM}(S(w)) = w$ , so (24) is valid. Since

$$\mathbf{J} \circ S = \mathbf{i} \circ \mathbf{j} \circ \mathcal{H} \circ \delta = \mathbf{i} \circ \delta = C,$$

condition (25) also holds. To check (26), observe first that the vector field  $[C, S] - S$  is vertical, and hence  $\mathbf{h}[C, S] = \mathbf{h}S$ . However,  $\mathbf{h}S = \mathcal{H} \circ \mathbf{j} \circ \mathcal{H} \circ \delta = \mathcal{H} \circ \delta =: S$ , so we get  $\mathbf{h}[C, S] = S$ . On the other hand, by the homogeneity of  $\mathcal{H}$ ,

$$0 = -\mathbf{it}(\delta) = -\mathbf{v}[\mathcal{H} \circ \delta, C] = \mathbf{v}[C, S],$$

therefore  $\mathbf{h}[C, S] = [C, S]$  and  $[C, S] = S$ . Finally, the  $C^1$  differentiability of  $S$  can be shown using the ‘Observation’ in 3.11 (p. 1378) of [16].

Thus sprays exist in abundance for a manifold. Conversely, if  $S$  is a spray for  $M$ , then there exists a unique torsion-free homogeneous Ehresmann connection  $\mathcal{H}$  in  $TM$  such that the horizontal lifts with respect to  $\mathcal{H}$  are given by

$$X^{\mathbf{h}} := \mathcal{H}(\widehat{X}) = \frac{1}{2}(X^{\mathbf{c}} + [X^{\mathbf{v}}, S]), \quad X \in \mathfrak{X}(M). \tag{27}$$

For a proof of this fundamental fact we refer to [16], 3.3, or to the original source [5]. The Ehresmann connection specified by (27) is said to be the *Ehresmann connection induced by the spray  $S$* .

### 3.3

Let  $(M, S)$  be a spray manifold. We say that a vector field  $X$  on  $M$  is a *projective vector field* for  $(M, S)$  (or for the spray  $S$ ) if there is a continuous function  $\varphi$  on  $TM$ , smooth on  $\overset{\circ}{TM}$ , such that

$$[X^{\mathbf{c}}, S] = \varphi C. \tag{28}$$

If, in particular,  $\varphi$  is the zero function, then we say that  $X$  is an *affine vector field* for  $(M, S)$ , or a *Lie symmetry* of  $S$ .

**Theorem 1.** *Suppose  $(M, S)$  is a spray manifold. Let  $\mathcal{H}$  be the Ehresmann connection induced by  $S$ , and let  $\nabla$  be the Berwald derivative arising from  $\mathcal{H}$ . For a vector field  $X$  on  $M$ , the following conditions are equivalent:*

- (i)  $X$  is a Lie symmetry of  $S$ ;
- (ii)  $[\mathbf{h}, X^{\mathbf{c}}] = 0$ ;
- (iii)  $[\mathbf{v}, X^{\mathbf{c}}] = 0$ ;

- (iv)  $\tilde{\mathcal{L}}_{X^c}\nabla = 0$ ;
- (v)  $[X^c, Y^h] = [X, Y]^h$ , for any vector field  $Y$  on  $M$ ;
- (vi)  $[\tilde{\mathcal{L}}_{X^c}, \tilde{\mathcal{L}}_{Y^h}] = \tilde{\mathcal{L}}_{[X, Y]^h}$ ,  $Y \in \mathfrak{X}(M)$ ;
- (vii)  $\tilde{\mathcal{L}}_{X^c} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{L}_{X^c}$ .

*Proof.* The equivalence of conditions (i), (ii) and (iv) has already been proved in [12].

(ii)  $\iff$  (iii) This is evident, since  $\mathbf{v} = \mathbf{1} - \mathbf{h}$  ( $\mathbf{1} := 1_{\mathfrak{X}(TM)}$ ) and  $[\mathbf{1}, \xi] = 0$  for all  $\xi \in \mathfrak{X}(TM)$ .

(ii)  $\iff$  (v) For any vector field  $Y$  on  $M$ ,

$$[\mathbf{h}, X^c]Y^c = [\mathbf{h}Y^c, X^c] - \mathbf{h}[Y^c, X^c] = [Y^h, X^c] - \mathbf{h}[Y, X]^c = [Y^h, X^c] - [Y, X]^h,$$

so the vanishing of  $[\mathbf{h}, X^c]$  implies that  $[X^c, Y^h] = [X, Y]^h$ . The converse is also true, since  $[\mathbf{h}, X^c]$  annihilates the module of vector fields: for any vector field  $\xi$  on  $TM$  we have

$$[\mathbf{h}, X^c]\mathbf{J}\xi = [\mathbf{h} \circ \mathbf{J}(\xi), X^c] - \mathbf{h}[\mathbf{J}\xi, X^c] = 0.$$

(v)  $\iff$  (vi) This is an immediate consequence of the identity

$$[\tilde{\mathcal{L}}_{X^c}, \tilde{\mathcal{L}}_{Y^h}] = \tilde{\mathcal{L}}_{[X^c, Y^h]}$$

(see Lemma 1).

(iii)  $\iff$  (vii) For any vector field  $\xi$  on  $TM$ ,

$$\mathbf{i}\tilde{\mathcal{L}}_{X^c}(\mathcal{V}\xi) = [X^c, \mathbf{v}\xi], \quad \mathbf{i}\mathcal{V}(\mathcal{L}_{X^c}\xi) = \mathbf{v}[X^c, \xi],$$

hence  $\tilde{\mathcal{L}}_{X^c}(\mathcal{V}\xi) = \mathcal{V}(\mathcal{L}_{X^c}\xi)$  if, and only if,

$$0 = [\mathbf{v}\xi, X^c] - \mathbf{v}[\xi, X^c] = [\mathbf{v}, X^c]\xi. \quad \square$$

## 4 Conformal vector fields on a Finsler manifold

### 4.1

Let  $(M, F)$  be a *Finsler manifold*. We recall that the *Finsler function*  $F: TM \rightarrow \mathbf{R}$  here is assumed to be *smooth* on  $\overset{\circ}{TM}$ , *positive* ( $F(v) > 0$ , if  $v \in \overset{\circ}{TM}$ ), *positive-homogeneous of degree 1* ( $F(\lambda v) = \lambda F(v)$  for all  $v \in TM$  and positive real number  $\lambda$ ), and it is also required that the *metric tensor*

$$g := \frac{1}{2}\nabla^v\nabla^v F^2$$

is *fibrewise non-degenerate*. The function  $E := \frac{1}{2}F^2$  is the *energy function* of  $(M, F)$ . The homogeneity of  $F$  implies that over  $\overset{\circ}{TM}$  we have

$$CF = F, \quad CE = 2E.$$

The *Hilbert 1-form* of  $(M, F)$  is

$$\begin{aligned}\tilde{\theta} &:= \nabla^\vee E = F\nabla^\vee F - \text{in the pull-back formalism,} \\ \theta &:= d_{\mathbf{J}}E - \text{in the } \tau_{TM} \text{ formalism.}\end{aligned}$$

It is easy to check that

$$\tilde{\theta}(\tilde{X}) = g(\tilde{X}, \delta) \text{ for each } \tilde{X} \in \text{Sec}(\overset{\circ}{\pi}).$$

$\tilde{\theta}$  and  $\theta$  are related by

$$\theta = \tilde{\theta} \circ \mathbf{j}. \quad (29)$$

The 2-form

$$\omega := d\theta = dd_{\mathbf{J}}E$$

on  $\overset{\circ}{TM}$  is said to be the *fundamental 2-form* of  $(M, F)$ . Its relation to the metric tensor is given by

$$\omega(\mathbf{J}\xi, \eta) = g(\mathbf{j}\xi, \mathbf{j}\eta); \quad \xi, \eta \in \mathfrak{X}(\overset{\circ}{TM}). \quad (30)$$

The non-degeneracy of  $g$  implies the non-degeneracy of  $\omega$  – and vice versa.

**Lemma 3.** *With the notations introduced above, let  $(M, F)$  be a Finsler manifold, and let  $X$  be a vector field on  $M$ . Then*

$$(\tilde{\mathcal{L}}_{X^c}\tilde{\theta}) \circ \mathbf{j} = \mathcal{L}_{X^c}\theta; \quad (31)$$

$$(\tilde{\mathcal{L}}_{X^c}g)(\mathbf{j}\xi, \mathbf{j}\eta) = (\mathcal{L}_{X^c}\omega)(\mathbf{J}\xi, \eta); \quad \xi, \eta \in \mathfrak{X}(\overset{\circ}{TM}). \quad (32)$$

*Proof.* We check only the less trivial second relation:

$$\begin{aligned}(\mathcal{L}_{X^c}\omega)(\mathbf{J}\xi, \eta) &= X^c\omega(\mathbf{J}\xi, \eta) - \omega(\mathcal{L}_{X^c}\mathbf{J}\xi, \eta) - \omega(\mathbf{J}\xi, \mathcal{L}_{X^c}\eta) \\ &\stackrel{(23), (30)}{=} X^c g(\mathbf{j}\xi, \mathbf{j}\eta) - \omega(\mathcal{L}_{X^c}\mathbf{J}\xi, \eta) - g(\mathbf{j}\xi, \tilde{\mathcal{L}}_{X^c}\mathbf{j}\eta).\end{aligned}$$

Since  $\mathcal{L}_{X^c}\mathbf{J}\xi = [X^c, \mathbf{J}\xi] = -[\mathbf{J}, X^c]\xi + \mathbf{J}[X^c, \xi] = \mathbf{J}\mathcal{L}_{X^c}\xi$ , the second term at the right-hand side of the above relation takes the form

$$\omega(\mathcal{L}_{X^c}\mathbf{J}\xi, \eta) = \omega(\mathbf{J}\mathcal{L}_{X^c}\xi, \eta) \stackrel{(30)}{=} g(\mathbf{j}\mathcal{L}_{X^c}\xi, \mathbf{j}\eta) \stackrel{(23)}{=} g(\tilde{\mathcal{L}}_{X^c}\mathbf{j}\xi, \mathbf{j}\eta).$$

So we obtain

$$(\mathcal{L}_{X^c}\omega)(\mathbf{J}\xi, \eta) = X^c g(\mathbf{j}\xi, \mathbf{j}\eta) - g(\tilde{\mathcal{L}}_{X^c}\mathbf{j}\xi, \mathbf{j}\eta) - g(\mathbf{j}\xi, \tilde{\mathcal{L}}_{X^c}\mathbf{j}\eta) = (\tilde{\mathcal{L}}_{X^c}g)(\mathbf{j}\xi, \mathbf{j}\eta). \quad \square$$

## 4.2

We continue to assume that  $(M, F)$  is a Finsler manifold. The  $2n$ -form

$$\sigma := \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \omega^n,$$

where  $\omega^n = \omega \wedge \cdots \wedge \omega$  ( $n$  factors) is a volume form on  $\overset{\circ}{TM}$ , called the *Dazord volume form* of  $(M, F)$ . By the *divergence* of a vector field  $\xi$  on  $\overset{\circ}{TM}$  (with respect to  $\sigma$ ) we mean the unique function  $\text{div } \xi \in C^\infty(\overset{\circ}{TM})$  such that

$$\mathcal{L}_\xi \sigma = (\text{div } \xi)\sigma.$$

**Lemma 4.** *If  $(M, F)$  is a Finsler manifold, then the divergence of the Liouville vector field  $C$  on  $\overset{\circ}{T}M$  with respect to the Dazord volume form is  $n = \dim M$ .*

*Proof.*  $\mathcal{L}_C \omega = \mathcal{L}_C dd_{\mathbf{J}}E = d\mathcal{L}_C d_{\mathbf{J}}E \stackrel{(13)}{=} dd_{\mathbf{J}}\mathcal{L}_C E - dd_{\mathbf{J}}E = 2dd_{\mathbf{J}}E - dd_{\mathbf{J}}E = \omega$ . From this it follows by induction that  $\mathcal{L}_C \omega^n = n\omega^n$ , whence our claim.  $\square$

### 4.3

If  $(M, F)$  is a Finsler manifold, then there exists a unique spray  $S$  for  $M$  such that

$$i_S dd_{\mathbf{J}}E = -dE \quad \text{over } \overset{\circ}{T}M, \text{ and } S \upharpoonright o(M) = 0. \quad (33)$$

We say that  $S$  is the *canonical spray* of  $(M, F)$ ; the Ehresmann connection induced by  $S$  according to (27) is said to be the *canonical connection* of  $(M, F)$ . It may be characterized as *the unique torsion-free homogeneous Ehresmann connection  $\mathcal{H}$  for  $M$  which is compatible with the Finsler function* in the sense that  $dF \circ \mathcal{H} = 0$ , or, equivalently,

$$X^h F = 0 \quad \text{for all } X \in \mathfrak{X}(M).$$

With the help of the canonical connection, we define the *Sasaki extension*  $G$  of the metric tensor  $g$  of  $(M, F)$  by the rule

$$G(\xi, \eta) := g(\mathbf{j}\xi, \mathbf{j}\eta) + g(\mathcal{V}\xi, \mathcal{V}\eta); \quad \xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M), \quad (34)$$

where  $\mathcal{V}$  is the vertical mapping associated to  $\mathcal{H}$ . Then  $G$  is a Riemannian metric tensor on  $\overset{\circ}{T}M$ .

For subsequent applications, we collect here some further technical results.

**Lemma 5.** *For any section  $\tilde{X}$  in  $\text{Sec}(\pi)$ , we have*

$$\nabla_{\tilde{X}}^{\mathcal{V}} \delta = \tilde{X}. \quad (35)$$

*Proof.* Let  $\mathcal{H}$  be a homogeneous Ehresmann connection for  $M$  and let  $S := \mathcal{H} \circ \delta$  be the spray associated to  $\mathcal{H}$  (3.2). Then, applying the so-called Grifone identity ([8], Prop. I.7), we find that

$$\nabla_{\tilde{X}}^{\mathcal{V}} \delta := \mathbf{j}[\mathbf{i}\tilde{X}, \mathcal{H}\delta] = \mathbf{j}[\mathbf{i}\tilde{X}, S] = \tilde{X}. \quad \square$$

**Lemma 6.** *The energy function of a Finsler manifold can be obtained from the metric tensor by*

$$g(\delta, \delta) = 2E; \quad (36)$$

*from the fundamental 2-form by*

$$\omega(C, S) = 2E, \quad (37)$$

*where  $S$  is a spray for the base manifold.*

*Proof.*

$$\begin{aligned}
g(\delta, \delta) &= \nabla^\vee(\nabla^\vee E)(\delta, \delta) = \nabla_\delta^\vee(\nabla^\vee E)(\delta) = \nabla_\delta^\vee(\nabla^\vee E(\delta)) - \nabla^\vee E(\nabla_\delta^\vee \delta) \\
&\stackrel{(35)}{=} \nabla_\delta^\vee(CE) - \nabla^\vee E(\delta) = C(CE) - CE = 4E - 2E = 2E; \\
\omega(C, S) &= dd_{\mathbf{J}}E(C, S) = Cd_{\mathbf{J}}E(S) - S(d_{\mathbf{J}}E(C)) - d_{\mathbf{J}}E([C, S]) \\
&= C(CE) - d_{\mathbf{J}}E(S) = 4E - 2E = 2E. \quad \square
\end{aligned}$$

**Lemma 7.** *The divergence of the canonical spray of a Finsler manifold vanishes.*

*Proof.*  $\mathcal{L}_S \omega = \mathcal{L}_S dd_{\mathbf{J}}E \stackrel{(1)}{=} i_S ddd_{\mathbf{J}}E + di_S dd_{\mathbf{J}}E \stackrel{(33)}{=} -ddE = 0$ , which implies our claim.  $\square$

#### 4.4

Let  $(M, F)$  be a Finsler manifold. We say that a vector field  $X$  on  $M$  is a *projective*, resp. an *affine vector field* of  $(M, F)$ , if it is a projective vector field, resp. a Lie symmetry for the canonical spray of  $(M, F)$ . A vector field  $X$  on  $M$  is said to be a *conformal vector field*, if the Lie derivative of the metric tensor of  $(M, F)$  with respect to the complete lift of  $X$  satisfies the relation

$$\tilde{\mathcal{L}}_{X^c} g = \varphi g \quad (38)$$

for a continuous function  $\varphi: TM \rightarrow \mathbf{R}$ , of class  $C^1$  on  $\overset{\circ}{TM}$ , called the *conformal factor* of  $X$ . Particular cases of conformal vector fields are *homothetic vector fields* for which the conformal factor is a constant function and *isometric vector fields*, also called *Killing vector fields*, for which the conformal factor is the zero function on  $TM$ .

**Lemma 8.** *If  $X$  is a conformal vector field on a Finsler manifold  $(M, F)$  with conformal factor  $\varphi$ , then  $X^c E = \varphi E$ .*

*Proof.*

$$\begin{aligned}
2X^c E &\stackrel{(36)}{=} X^c(g(\delta, \delta)) = (\tilde{\mathcal{L}}_{X^c} g)(\delta, \delta) + 2g(\tilde{\mathcal{L}}_{X^c} \delta, \delta) \stackrel{(21)}{=} (\tilde{\mathcal{L}}_{X^c} g)(\delta, \delta) \\
&\stackrel{(38)}{=} \varphi g(\delta, \delta) \stackrel{(36)}{=} 2\varphi E. \quad \square
\end{aligned}$$

**Lemma 9.** *If  $X$  is a conformal vector field on a Finsler manifold  $(M, F)$ , then the conformal factor of  $X$  is the vertical lift of a smooth function on  $M$ .*

*Proof.* In view of the previous lemma,  $X^c E = \varphi E$ , where  $\varphi \in C^0(TM) \cap C^1(\overset{\circ}{TM})$ . Acting on both sides of this relation by the Liouville vector field, we get on the one hand

$$C(X^c E) = C(\varphi E) = (C\varphi)E + 2\varphi E,$$

on the other hand

$$C(X^c E) = [C, X^c]E + X^c(CE) = 2X^c E = 2\varphi E,$$

so it follows that  $(C\varphi)E = 0$ , and hence  $C\varphi = 0$ . This means that  $\varphi$  is positive-homogeneous of degree 0, which implies (see, e.g., [16], 2.6, Lemma 2) that  $\varphi$  is of the form  $\varphi = f \circ \tau$ ,  $f \in C^\infty(M)$ .  $\square$

**Theorem 2.** *Let  $(M, F)$  be a Finsler manifold. For a vector field  $X$  on  $M$ , the following conditions are equivalent:*

- (i)  $X$  is a conformal vector field with conformal factor  $\varphi$ ;
- (ii)  $X^c E = \varphi E$ ;
- (iii)  $\mathcal{L}_{X^c} \theta = \varphi \theta$ ;
- (iv)  $\tilde{\mathcal{L}}_{X^c} \tilde{\theta} = \varphi \tilde{\theta}$ ;
- (v)  $\mathcal{L}_{X^c} \omega = \varphi \omega + d\varphi \wedge d_{\mathbf{J}} E$ ;  $\varphi = f \circ \tau$ ,  $f \in C^\infty(M)$ .

In conditions (ii)–(iv),  $\varphi \in C^0(TM) \cap C^1(\overset{\circ}{T}M)$ .

*Proof.* The arrangement of our reasoning follows the scheme

$$\begin{array}{ccccc} (i) & \implies & (ii) & & \\ \uparrow & & \downarrow & & \\ (v) & \longleftarrow & (iii) & \iff & (iv). \end{array}$$

(i)  $\implies$  (ii) This is just a restatement of Lemma 8.

(ii)  $\implies$  (iii) Let  $Y$  be a vector field on  $M$ . We have on the one hand

$$\begin{aligned} (\mathcal{L}_{X^c} \theta)(Y^\vee) &= X^c(\theta(Y^\vee)) - \theta([X^c, Y^\vee]) \stackrel{(6b)}{=} X^c(\theta(Y^\vee)) - \theta([X, Y]^\vee) = 0 \\ &= (\varphi \theta)(Y^\vee), \end{aligned}$$

since the vertical vector fields are annihilated by the 1-form  $\theta = d_{\mathbf{J}} E$ . On the other hand,

$$\begin{aligned} (\mathcal{L}_{X^c} \theta)(Y^c) &= X^c(d_{\mathbf{J}} E(Y^c)) - d_{\mathbf{J}} E([X^c, Y^c]) \stackrel{(6c)}{=} X^c(Y^\vee E) - [X, Y]^\vee E \\ &\stackrel{(6b)}{=} X^c(Y^\vee E) - [X^c, Y^\vee] E = Y^\vee(X^c E) \stackrel{(ii)}{=} Y^\vee(\varphi E) \stackrel{(*)}{=} \varphi(Y^\vee E) \\ &= (\varphi d_{\mathbf{J}} E)(Y^c) = (\varphi \theta)(Y^c). \end{aligned}$$

At step (\*) we used the fact that our condition  $X^c E = \varphi E$  implies, as it turns out from the proof of Lemma 9, that  $\varphi$  is a vertical lift. Thus  $\mathcal{L}_{X^c} \theta = \varphi \theta$ , as we claimed.

(iii)  $\implies$  (v)

$$\mathcal{L}_{X^c} \omega = \mathcal{L}_{X^c} d\theta = d\mathcal{L}_{X^c} \theta \stackrel{(iii)}{=} d(\varphi \theta) = d\varphi \wedge \theta + \varphi d\theta = \varphi \omega + d\varphi \wedge d_{\mathbf{J}} E.$$

To check that the function  $\varphi$  here is a vertical lift, we evaluate both sides of (iii) at a spray  $S$ . Then  $\theta(S) = d_{\mathbf{J}} E(S) = dE(C) = 2E$ , while

$$(\mathcal{L}_{X^c} \theta)(S) = X^c(d_{\mathbf{J}} E(S)) - d_{\mathbf{J}} E([X^c, S]) = 2X^c E - \mathbf{J}[X^c, S]E = 2X^c E,$$

since  $[X^c, S]$  is vertical (see, e.g., [16], p. 1350). Thus we obtain that  $X^c E = \varphi E$ , which implies, as we have just remarked, that  $\varphi = f \circ \tau$ ,  $f \in C^\infty(M)$ .

(v)  $\implies$  (i) For any vector fields  $\xi, \eta$  on  $\mathring{T}M$ ,

$$\begin{aligned} (\tilde{\mathcal{L}}_{X^c}g)(\mathbf{j}\xi, \mathbf{j}\eta) &\stackrel{(32)}{=} (\mathcal{L}_{X^c}\omega)(\mathbf{J}\xi, \eta) \stackrel{(v)}{=} (\varphi\omega + d\varphi \wedge d_{\mathbf{J}}E)(\mathbf{J}\xi, \eta) \\ &= \varphi\omega(\mathbf{J}\xi, \eta) + d_{\mathbf{J}}\varphi(\xi)d_{\mathbf{J}}E(\eta) - d\varphi(\eta)d_{\mathbf{J}}E(\mathbf{J}\xi) \\ &\stackrel{d_{\mathbf{J}}\varphi=0}{=} \varphi\omega(\mathbf{J}\xi, \eta) \stackrel{(30)}{=} (\varphi g)(\mathbf{j}\xi, \mathbf{j}\eta), \end{aligned}$$

hence  $\tilde{\mathcal{L}}_{X^c}g = \varphi g$ .

(iii)  $\iff$  (iv) If  $\mathcal{L}_{X^c}\theta = \varphi\theta$ , then for any vector field  $\xi$  on  $\mathring{T}M$ ,

$$(\tilde{\mathcal{L}}_{X^c}\tilde{\theta})(\mathbf{j}\xi) \stackrel{(31)}{=} (\mathcal{L}_{X^c}\theta)(\xi) \stackrel{(iii)}{=} (\varphi\theta)(\xi) \stackrel{(29)}{=} \varphi\tilde{\theta}(\mathbf{j}\xi),$$

whence  $\tilde{\mathcal{L}}_{X^c}\tilde{\theta} = \varphi\tilde{\theta}$ . The converse may be checked in the same way.  $\square$

We note that relation (v), as a characterization of conformal vector fields on a Finsler manifold, was announced first by J. Grifone [10].

**Corollary 1.** *Let  $(M, F)$  be a Finsler manifold. For a vector field  $X$  on  $M$ , the following conditions are equivalent:*

- (i)  $X$  is a homothetic vector field, i.e.,  $\tilde{\mathcal{L}}_{X^c}g = \alpha g$ , where  $\alpha$  is a real number;
- (ii) the energy function is an eigenfunction of  $X^c$  with eigenvalue  $\alpha$ , i.e.,  $X^c E = \alpha E$ ;
- (iii)  $\mathcal{L}_{X^c}\theta = \alpha\theta$ ;
- (iv)  $\tilde{\mathcal{L}}_{X^c}\tilde{\theta} = \alpha\tilde{\theta}$ ;
- (v)  $\mathcal{L}_{X^c}\omega = \alpha\omega$ .

In conditions (iii)–(v)  $\alpha$  is a real number. With the choice  $\alpha := 0$  we obtain criteria that a vector field  $X$  on  $M$  be a Killing vector field of  $(M, F)$ .

**Theorem 3.** *Let  $(M, F)$  be a Finsler manifold. If a vector field  $X$  on  $M$  is both affine and conformal, then  $X^c$  is a conformal vector field on the Riemannian manifold  $(\mathring{T}M, G)$ , i.e.,  $\mathcal{L}_{X^c}G = \varphi G$ , where  $\varphi \in C^0(TM) \cap C^1(\mathring{T}M)$  and  $G$  is the Sasaki extension of the metric tensor of  $(M, F)$ .*

*Conversely, if  $X^c$  is a conformal vector field of  $(\mathring{T}M, G)$ , then  $X$  is a conformal vector field on the Finsler manifold  $(M, F)$ .*

*Proof.* Suppose first that  $X$  is both an affine and a conformal vector field on  $(M, F)$ . Applying (34), (23) and Proposition 1/(vii), for any vector fields  $\xi, \eta$  on  $\mathring{T}M$  we



have

$$\begin{aligned}
 (\mathcal{L}_{X^c}G)(\xi, \eta) &= \mathcal{L}_{X^c}(G(\xi, \eta)) - G(\mathcal{L}_{X^c}\xi, \eta) - G(\xi, \mathcal{L}_{X^c}\eta) \\
 &= \mathcal{L}_{X^c}(g(\mathbf{j}\xi, \mathbf{j}\eta)) + \mathcal{L}_{X^c}(g(\mathcal{V}\xi, \mathcal{V}\eta)) - g(\mathbf{j}\mathcal{L}_{X^c}\xi, \mathbf{j}\eta) \\
 &\quad - g(\mathcal{V}\mathcal{L}_{X^c}\xi, \mathcal{V}\eta) - g(\mathbf{j}\xi, \mathbf{j}\mathcal{L}_{X^c}\eta) - g(\mathcal{V}\xi, \mathcal{V}\mathcal{L}_{X^c}\eta) \\
 &= \tilde{\mathcal{L}}_{X^c}(g(\mathbf{j}\xi, \mathbf{j}\eta)) + \tilde{\mathcal{L}}_{X^c}(g(\mathcal{V}\xi, \mathcal{V}\eta)) - g(\tilde{\mathcal{L}}_{X^c}(\mathbf{j}\xi), \mathbf{j}\eta) \\
 &\quad - g(\tilde{\mathcal{L}}_{X^c}(\mathcal{V}\xi), \mathcal{V}\eta) - g(\mathbf{j}\xi, \tilde{\mathcal{L}}_{X^c}(\mathbf{j}\eta)) - g(\mathcal{V}\xi, \tilde{\mathcal{L}}_{X^c}(\mathcal{V}\eta)) \\
 &= (\tilde{\mathcal{L}}_{X^c}g)(\mathbf{j}\xi, \mathbf{j}\eta) + (\tilde{\mathcal{L}}_{X^c}g)(\mathcal{V}\xi, \mathcal{V}\eta) \\
 &= \varphi g(\mathbf{j}\xi, \mathbf{j}\eta) + \varphi g(\mathcal{V}\xi, \mathcal{V}\eta) = \varphi G(\xi, \eta).
 \end{aligned}$$

This proves that  $X^c$  is a conformal vector field on  $(\overset{\circ}{T}M, G)$ . Conversely, under this condition we find that

$$\begin{aligned}
 2\varphi E &= \varphi g(\delta, \delta) = \varphi g(\mathcal{V}C, \mathcal{V}C) = \varphi G(C, C) = (\mathcal{L}_{X^c}G)(C, C) \\
 &= X^c(G(C, C)) - G([X^c, C], C) - G(C, [X^c, C]) = X^c(G(C, C)) \\
 &= X^c g(\delta, \delta) = 2X^c E,
 \end{aligned}$$

so, by Proposition 2,  $X$  is a conformal vector field on  $(M, F)$ . □

**Theorem 4.** *Any homothetic vector field on a Finsler manifold is an affine vector field.*

*Proof.* Let  $(M, F)$  be a Finsler manifold, and let  $S$  be the canonical spray for  $(M, F)$ . Suppose that  $X$  is a homothetic vector field of  $(M, F)$ . Then, by Corollary 1, there is a real number  $\alpha$  such that  $X^c E = \alpha E$ , or, equivalently,  $\mathcal{L}_{X^c}\omega = \alpha\omega$ , so we have

$$\begin{aligned}
 \mathcal{L}_{X^c}dE &= d(X^c E) = \alpha dE \stackrel{(33)}{=} -\alpha i_S \omega = -i_S(\alpha\omega) = -i_S(\mathcal{L}_{X^c}\omega) \\
 &= -\mathcal{L}_{X^c}i_S \omega + i_{[X^c, S]}\omega = \mathcal{L}_{X^c}dE + i_{[X^c, S]}\omega.
 \end{aligned}$$

Thus  $i_{[X^c, S]}\omega = 0$ , and hence – by the non-degeneracy of  $\omega$  –  $[X^c, S] = 0$ . This means that  $X$  is a Lie symmetry of the canonical spray of  $(M, F)$ . □

**Lemma 10.** *If  $X$  is a conformal vector field on an  $n$ -dimensional Finsler manifold, then (with respect to the Dazord volume form)  $\operatorname{div} X^c = n\varphi$ , where  $\varphi$  is the conformal factor of  $X$ .*

*Proof.* Choose a local frame  $(X_i)_{i=1}^n$  for  $TM$  over an open subset  $U$  of  $M$ . Then the family  $(X_i^v, X_i^c)_{i=1}^n$  is a local frame for  $TTM$  over  $\tau^{-1}(U)$ . It may be shown by a little lengthy inductive argument that

$$(\mathcal{L}_{X^c}\omega)(X_1^v, X_1^c, \dots, X_n^v, X_n^c) = n\varphi\omega(X_1^v, X_1^c, \dots, X_n^v, X_n^c),$$

which implies our claim. □

**Theorem 5.** *If a vector field is both a projective and a conformal vector field on a Finsler manifold, then it is a homothetic vector field.*

*Proof.* Let  $(M, F)$  be an  $n$ -dimensional Finsler manifold. Suppose that a vector field  $X$  on  $M$  is both projective and conformal. Then, on the one hand,

$$[X^c, S] = \psi C, \quad \psi \in C^0(TM) \cap C^1(TM),$$

where  $S$  is the canonical spray of  $(M, F)$ . On the other hand, by Proposition 2,

$$X^c E = f^\nu E, \quad f \in C^\infty(M).$$

Thus we get

$$\begin{aligned} 2\psi E &= \psi(CE) = [X^c, S]E = X^c(SE) - S(X^c E) = -S(f^\nu E) \\ &= -(Sf^\nu)E - f^\nu(SE) = -f^c E, \end{aligned}$$

taking into account that  $S$  is horizontal with respect to the canonical connection of  $(M, F)$  and hence  $SE = \frac{1}{2}SF^2 = F(SF) = 0$  (see 4.3), applying furthermore the relation  $Sf^\nu = f^c(f \in C^\infty(M))$ , whose verification is routine. It follows that

$$\psi = -\frac{1}{2}f^c.$$

Now we determine the divergence (with respect to the Dazord volume form) of both sides of the relation  $[X^c, S] = -\frac{1}{2}f^c C$ . Applying the well-known rules for calculation (see, e.g., [1], §6.5 or [11], XV, §1) we find that

$$\operatorname{div}[X^c, S] = X^c \operatorname{div} S - S \operatorname{div} X^c \stackrel{\text{Lemmas 7, 10}}{=} -S(nf^\nu) = -nf^c$$

and

$$\operatorname{div}\left(-\frac{1}{2}f^c C\right) = -\frac{1}{2}(Cf^c + f^c \operatorname{div} C) \stackrel{\text{Lemma 4}}{=} -\frac{1}{2}(n+1)f^c.$$

So  $(n-1)f^c = 0$ , where  $n \geq 2$  (1.1 (a)), whence  $f^c = 0$ . This implies by the connectedness of  $M$  that  $f$  is a constant function, and therefore the conformal factor of  $X$  is constant. □

We note that this result is an infinitesimal version of Theorem 2 in [17].

**Theorem 6.** *Let  $(M, F)$  be a Finsler manifold. Suppose that a vector field  $X$  on  $M$  preserves the Dazord volume form of  $(M, F)$ , i.e.,  $\mathcal{L}_{X^c}\sigma = 0$ . If, in addition,*

- (i)  *$X$  is a projective vector field, then  $X$  is affine;*
- (ii)  *$X$  is a conformal vector field, then  $X$  is isometric.*

*Proof.* First we note that our condition  $\mathcal{L}_{X^c}\sigma = 0$  implies that  $\operatorname{div} X^c = 0$ .

(i) Suppose that  $X$  is also a projective vector field, i.e.,

$$[X^c, S] = \bar{\psi} C, \quad \bar{\psi} \in C^0(TM) \cap C^1(\overset{\circ}{T}M).$$

Observe that over  $\overset{\circ}{T}M$  the function  $\psi$  satisfies the relation  $C\psi = \psi$ . Indeed, by the Jacobi identity

$$0 = [C, [X^c, S]] + [X^c, [S, C]] + [S, [C, X^c]] = [C, [X^c, S]] - [X^c, S],$$

hence

$$[X^c, S] = [C, [X^c, S]] = [C, \psi C] = (C\psi)C,$$

therefore  $(C\psi)C = \psi C$ , and so  $C\psi = \psi$ .

Now, as in the previous proof, we calculate the divergence of both sides of the relation  $[X^c, S] = \psi C$ . Since  $\operatorname{div} X^c = \operatorname{div} S = 0$ , we have

$$\operatorname{div}[X^c, S] = X^c \operatorname{div} S - S \operatorname{div} X^c = 0.$$

On the other hand, by our above remark,

$$\operatorname{div}(\psi C) = \psi \operatorname{div} C + C\psi = (n + 1)\psi.$$

So it follows that  $\psi = 0$ , hence  $[X^c, S] = 0$ . Thus  $X$  is an affine vector field on  $(M, F)$ .

(ii) Now suppose that ( $\operatorname{div} X^c = 0$  and)  $X$  is also a conformal vector field. Then, by Proposition 2,  $X^c E = f^\nu E$ ,  $f \in C^\infty(M)$ . Since

$$nf^\nu \stackrel{\text{Lemma 10}}{=} \operatorname{div} X^c \stackrel{\text{cond.}}{=} 0,$$

it follows that  $X^c E = 0$ . Thus, by Corollary 1,  $X$  is an isometric vector field on  $(M, F)$ . □

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