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On Finsler Connections*

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1. In this paper we shall discuss some new basic ideas concerning Finsler connections. Our approach uses two key notions: *horizontal structure* and (general) *pseudoconnection.* We hope that this new approach will result in a real conceptual and technical simplification of M. Matsumoto's well-known theory [9]. Also, it may be helpful in better understanding the "pure essence" of Finsler connections.

Our ideas also have natural relations with R. Miron's recent theory [10]. Miron's theory is more general than ours in the sense that it works on the whole tangent bundle of the total space of a vector bundle, while we limit ourselves to the vertical subbundle. However, this restriction is perfectly adequate with the demands of classical Finsler geometry.

We shall always be working in the category of the finite dimensional, second countable, smooth manifolds. We are concerned with global properties, but for the sake of a comparison with other (mainly the classical) treatments, we shall systematically give coordinate expressions. As for the notations and basic conventions we refer to the monographs [4], [5] and the paper [14].

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* This paper is in final form and no version of it will be submitted for publication elsewhere.

2. To begin with, let us consider a fibered space $\xi = (E, \pi, M)$. (Then the only requirement is that $\pi : E \longrightarrow M$ is a surjective submersion.) As in the case of vector bundles, one can construct the vertical subbundle $V\xi$ of the tangent bundle τ_E of the total space E of ξ. Assigning a Whitney-complement $H\xi$ to $V\xi$, we say that a *horizontal structure* H has been given in ξ. This will be written as follows:

$$(2.1) \qquad H : \quad \tau_E = V\xi \oplus H\xi.$$

H naturally determines the *vertical projector* $v : \tau_E \longrightarrow V\xi$ and the *horizontal projector* $h : \tau_E \longrightarrow H\xi$. The decomposition (2.1) also induces a direct decomposition

$$\mathfrak{X}(E) = \mathfrak{X}_V E \oplus \mathfrak{X}_H E$$

of the module of the vector fields on E onto the submodule of the vertical vector fields and that of the horizontal ones. For the sake of simplicity, the projectors $\mathfrak{X}(E) \longrightarrow \mathfrak{X}_V E$ and $\mathfrak{X}(E) \longrightarrow \mathfrak{X}_H(E)$ also will be denoted by v and h, resp. Every horizontal structure (2.1) determines a unique mapping

$$\ell^H : \quad \mathfrak{X}(M) \longrightarrow \mathfrak{X}_H E, \quad X \longmapsto X^H$$

where X and X^H are π-related. If $A^1_P(M, \tau_E)$ denotes the graded Lie algebra of the *projectable vector* 1*-forms on* M [8], then ℓ^H can be interpreted as such an element of $A^1_P(M, \tau_E)$ which is projectable onto the identity (1,1) tensor field on M. Conversely, if $\Gamma \in A^1_P(M, \tau_E)$ and it is projectable onto $1_{\mathfrak{X}^1_1(M)}$, then Γ determines a unique horizontal structure in ξ. Such a vector 1-form Γ will be mentioned as a *lifting form* in the sequel.

In particular, let $\xi = (E, \pi, M)$ be a *vector bundle*. We shall denote by Z the Liouville vector field, that is the vertical vector field generated by the homothetic transformations of E.

(2.2) Definition. *A horizontal structure H in the vector bundle ξ is said to be satisfying the homogeneity condition (HC) if the Lie derivative $\mathcal{L}_Z v$ of the vertical projector vanishes.*

For other formulations of (HC) and a discussion see [13]. Here we only mention the useful criterion

$$(2.3) \qquad (HC) \Longleftrightarrow \forall X \in \mathfrak{X}(M) : \quad [X^H, Z] = 0.$$

3. Suppose that H is a horizontal structure in the fibered space ξ. Let λ be a vertical vector 1-form on M, symbolically $\lambda \in A^1(M, V\xi)$ (cf. [3]). In this case the pair (H, λ) — or (ℓ^H, λ) — is said to be an *affine structure* in ξ. Obviously, $\ell^H - \lambda \in A^1_P(M, \tau_E)$; this projectable vector 1-form is said to be the *affine lifting form* of the affine structure (H, λ). In the special case $\xi := \tau_M$ there exists a canonical vertical vector 1-form, the vertical lift

$$\ell^v : \quad \mathfrak{X}(M) \longrightarrow \mathfrak{X}_V TM, \quad X \longmapsto X^v.$$

Then an affine structure (H, ℓ^v) is nothing but an immediate (nonlinear) generalization of the classical affine structures (cf. [12] 2.108).

We shall see soon that our general concept of an affine structure proves to be useful in the theory of Finsler connections, too.

4. We briefly recall the definition of the (generalized) pseudoconnections, which was proposed by the present author and Z. Kovács in [14]. — Let ξ and $\widetilde{\xi}$ be *vector bundles* over the same manifold M. A pair (∇, A) is called a *pseudoconnection* in ξ with respect to (w.r.t.) $\widetilde{\xi}$ if $A : \widetilde{\xi} \longrightarrow \tau_M$ is an M-morphism and

$$\nabla : \quad \operatorname{Sec} \widetilde{\xi} \times \operatorname{Sec} \xi \longrightarrow \operatorname{Sec} \xi, \quad (\widetilde{\sigma}, \sigma) \longmapsto \nabla_{\widetilde{\sigma}} \sigma$$

is a mapping which is $C^\infty(M)$-linear in its first variable, additive in its second variable and has the following characteristic property:

$$\forall \widetilde{\sigma} \in \operatorname{Sec} \widetilde{\xi}, \quad \sigma \in \operatorname{Sec} \xi, \quad f \in C^\infty(M) : \quad \nabla_{\widetilde{\sigma}} f\sigma = [(A \circ \widetilde{\sigma})f]\sigma + f \nabla_{\widetilde{\sigma}} \sigma.$$

A trivial but surprisingly important example of pseudoconnections is given in the following

(4.1) Lemma. *Let ι denote the identity morphism of $V\xi$. — There exists a unique pseudoconnection (∇^i, ι) (briefly ∇^i) in $V\xi$ w.r.t. itself (briefly in $V\xi$) satisfying the following condition:*

$$\forall \sigma \in \operatorname{Sec} \xi, \quad X \in \mathfrak{X}_V E : \quad \nabla^i_X \sigma^v = 0,$$

where σ^v is the vertical lift of the section σ (cf. [14], Lemma 3). ■

For a *proof* see again [14], Lemma 3 and 4. (Another, coordinate-free, proof has been communicated to the author by Z. Kovács. His reasoning is based on elementary homological algebra.) We mention in advance that the trivial pseudoconnection ∇^i will play the same role in our theory as the "flat vertical connection" ([9], p. 57) in Matsumoto's theory.

5. Now we are in a position to formulate some of the indispensable definitions concerning Finsler connections. But what is a Finsler connection? We begin with an answer to this question. — Suppose that $\xi = (E, \pi, M)$ is a fixed vector bundle of rank r with n-dimensional base manifold M.

(5.1) Definition. *A Finsler connection is a linear connection in the vertical bundle $V\xi$, i.e. a linear connection of type*

$$\mathfrak{X}(E) \times \mathfrak{X}_V E \longrightarrow \mathfrak{X}_V E, \quad (X, Y) \longmapsto \nabla_X Y$$

(cf. [6], [11]). If H is a horizontal structure in ξ and ∇ is a Finsler connection, then the pair (∇, H) is said to be a Matsumoto pair.

Let $\left(U, (u^i)_{i=1}^n\right)$ be a chart in M. Consider and fix the vector bundle chart $\left(\pi^{-1}(U); (x^i)_{i=1}^n, (y^\alpha)_{\alpha=1}^r\right)$ — where $x^i := u^i \circ \pi$ — for ξ as described in [14], p. 1168. If H is a horizontal structure in ξ, then there exists a unique family of functions

$$N_i^\alpha : \quad \pi^{-1}(U) \longrightarrow \mathbb{R} \qquad (1 \leq i \leq n, 1 \leq \alpha \leq r)$$

such that

$$\forall i \in \{1, \dots, n\} : \quad \left(\frac{\partial}{\partial u^i}\right)^H = \frac{\partial}{\partial x^i} - N_i^\alpha \frac{\partial}{\partial y^\alpha}.$$

The functions N_i^α are called the (local) *parameters* of H w.r.t. the fixed vector bundle chart. It is easy to see that the lifts

$$\frac{\delta}{\delta x^i} := \left(\frac{\partial}{\partial u^i}\right)^H, \qquad 1 \leq i \leq n$$

constitute a local basis for the horizontal subbundle $H\xi$. — Returning to the Finsler connection ∇, its parameters over $\pi^{-1}(U)$ are defined by the relations

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^\alpha} = \Gamma_{i\alpha}^\beta \frac{\partial}{\partial y^\beta}, \quad \nabla_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta} = C_{\alpha\beta}^\gamma \frac{\partial}{\partial y^\gamma} \tag{5.2}$$

$$(1 \leq i \leq n; \quad 1 \leq \alpha, \beta, \gamma \leq r)$$

as usual in case of linear connections (our notation is the traditional one).

(5.3) Definition. *A Finsler connection ∇ is said to be regular if the kernel of the mapping $L: \quad X \in \mathfrak{X}(E) \longmapsto L(X) := \nabla_X Z$ is a horizontal structure in ξ, i.e. if $\mathfrak{X}(E) = \mathfrak{X}_V E \oplus \operatorname{Ker} L$ (cf. [6], def. 1.6.1).*

(5.4) Proposition. *A Finsler connection ∇ is regular iff*

(5.5) $$\det\left(\delta_\alpha^\beta + y^\lambda C_{\alpha\lambda}^\beta\right) \neq 0.$$

Proof. $\mathfrak{X}(E) = \mathfrak{X}_V E \oplus \operatorname{Ker} L \iff L \restriction \mathfrak{X}_V E$ is an isomorphism $\iff \left(L\left(\frac{\partial}{\partial y^\alpha}\right)\right)_{\alpha=1}^n$ is a local basis of $\mathfrak{X}_V E \iff$ (5.5), because

$$\forall\, \alpha \in \{1, \ldots, r\}: \quad L\left(\frac{\partial}{\partial y^\alpha}\right) = \nabla_{\frac{\partial}{\partial y^\alpha}} y^\lambda \frac{\partial}{\partial y^\lambda} = \frac{\partial}{\partial y^\alpha} +$$
$$+ y^\lambda C_{\alpha\lambda}^\beta \frac{\partial}{\partial y^\beta} = \left(\delta_\alpha^\beta + y^\lambda C_{\alpha\lambda}^\beta\right) \frac{\partial}{\partial y^\beta}.$$

■

(5.5) Corollary. *If ∇ is a regular Finsler connection, then the parameters of the horizontal structure* Ker L *are the unique functions L_i^α satisfying the equations*

$$y^\beta \Gamma_{i\beta}^\alpha - L_i^\lambda \left(\delta_\lambda^\alpha + y^\beta C_{\lambda\beta}^\alpha\right) = 0.$$

Proof. If $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - L_i^\alpha \frac{\partial}{\partial y^\alpha}$ $(1 \leq i \leq n)$ is a local basis for Ker L, then

$$\forall\, i \in \{1, \ldots, n\}: \quad 0 = L\left(\frac{\delta}{\delta x^i}\right) = \nabla_{\frac{\partial}{\partial x^i} - L_i^\lambda \frac{\partial}{\partial y^\lambda}} y^\beta \frac{\partial}{\partial y^\beta} =$$
$$= \nabla_{\frac{\partial}{\partial x^i}} y^\beta \frac{\partial}{\partial y^\beta} - L_i^\alpha \nabla_{\frac{\partial}{\partial y^\alpha}} y^\beta \frac{\partial}{\partial y^\beta} = \left(y^\beta \Gamma_{i\beta}^\alpha - L_i^\lambda \left(\delta_i^\alpha + y^\beta C_{\lambda\beta}^\alpha\right)\right) \frac{\partial}{\partial y^\alpha},$$

so (5.6) holds. At the same time, (5.5) guarantees that the functions L_i^α are uniquely determined by (5.6). ■

6. Recall that any horizontal structure H in the vector bundle ξ gives rise to a splitting $\mathcal{H}: \quad \pi^* \tau_M \longrightarrow \tau_E$ of the canonical short exact sequence

$$0 \longrightarrow V\xi \xrightarrow{\lambda} \tau_E \xrightarrow{\mu} \pi^* \tau_M \longrightarrow 0$$

(λ is the natural inclusion, $\mu: \quad a \in T_z E \longmapsto (z, T_z \pi(a))$) such that $H\xi = \operatorname{Im} \mathcal{H}$ — and conversely.

(6.1) Definition. *Let H be a horizontal structure in the vector bundle ξ and consider the splitting $\mathcal{H}$ derived by H. — A triplet (∇^h, ∇^v, H) is said to be a Cartan triad if $(\nabla^h, \mathcal{H})$ is a pseudoconnection in $V\xi$ w.r.t. $\pi^*\tau_M$ and (∇^v, ι) is a pseudoconnection in $V\xi$ w.r.t. itself. ∇^h and ∇^v are called the h-connection term and the v-connection term of the triad, resp.*

We represent a Cartan triad locally by the family

$$\left(F^{\beta}_{i\alpha}, C^{\gamma}_{\alpha\beta}, N^{\alpha}_i\right)$$

where the functions N^{α}_i are the parameters of H, the $C^{\gamma}_{\alpha\beta}$-s are defined by (5.2) (replacing ∇ with ∇^v) and the functions $F^{\beta}_{i\alpha}$ are determined by the relations

$$\nabla^h_{\widehat{\frac{\partial}{\partial u^i}}} \frac{\partial}{\partial y^\alpha} = F^{\beta}_{i\alpha} \frac{\partial}{\partial y^\beta}. \tag{6.2}$$

In (6.2) $\widehat{\frac{\partial}{\partial u^i}}: \ z \in \pi^{-1}(U) \longmapsto \left(z, \frac{\partial}{\partial u^i} \circ \pi(z)\right) \in E \times_M TM$. More generally, we shall need the following convention: if $X \in \mathfrak{X}(M)$, $\widehat{X}$ denotes the section

$$z \in E \longmapsto \widehat{X}(z) := (z, X \circ \pi(z)) \in E \times_M TM.$$

The next result (cf. [14], Theorem) provides a simple but important connection between Cartan triads and Matsumoto pairs.

(6.3) Theorem. *Let (∇^h, ∇^v, H) be a Cartan triad. If the mapping $\nabla: \ \mathfrak{X}(E) \times \mathfrak{X}_V E \longrightarrow \mathfrak{X}_V E$ is given by the formula*

$$\nabla_X Y = \nabla^v_{vX} Y + \nabla^h_{\mu \circ X} Y,$$

then ∇ is a Finsler connection, consequently (∇, H) is a Matsumoto pair which is said to be associated with (∇^h, ∇^v, H).

Conversely, suppose that (∇, H) is a Matsumoto pair and form the mappings

$$\nabla^h: \ \mathrm{Sec}\,\pi^*\tau_M \times \mathfrak{X}_V E \longrightarrow \mathfrak{X}_V E, \quad (\widetilde{X}, Y) \longmapsto \nabla_{\mathcal{H} \circ \widetilde{X}} Y,$$
$$\nabla^v: \ \mathfrak{X}_V E \times \mathfrak{X}_V E \longrightarrow \mathfrak{X}_V E, \quad (X, Y) \longmapsto \nabla_X Y.$$

Then the triad (∇^h, ∇^v, H) is a Cartan triad. ∎

The (not too hard) *proof* follows the same line as the proof of the above cited theorem of [14], so we omit it. Similarly, a quite straightforward calculation yields the

(6.4) Proposition. *If* (∇, H) *is a Matsumoto pair with parameters* $\left(\Gamma^\alpha_{i\beta}, C^\alpha_{\beta\gamma}, N^\alpha_i\right)$ *— see (5.2) — and* (∇^h, ∇^v, H) *is the associated Cartan triad of* (∇, H) *with parameters* $\left(F^\alpha_{i\beta}, \widetilde{C}^\alpha_{\beta\gamma}, N^\alpha_i\right)$, *then*

$$F^\alpha_{i\beta} = \Gamma^\alpha_{i\beta} - N^\lambda_i C^\alpha_{\lambda\beta}, \qquad \widetilde{C}^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma}$$

$$(1 \leq i \leq n; \quad 1 \leq \alpha, \beta, \gamma \leq r).$$

■

(6.5) Definition. *A Cartan triad* (∇^h, ∇^v, H) *is said to be satifying the* C_2*-condition if*

$$\forall X \in \mathfrak{X}_V E: \quad \nabla^v_X Z = \nabla^i_X Z$$

(cf. [7] and [9] def. 13.1).

(6.6) Proposition. *If the Cartan triad* (∇^h, ∇^v, H) *satisfies the* C_2*-condition, then the connection term* ∇ *in its associated Matsumoto pair is regular.*

Proof. Due to the proof of (5.4) it is enough to show that $L\left(\frac{\partial}{\partial y^\alpha}\right)$, $1 \leq \alpha \leq r$ is a local basis of $\mathfrak{X}_V E$. But this follows immediately by (C_2), because

$$L\left(\frac{\partial}{\partial y^\alpha}\right) := \nabla_{\frac{\partial}{\partial y^\alpha}} Z \overset{(6.3)}{=} \nabla^v_{\frac{\partial}{\partial y^\alpha}} Z + \nabla^h_{\mu \circ \frac{\partial}{\partial y^\alpha}} Z \overset{(C_2)}{=} \nabla^i_{\frac{\partial}{\partial y^\alpha}} y^\beta \frac{\partial}{\partial y^\beta} =$$
$$= \frac{\partial}{\partial y^\alpha} + y^\beta \nabla^i_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta} \overset{(4.1)}{=} \frac{\partial}{\partial y^\alpha}.$$

■

7. In this section we introduce and briefly discuss a further important notion: the notion of *deflection.*

(7.1) Proposition and Definition. *(cf. [9], p. 67)*

(a) Let (∇^h, ∇^v, H) *be a Cartan triad. The mapping*

$$D: \quad \mathfrak{X}(M) \longrightarrow \mathfrak{X}_V E, \quad X \longmapsto \nabla^h_{\widehat{X}} Z$$

is a vertical vector 1-form (i.e. $D \in A^1(M, V\xi)$) whose coordinate expression is

$$D = \left(y^\lambda F^\alpha_{i\lambda} - N^\alpha_i\right) \frac{\partial}{\partial y^\alpha} \otimes dx^i.$$

D is said to be the deflection of the given Cartan triad.

(b) *The deflection of a Matsumoto pair (∇, H) is the $(1,1)$ tensor field*

$$\widetilde{D}: \quad X \in \mathfrak{X}(E) \longmapsto \widetilde{D}(X) := \nabla_{hX} Z.$$

It appears in coordinates thus:

$$\widetilde{D} = y^\lambda \left(\Gamma^\alpha_{i\lambda} - N^\nu_i C^\alpha_{\nu\lambda}\right) \frac{\partial}{\partial y^\alpha} \otimes dx^i.$$

■

(7.2) Proposition. *If ∇ is a regular Finsler connection, then the Matsumoto pair $(\nabla, \operatorname{Ker} L)$ and the Cartan triad associated with $(\nabla, \operatorname{Ker} L)$ have no deflection.*

Proof. From the meaning of the horizontal structure $\operatorname{Ker} L$, $\forall X \in \mathfrak{X}(E)$: $hX \in \operatorname{Ker} L$, hence

$$\widetilde{D}(X) := \nabla_{hX} Z \overset{(5.3)}{=} L(hX) = 0.$$

Let us denote by $\mathcal{L}$ the splitting belonging to $\operatorname{Ker} L$. Taking into account (6.3), the deflection of $(\nabla^h, \nabla^v, \operatorname{Ker} L)$ is the mapping

$$D: \quad X \in \mathfrak{X}(M) \longmapsto \nabla^h_{\widehat{X}} Z = \nabla_{\mathcal{L} \circ \widehat{X}} Z.$$

Here $\mathcal{L} \circ \widehat{X} \in \operatorname{Ker} L$, thus

$$0 = L(\mathcal{L} \circ \widehat{X}) := \nabla_{\mathcal{L} \circ \widehat{X}} Z \Longleftrightarrow D = 0.$$

■

From (7.1) we immediately get the

(7.3) Proposition. *If D is the deflection of the Cartan triad (∇^h, ∇^v, H), then (H, D) is an affine structure in the vector bundle ξ whose affine lifting form has the coordinate expression*

(7.4)
$$\ell^H - D = \left(\frac{\partial}{\partial x^i} - y^\beta F^\alpha_{i\beta}\right)\frac{\partial}{\partial y^\alpha} \otimes dx^i.$$

■

Comparing (7.4) with the formula (8.17') of [9], we can conclude that the quite natural affine structure (H, D) plays the same role in our theory as Matsumoto's somewhat mysterious "non-linear connection associated with a V-connection" in his theory. (See also Matsumoto's instructive Remark 8.2 in [9].)

8.

(8.1) Definition. *The mapping*

$$P^1 : \ \mathrm{Sec}\,\pi^*\tau_M \times \mathfrak{X}_V E \longrightarrow \mathfrak{X}_V E, \quad (\widetilde{X}, Y) \longmapsto v[\mathcal{H} \circ \widetilde{X}, Y] - \nabla^h_{\widetilde{X}} Y$$

is said to be the P^1-torsion of the Cartan triad (∇^h, ∇^v, H).

Note that if $X \in \mathfrak{X}(M)$, then $\mathcal{H} \circ \widehat{X} = \ell^H(X) = X^H$, hence

$$\forall X \in \mathfrak{X}(M),\ Y \in \mathfrak{X}_V E : \quad P^1(\widehat{X}, Y) = [X^H, Y] - \nabla^h_{\widehat{X}} Y.$$

Thus $\forall i \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, r\}$:

$$P^1\left(\widehat{\frac{\partial}{\partial u^i}}, \frac{\partial}{\partial y^\alpha}\right) = \left(\frac{\partial N^\beta_i}{\partial y^\alpha} - F^\beta_{i\alpha}\right)\frac{\partial}{\partial y^\beta};$$

the functions

$$P^\beta_{i\alpha} := \frac{\partial N^\beta_i}{\partial y^\alpha} - F^\beta_{i\alpha}$$

are said to be the *coordinate functions* of the P^1-torsion.

(8.2) Proposition. *Under the homogeneity condition (HC), $\forall X \in \mathfrak{X}(M) : \ P^1(\widehat{X}, Z) = -D(X)$; if, moreover, the deflection vanishes, we get the relation*

$$y^\beta P^\alpha_{i\beta} = 0.$$

Proof. From the preceding remark

$$P^1(\widehat{X}, Z) = [X^H, Z] - \nabla^h_{\widehat{X}} Z \overset{(2.3)}{=} -\nabla^h_{\widehat{X}} Z \overset{(7.1)}{=} -D(X);$$

$$y^\beta P^\alpha_{i\beta} = y^\beta \frac{\partial N^\alpha_i}{\partial y^\beta} - y^\beta F^\alpha_{i\beta} \overset{(HC)}{=} N^\alpha_i - y^\beta F^\alpha_{i\beta} \overset{(7.1)}{=} 0,$$

if D vanishes. ■

With the help of the P^1-torsion and the trivial pseudoconnection ∇^i (lemma (4.1)) one can construct some important special Cartan triads (cf. [8], p. 120).

(8.3) Definition. *A Cartan triad* (∇^h, ∇^v, H) *is said to be a Hashiguchi triad if* $P^1 = 0$,
a Rund triad if $\nabla^v = \nabla^i$,
a Berwald triad if it is a Rund triad and a Hashiguchi triad at the same time.

So we have the following "commutative diagram":

$$\begin{array}{ccc} \dfrac{(\nabla^h, \nabla^v, H)}{\text{Cartan triad}} & \xrightarrow{\quad \nabla^v := \nabla^i \quad} & \textit{Rund triad} \\ P^1 = 0 \Big\downarrow & & \Big\downarrow P^1 = 0 \\ \textit{Hashiguchi triad} & \xrightarrow{\quad \nabla^v := \nabla^i \quad} & \textit{Berwald triad} \end{array}$$

(8.4) Definition. *If* (∇, H) *is a Matsumoto pair and its associated Cartan triad is the Berwaldian one, then the Finsler connection* ∇ *is said to be the Berwald connection.*

From our previous results one can deduce with a little extra work the

(8.5) Theorem. *Any horizontal structure* $H : \tau_E = V\xi \oplus H\xi$ *in a vector bundle* ξ *determines a unique Berwald triad and — consequently — a unique Berwald connection* ∇^B. H *satisfies the homogeneity condition* (HC) *iff* ∇^B *has no deflection.* ■

(Further details — from a slightly different point of view — can be found in [14] and [15].)

9. In this concluding section we shall have a look at a very special classical situation. — Suppose that (M, L) is a *Lagrange space* in Miron's sense [10], that is $L : TM \longrightarrow \mathbb{R}$ is a regular Lagrange function:

$$\det\left(\frac{\partial^2 L}{\partial y^i \partial y^i}\right) \neq 0.$$

Let X_L be the Lagrange vector field for L. It can be characterized by the well-known relation $i_{X_L}\omega_L = -dE$ ($E := Z(L) - L$, ω_L is the Lagrangian 2-form; cf. e.g. [1], def. 3.5.11). The coordinate expression of X_L is

$$X_L = y^i \frac{\partial}{\partial x^i} + G^i \frac{\partial}{\partial y^i},$$

where

$$G^i = g^{ij}\left(\frac{\partial^2 L}{\partial x^k \partial y^j} \cdot y^k - \frac{\partial L}{\partial x^j}\right), \quad (g^{ij}) := \left(\frac{\partial^2 L}{\partial y^i \partial y^j}\right)^{-1}.$$

Since the 25's of this century it has been a well-established folklore that the family

$$(G^i_j) := \left(\frac{1}{2}\frac{\partial G^i}{\partial y^j}\right)$$

determines — in our terminology — a horizontal structure and hence — owing to (8.5) — a Berwald connection in τ_M. This connection is called the *classical Berwald connection* of the Lagrange space (M, L) (cf. [9], Teorema 4.1). Utilizing now a clever observation of M. Crampin [2] we obtain the following elegant result:

(9.1) Theorem. *Let (M, L) be a Lagrange space and let X_L be the Lagrange vector field for L. The mapping*

$$\ell^B : \quad X \in \mathfrak{X}(M) \longmapsto \frac{1}{2}(X^c + [X^v, X_L])$$

(X^c and X^v are the obvious complete and vertical lifts of X) is a lifting form in the sense of section 2. The horizontal structure determined by ℓ^B induces just the classical Berwald connection.

Proof. Calculating by brute force, we get the coordinate expression

$$\ell^B\left(\frac{\partial}{\partial u^i}\right) = \frac{1}{2}\left(\left(\frac{\partial}{\partial u^i}\right)^c + \left[\frac{\partial}{\partial y^i}, y^j\frac{\partial}{\partial x^j} + G^j\frac{\partial}{\partial y^j}\right]\right) =$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \frac{\partial G^j}{\partial y^i}\frac{\partial}{\partial y^j}\right)$$

for ℓ^B, from which our assertions follow easily. ■

So at least three things of a seemengly quite different kind, namely horizontal structures, Berwald connection and Lagrangian mechanics have met in a fascinating manner.

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