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ON THE CURVATURE AND INTEGRABILITY OF HORIZONTAL MAPS

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1. Introduction. In the recent progress of connection theory (especially in Finsler geometry, see [9]) and its applications the vector bundle viewpoint plays a central role. Purely geometrically, from this point of view the notion of a (general) connection ([11], Definition 3) is based upon a Whitney decomposition of the tangent bundle of the total space of the considered vector bundle into the vertical subbundle and a *horizontal* one. Each of the horizontal subbundles can be obtained with the help of a (right, smooth) splitting of the canonical short exact sequence ([4], Vol. II, p. 335) constructed from the given vector bundle. These splittings are called *horizontal maps*. Hence the theory of connections may be developed starting from a horizontal map; a sketch of such an approach can be found in the author's paper [11], cf. also [2], [12]. In this note we are going to investigate mainly the integrability of a horizontal map. Here, *integrability* means that the image bundle of the horizontal map under discussion is an involutive distribution in the usual sense ([4], Vol. I, p. 134). Our main result gives a number of necessary and sufficient conditions for a horizontal map to be integrable. These criteria will be formulated in terms of *horizontal projection, vertical projection, almost product structure* and *curvature tensor field* induced by the horizontal map $\mathcal{H}$ in question. We shall also discuss some interesting relations between the *Dombrowski map K* belonging to $\mathcal{H}$ and the curvature tensor field, furthermore — in the linear case — between K and the usual curvature *form* induced by $\mathcal{H}$. These relations generalize some results of Dombrowski's important paper [1].

Grifone's and Vilms' works [5], [12] were very stimulating for the present investigations. A discussion of similar questions also occurs in Duc's excellent survey [2].

Notations, terminology and basic conventions are as in the monograph [4] and in the paper [11].

2. Preliminaries. For the convenience of the reader, we begin with some definitions and technical remarks needed in the subsequent considerations.

Let M be a manifold. $\mathcal{X}^p_q(M)$, $A^p(M, \tau_M)$, $\operatorname{End}\tau_M$ and $\operatorname{End}\mathcal{X}(M)$ denote the $C^\infty(M)$-module of (p, q) tensor fields on M, τ_M-valued p-forms on M, $\tau_M\to\tau_M$ (linear) bundle maps, and $\mathcal{X}(M)\to\mathcal{X}(M)$ $C^\infty(M)$-linear maps, respectively. It is known (e.g. from [7], Proposition 3.1) that the module of q-linear maps $\mathcal{X}(M)\times\dots\times\mathcal{X}(M)\to\mathcal{X}(M)$ is isomorphic to the module $\mathcal{X}^1_q(M)$; in particular $\mathcal{X}^1_1(M)\cong\operatorname{End}\mathcal{X}(M)$. Now suppose that $f\in\operatorname{End}\tau_M$. Then $\forall p\in M: f|T_pM=f_p\in\operatorname{End}T_pM$, so the map $\tilde f: p\in M\to\tilde f(p):=f_p$ is an element of $A^1(M, \tau_M)$ and the correspondence $f\to\tilde f$ defines a (natural) isomorphism $\operatorname{End}\tau_M\cong A^1(M, \tau_M)$. Because of $\operatorname{End}T_pM\cong(T_pM)^*\otimes T_pM$ one can also write $f\in\operatorname{Sec}(\tau^*_M\otimes\tau_M)=\mathcal{X}^1_1(M)$. Summarizing, we obtain the following.

LEMMA. $\mathscr{X}^1_1(M)\cong\operatorname{End}\mathscr{X}(M)\cong\operatorname{End}\tau_M\cong A^1(M,\tau_M)$.

We shall identify these isomorphic modules, without further reference.

DEFINITION 1. (See [3] or [7], Proposition 3.12). Let $f,g\in\operatorname{End}\tau_M$. Their *Nijenhuis-torsion* is the (1, 2) tensor field $[f,g]\colon \mathscr{X}(M)\times\mathscr{X}(M)\to\mathscr{X}(M)$ given by

$$[f,g](X,Y):=[fX,gY]+[gX,fY]-g[fX,Y]-f[gX,Y]-$$
$$-g[X,fY]-f[X,gY]+g\circ f[X,Y]+f\circ g[X,Y]$$

($[fX,gY]$ etc. are the usual Lie products of vector fields).

The introduction of the following useful concept was inspired by Haantjes's theorem ([3], Theorem III).

DEFINITION 2. Let $f\in\operatorname{End}\tau_M$. Suppose that $\forall p\in M\colon f_p=:f|T_pM$ has constant eigenvalues $\lambda_1,\dots,\lambda_k$ with constant geometric multiplicity $(1\leqq k\leqq n)$. Let us denote by $S(\lambda_i)$ those subbundles of τ_M whose fibers at a point $p\in M$ are the invariant subspaces belonging to λ_i. f is called *integrable* if the subbundles $S(\lambda_i)$ and the Whitney sums $S(\lambda_1)\oplus\dots\oplus S(\lambda_j)$ $(j=2,\dots,k)$ are involutive distributions.

In the sequel $\xi=(E,\pi,B,F)$ will always denote a fixed vector bundle over the n-dimensional base manifold B. The sequence of vector bundles $0\to V_\xi\xrightarrow{i}\tau_E\to$ $\xrightarrow{\widetilde{d\pi}}\pi^*(\tau_B)\to 0$ (where V_ξ denotes the vertical subbundle of the tangent bundle τ_E) is a short exact sequence, which is also called the canonical exact sequence starting from ξ. So a horizontal map is a bundle map $\mathscr{H}\colon\pi^*(\tau_B)\to\tau_E$, while the horizontal projection, vertical projection and almost product structure mentioned above are

$$h:=\mathscr{H}\circ\widetilde{d\pi},\quad v:=1-h,\quad P:=2h-1$$

respectively. If $\alpha\colon V_\xi\to\xi$ is the canonical bundle map described e.g. in [4], Vol. I p. 291, then $K:=\alpha\circ V$ is the Dombrowski-map induced by $\mathscr{H}$. (For details, see [11].) The endomorphisms $h,v,P\in\operatorname{End}\tau_E$ — according to the lemma — will also be interpreted as (1, 1) tensor fields. In this case — for better clarity — we shall sometimes write $\tilde h,\tilde v,\tilde P$ instead of h,v and P. Observe (cf. [11], Section 5) that these endomorphisms satisfy the conditions of Definition 2, hence we can speak about their integrability.

3. Curvature of horizontal maps. Following Grifone's idea, which in turn was inspired by [3], we define the *curvature tensor field* of a horizontal map $\mathscr{H}$ as the Nijenhuis-torsion $R:=-1/2\,[h,h]$. The geometric meaning of this construction will be clarified by the next results. There are, of course, other possibilities to define curvature for a horizontal map; see e.g. [8], [12] and — in the linear case — [10].

PROPOSITION 1. *Let $U\subset B$ be a trivializing neighbourhood for ξ and (x^i,y^α) the local coordinate system over $\pi^{-1}(U)$ described in* [11], *Section* 5. *If the functions $\Gamma^\alpha_i\colon\pi^{-1}(U)\to\mathbf{R}$ are the connection parameters of $\mathscr{H}$ with respect to (x^i,y^α) and $Z=Z^i\frac{\partial}{\partial x^i}+Z^\alpha\frac{\partial}{\partial y^\alpha}$, $V=V^i\frac{\partial}{\partial x^i}+V^\alpha\frac{\partial}{\partial y^\alpha}$ are vector fields over $\pi^{-1}(U)$, then $R(Z,V)=Z^iV^jR^\alpha_{ij}\frac{\partial}{\partial y^\alpha}$, where*

$$R^\alpha_{ij}=\frac{\partial\Gamma^\alpha_j}{\partial x^i}-\frac{\partial\Gamma^\alpha_i}{\partial x^j}+\Gamma^\beta_j\frac{\partial\Gamma^\alpha_i}{\partial y^\beta}-\Gamma^\beta_i\frac{\partial\Gamma^\alpha_j}{\partial y^\beta}.$$

COROLLARY. $\forall Z, V\in\mathfrak{X}(E)$: $R(Z, V)\in\mathfrak{X}_V(E):=\text{Sec}\, V_\xi$.

The proof is a standard, but lengthy calculation, so we omit it.

In view of $hX^h=X^h$, $hY^h=Y^h$ (where X^h and Y^h are the horizontal lifts of X and Y with respect to $\mathcal{H}$, see [11], Definition 2) we immediately obtain from the definition of R and of the Nijenhuis torsion

PROPOSITION 2 (cf. [13] and [2]). $\forall X, Y\in\mathfrak{X}(B)$: $R(X^h, Y^h)=h[X^h, Y^h]-[X^h, Y^h]$.

Using this proposition and the obvious relation $K\circ h=o$ we have the following generalization of formula (23) of [1]:

PROPOSITION 3. *If K is the Dombrowski map belonging to $\mathcal{H}$ then*

$$\forall X, Y\in\mathfrak{X}(B):\ \alpha\circ R(X^h, Y^h) = -K\circ[X^h, Y^h],$$

that is the diagram

$$\begin{array}{ccc} E & \xrightarrow{R(X^h,\,Y^h)} & VE \\ {\scriptstyle [X^h,\,Y^h]}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TE & \xrightarrow{-K} & E \end{array}$$

commutes.

PROOF. On the one hand

$$K\circ R(X^h, Y^h) = K\circ(h[X^h, Y^h]-[X^h, Y^h]) = -K\circ[X^h, Y^h],$$

on the other hand

$$K\circ R(X^h, Y^h) = \alpha\circ v\circ R(X^h, Y^h) = \alpha\circ R(X^h, Y^h),$$

because of $R(X^h, Y^h)\in\mathfrak{X}_V(E)$. □

Now we suppose that $\mathcal{H}$ satisfies the so-called homogeneity condition ([11], Definition 2). In this case $\mathcal{H}$ induces a ∇: $\text{Sec}\,\xi\to A^1(B,\xi)$ *linear connection.* It is well-known that the *curvature form* of ∇ is the mapping

$$\tilde{R}:\ \mathfrak{X}(B)\times\mathfrak{X}(B)\to\text{End}\ \ \text{Sec}\,\xi,$$

$$(X, Y)\mapsto\tilde{R}(X, Y):=\nabla_X\circ\nabla_Y-\nabla_Y\circ\nabla_X-\nabla_{[X,Y]},$$

where $\text{End}\,\text{Sec}\,\xi$ is the module of the $C^\infty(B)$-linear maps $\text{Sec}\,\xi\to\text{Sec}\,\xi$ (see [4], Vol. II). What is the relation between the curvature tensor field R and the curvature form $\tilde{R}$? The answer to this very natural question is given in the

THEOREM 1. $\forall X, Y\in\mathfrak{X}(B)$, $\sigma\in\text{Sec}\,\xi$: $\alpha\circ R(X^h, Y^h)\circ\sigma=\tilde{R}(X, Y)(\sigma)$, *so we have the following commutative diagram:*

$$\begin{array}{ccc} B & \xrightarrow{\sigma} & E \\ {\scriptstyle \tilde{R}(X,\,Y)(\sigma)}\downarrow & & \downarrow{\scriptstyle R(X^h,\,Y^h)} \\ E & \xleftarrow{\alpha} & VE \end{array}$$

PROOF. Choosing a trivializing neighbourhood $U\subset B$, we again work in the coordinate system (x^i, y^α). It induces a framing e_α: $U\to E$ (see [11], Section 5), so each section $\sigma: B\to E$ can be written locally in the form $\sigma^\alpha e_\alpha$. A straightforward (lengthy) calculation shows that over U

$$(\nabla_X\circ\nabla_Y-\nabla_Y\circ\nabla_X-\nabla_{[X,Y]})\sigma = X^iY^j\tilde{R}^\alpha_{ij\beta}\sigma^\beta e_\alpha,$$

where

$$X = X^i\frac{\partial}{\partial u^i},\quad Y = Y^j\frac{\partial}{\partial u^j},\quad \tilde{R}^\alpha_{ij\beta} = \frac{\partial\Gamma^\alpha_{j\beta}}{\partial u^i}-\frac{\partial\Gamma^\alpha_{i\beta}}{\partial u^j}+\Gamma^\alpha_{i\delta}\Gamma^\delta_{j\beta}-\Gamma^\alpha_{j\delta}\Gamma^\delta_{i\beta},$$

and in the last expression the functions $\Gamma^\alpha_{i\beta}$ are defined by $\Gamma^\alpha_{i\beta}:=\frac{\partial\Gamma^\alpha_i}{\partial y^\beta}$ (see [11], Th. 1). On the other hand,

$$R(X^h, Y^h) = (X^iY^j\circ\pi)y^\beta\left[\frac{\partial\Gamma^\alpha_{j\beta}}{\partial u^i}\circ\pi-\frac{\partial\Gamma^\alpha_{i\beta}}{\partial u^j}\circ\pi+(\Gamma^\alpha_{i\delta}\circ\pi)(\Gamma^\delta_{j\beta}\circ\pi)-\right.$$

$$\left.-(\Gamma^\alpha_{j\delta}\circ\pi)(\Gamma^\delta_{i\beta}\circ\pi)\right]\frac{\partial}{\partial y^\alpha}.$$

The value of this vector field at the point $\sigma(x)\in\pi^{-1}(U)$ is

$$R(X^h, Y^h)[\sigma(x)] = X^iY^j\sigma^\beta(x)\tilde{R}^\alpha_{ij\beta}(x)\left(\frac{\partial}{\partial y^\alpha}\right)_{\sigma(x)}\in T_{\sigma(x)}E,$$

consequently

$$\alpha_{\sigma(x)}[\tilde{R}(X^h, Y^h)[\sigma(x)]] = X^iY^j\sigma^\beta(x)\tilde{R}^\alpha_{ij\beta}(x)\left[\alpha_{\sigma(x)}\left(\frac{\partial}{\partial y^\alpha}\right)_{\sigma(x)}\right] =$$

$$= X^iY^j\tilde{R}^\alpha_{ij\beta}(x)\sigma^\beta(x)e_\alpha(x).$$

This is exactly a coordinate statement of the conclusion of the Theorem. □

COROLLARY. *In the linear case* $\forall X, Y\in\mathscr{X}(B)$, $\sigma\in\mathrm{Sec}\,\xi$: $K\circ[X^h, Y^h]\circ\sigma= =-[\tilde{R}(X, Y)](\sigma)$.

This last relation is the immediate generalization of the above mentioned formula of Dombrowski's paper.

4. Integrability of horizontal maps. Now we turn to the study of the integrability of a horizontal map $\mathscr{H}$. The next result summarizes the situation.

THEOREM 2. *For a horizontal map* $\mathscr{H}: \pi^*(\tau_B)\to\tau_E$ *the following are equivalent:*

(1) $\mathscr{H}$ *is integrable.*

(2) *The curvature tensor field of* $\mathscr{H}$ *vanishes, that is* $\forall Z, V\in\mathscr{X}(E)$: $R(Z, V)=0$.

(3) *The component functions* R^α_{ij} *of* R *vanish.*

(4) *The horizontal projection* h *is integrable.*

(5) $\tilde{v}\circ[\tilde{h}, \tilde{h}]=0$.

(6) *The vertical projection* v *is integrable.*

(7) $[\tilde{v}, \tilde{v}]=0$.

(8) $\tilde{v}\circ[\tilde{v}, \tilde{v}]=0$.

(9) $[\tilde{P}, \tilde{P}]=0$.

(10) *The almost product structure* P *is integrable.*

PROOF. (a) We begin with the following technical remarks:

$$[\tilde{h}, \tilde{h}] = [\tilde{v}, \tilde{v}], \quad \operatorname{Im} \tilde{h} = \operatorname{Ker} \tilde{v} = \operatorname{Sec}\ \operatorname{Im} \mathscr{H}.$$

Indeed, $[\tilde{v}, \tilde{v}]=[1-\tilde{h}, 1-\tilde{h}]=[1, 1]-[\tilde{h}, 1]-[1, \tilde{h}]+[\tilde{h}, \tilde{h}]$. Here $[1, 1]=0$, as it immediately follows from the definition of the operation $[\ ,\ ]$, while $\forall Z_1, Z_2\in\mathscr{X}(E)$:

$$[\tilde{h}, 1](Z_1, Z_2) = [\tilde{h}Z_1, Z_2]+[Z_1, \tilde{h}Z_2]-[\tilde{h}Z_1, Z_2]-\tilde{h}[Z_1, Z_2]-[Z_1, \tilde{h}Z_2]-$$
$$-\tilde{h}[Z_1, Z_2]+\tilde{h}[Z_1, Z_2]+\tilde{h}[Z_1, Z_2] = 0 \Rightarrow [\tilde{h}, 1] = 0.$$

Similarly $[1, \tilde{h}]=0$, thus $[\tilde{v}, \tilde{v}]=[\tilde{h}, \tilde{h}]$. The second assertion is clear.

(b) We show: (2)$\Leftrightarrow$(1)$\Leftrightarrow$(5)$\Leftrightarrow$(8). Let $Z_1, Z_2\in\mathscr{X}(E)$. From the definition of the curvature we have:

$$[\tilde{h}Z_1, \tilde{h}Z_2] = -R(Z_1, Z_2)-\tilde{h}[Z_1, Z_2]+\tilde{h}[\tilde{h}Z_1, Z_2]+\tilde{h}[Z_1, \tilde{h}Z_2].$$

On the right hand side $-R(Z_1, Z_2)\in\mathscr{X}_V(E)$, while the other terms belong to $\mathscr{X}_H(E):=\operatorname{Sec}\operatorname{Im}\mathscr{H}$ because of (a). It implies that $[\tilde{h}Z_1, \tilde{h}Z_2]\in\mathscr{X}_H(E)\Leftrightarrow R(Z_1, Z_2)=0$ proving the equivalence (2)$\Leftrightarrow$(1). From this the implication (1)$\Rightarrow$(5) also follows. If — conversely — $\tilde{v}\circ[\tilde{h}, \tilde{h}]=0$, then (applying (a) again) $\operatorname{Im}[\tilde{h}, \tilde{h}]\subset\operatorname{Ker}\tilde{v}=$ $=\operatorname{Im}\tilde{h}=\mathscr{X}_H(E)$ hence

$$\frac{1}{2}[\tilde{h}, \tilde{h}](Z_1, Z_2) = [\tilde{h}Z_1, \tilde{h}Z_2]+\tilde{h}[Z_1, Z_2]-\tilde{h}[\tilde{h}Z_1, Z_2]-\tilde{h}[Z_1, \tilde{h}Z_2]\in\operatorname{Sec}\operatorname{Im}\mathscr{H}\Rightarrow$$
$$\Rightarrow [\tilde{h}Z_1, \tilde{h}Z_2]\in\operatorname{Sec}\operatorname{Im}\mathscr{H},$$

that is (1) holds. The equivalence (5)$\Leftrightarrow$(8) is obvious from $[\tilde{h}, \tilde{h}]=[\tilde{v}, \tilde{v}]$.

(c) We verify that the assertions (3), (4), (6), (7), (9) and (10) are equivalent to (2). Let us first observe that in case of the endomorphisms $h, v, P\in\operatorname{End}\tau_E$ the subbundles $S(\lambda_i)$ mentioned in Definition 2 are the following:

$$h\colon\ S(1) = \operatorname{Im} h = \operatorname{Im}\mathscr{H}, \quad S(0) = \operatorname{Im} v = V_\xi;$$
$$v\colon\ S(1) = V_\xi, \quad S(0) = \operatorname{Im}\mathscr{H}; \quad P\colon S(1) = \operatorname{Im}\mathscr{H}, \quad S(-1) = V_\xi.$$

Here, of course, the subbundle V_ξ is an involutive distribution, so we easily get the equivalences (2)$\Leftrightarrow$(4), (2)$\Leftrightarrow$(6), (2)$\Leftrightarrow$(10). The equivalence (2)$\Leftrightarrow$(3) is evident, (2)$\Leftrightarrow$(7) follows from (a). Finally — applying also (a) —

$$[\tilde{P}, \tilde{P}] = [2\tilde{h}-1, 2\tilde{h}-1] = 4[\tilde{h}, \tilde{h}]-2[1, \tilde{h}]-2[\tilde{h}, 1]+[1, 1] = -8R,$$

this establishes the equivalence of (2) and (9).

REMARK. From the proof we find that $R=-1/8\,[P, P]$. In a more special situation this was proved also by Stere Ianus [6].

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