ON THE FINSLER-METRIZABILITIES OF SPRAY MANIFOLDS

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Abstract. In this essentially selfcontained paper *first* we establish an intrinsic version and present a coordinate-free deduction of the so-called Rapcsák equations, which provide, in the form of 2nd order PDE-s, necessary and sufficient conditions for a Finsler structure to be projectively related to a spray. From another viewpoint, the Rapcsák equations are the conditions for the Finsler-metrizability of a spray in a broad sense. *Second*, we give a reformulation in terms of 0-homogeneous Hilbert 1-forms of both this and another metrizability problem, called Finsler-metrizability in a natural sense. (The latter is just a Finslerian version of the classical inverse problem of the calculus of variations.) *Finally*, in our main theorem we provide a reduction of the Rapcsák equations to a 1st order PDE with an algebraic condition. — The preparatory parts of the paper are devoted to a careful elaboration of the necessary technical tools, while in an Appendix the computational background is summarized.

Introduction

Two sprays on a manifold are *projectively equivalent* if the sets of their geodesics, as *geometric arcs* in the sense of [2], 8.1.4, are the same. A more formal and detailed definition may be stated as follows: two sprays S_1 and S_2 on a manifold M are called projectively equivalent, if for any geodesic $c_1 : I_1 \subset \mathbb{R} \to M$ of S_1 there exists a parameter transformation $\theta : I_2 \to I_1$ (which may be choosen to be strictly increasing) such that $c_2 := c_1 \circ \theta$ is a geodesic of S_2 — and *vice versa*. (Later we are going to present a more efficient formulation of this idea.) Having defined the concept of projective equvivalence of sprays, and keeping in mind that any Finsler manifold has a canonical spray, we can also speak of the projective equivalence of a spray manifold and a Finsler manifold, and of that of two Finsler manifolds (which have, of course, a common carrier manifold). A spray manifold (M, S) is said to be *Finsler-metrizable in a broad sense* or — following Shen's terminology [27] — *projectively Finsler*, if there exists a Finsler-Lagrangian $L : TM \to \mathbb{R}$ such that the Finsler manifold (M, L) is projectively equivalent to (M, S). If, in particular, the canonical spray of (M, L) coincides with the given spray S, then we say that (M, S) is *Finsler-metrizable in a natural sense* or that S is a *Finsler-variational*

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spray. The latter concept is a faithful analogue of the variationality of a spray (or a semispray) used in the classical inverse problem of the calculus of variations ([9], [15], [16], [18]).

The following problems naturally arise:

(1) *Find criteria for the Finsler-metrizability of a spray both in a broad sense and in a natural sense.*

(2) *Search for an "algorithm" to decide whether or not a given spray is projectively Finsler /Finsler-variational.*

In this paper we would like to contribute to a better geometrical understanding of problem (1), hoping that our results (combined e.g. with the technique applied in [15] and [16]) will serve as a good starting point for the attack on the much harder problem (2). As for problem (1), the key ingredients will be *the fundamental equations of projective equivalence*. These provide equivalent (and, in our presentation, intrinsically formulated) second order partial differential equations for the Finsler-Lagrangian to be determined. Their coordinate version was discovered by A. Rapcsák in the early sixties (see [23], [24], [25], [26] and, for a recent account, [27]), hence we also call them *Rapcsák's equations*. (Let us remark that their present form, as well as problem (2), have already been announced by the first author in [1], p. 186.)

The main theorem of this paper (baptized *alternative theorem*) gives a characterization of projectively Finsler spray manifolds through the existence of a $(0, 2)$ tensor field on the tangent manifold, satisfying some, partly quite complicated, algebraic conditions. It seems to us that this result is a (perhaps not so distant) relative of M. Crampin's stimulating intrinsic reformulation of the Helmholtz conditions from the classical inverse problem of the calculus of variations in [6]. Nevertheless, the real significance of our alternative theorem lies in the fact that it reduces the problem of Finsler-metrizabilty in a broad sense to a *first order* partial differential equation.

In view of the importance of the field we have tried to make the paper as self-explanatory as is consistent with a reasonable length. Technically we need tools from tangent bundle differential geometry and from the Frölicher-Nijenhuis calculus of vector-valued differential forms. In addition we apply Berwald-type covariant differentiation induced by a horizontal endomorphism, or in particular, a spray. Therefore in *Section 1* we give a brief resumé of background material mainly at a conceptual level, while in an Appendix we collect those rules for calculation and those formulas which we refer to throughout the paper. (A few of our formulas may be new, but "easy to prove".) As for the sources, the reader is referred to the monographs [11], [20], [31], the epoch-making papers [12], [13], [14] and the excellent survey article [19]; moreover, our earlier works [28], [29], [30] may also be useful. *Section 2* is devoted partly to a sketchy review of Berwald-type connections. (Some formulas can also be found in the Appendix.) However, the main point of this section is the derivation of two important auxiliary results (2.7, 2.8), which are crucial in the deduction of the main theorem. *Section 3* has a similar character. It seemed to us unavoidable to deal with the different types of Lagrangians in some depth, because e.g. such a fundamental notion as a Finsler structure is burdened with an abundance of confusing slight differences in different publications. The most important observations of this section are Propositions 3.3 and 3.9; the latter is the key to obtaining Rapcsák equations in an illuminating way.

After these prepartions, which certainly have interest also in their own right, the main results, including the alternative theorem, are derived in *Section 4*, which is the core of the paper.

1. Some preliminaries

1.1. Conventions. (1) Throughout this paper the letter M will stand for a connected, second countable, n-dimensional smooth manifold ($n \geq 2$). $C^\infty(M)$ denotes the ring of the real-valued *smooth functions* on M, $\mathfrak{X}(M)$ and $\mathcal{T}^r_s(M)$ are the $C^\infty(M)$-modules of *vector fields* and (r, s) *tensor fields* on M, respectively. We denote by $\Omega^k(M)$ the space of *differential k-forms* on M, $\Omega(M) := \overset{n}{\underset{k=0}{\oplus}} \Omega^k(M)$ is the *exterior algebra* of the manifold M. The distinguished derivations of the exterior algebra, the *exterior differentiation*, the *Lie differentiation* (with respect to a vector field X on M) and the *substitution operator* (induced by X) are denoted by d, $\mathcal{L}_X$ and i_X, respectively.

(2) $\pi : TM \to M$ is the *tangent bundle* of M, $\mathcal{T}M$ is the (open) set of the nonzero tangent vectors. We will remain in the smooth category, but the smoothness of some objects given on the tangent bundle will be assured only over $\mathcal{T}M$. (This is a characteristic feature of Finsler geometry.) $\mathfrak{X}^{\mathrm{v}}(TM)$ is the module of *vertical vector fields* on TM, i.e. of vector fields π-equivalent to the zero vector field on M. C denotes the *Liouville vector field* [3].

(3) A *vector k-form* on M is a k-linear (over $C^\infty(M)$) skew-symmetric map $[\mathfrak{X}(M)]^k \to \mathfrak{X}(M)$ ($k \in \mathbb{N}^*$); the $C^\infty(M)$-module of vector k-forms is denoted by $\Psi^k(M)$. $\Psi^\circ(M) := \mathfrak{X}(M)$, $\Psi^1(M) = \mathcal{T}^1_1(M)$; $\Psi(M) := \overset{n}{\underset{k=0}{\oplus}} \Psi^k(M)$. We are going to use the Frölicher-Nijenhuis calculus of vector forms and graded derivations attached to them. All the necessary practical information about this apparatus is summarized in 5.2 and 5.3; for a systematic treatment we refer to [12] and [20].

1.2. A local basis principle. (1) The *complete* and the *vertical lift* of a vector field X on M is denoted by X^c and X^{v} respectively. X^c is defined by (5.1.2a), while X^{v} is uniquely determined by (5.1.3a). It can readily be seen that

> *if* $(X_i)_{i=1}^n$ *is a local basis for the* $C^\infty(M)$*-module* $\mathfrak{X}(M)$*, then* $(X_i^{\mathrm{v}}, X_i^c)_{i=1}^n$ *is a local basis for the* $C^\infty(TM)$*-module* $\mathfrak{X}(TM)$*.*

It is therefore sufficient, in order to determine a tensor field on TM, to specify its action on the vertical and the complete lifts of vector fields on M [8].

(2) We shall apply the "*local basis principle*" formulated in (1) as a conditioned reflex. As a first, very important example, we define a $(1, 1)$ tensor field on TM by

$$JX^{\mathrm{v}} := 0, \ JX^c := X^{\mathrm{v}} \quad (X \in \mathfrak{X}(M)). \tag{1.2.1a,b}$$

J is called the *vertical endomorphism* of TTM.

(3) A useful further example: the *vertical lift* of a 1-form $\alpha \in \Omega^1(M)$ is the 1-form $\alpha^{\mathrm{v}} \in \Omega^1(TM)$ given by

$$\alpha^{\mathrm{v}}(X^{\mathrm{v}}) := 0, \quad \alpha^{\mathrm{v}}(X^c) := \big[\alpha(X)\big]^{\mathrm{v}} \qquad (X \in \mathfrak{X}(M)). \tag{1.2.2a,b}$$

More generally, the vertical lift of a k-form $\beta \in \Omega^k(M)$ $(k \geq 1)$ is defined by

$$i_{X^{\mathrm{v}}}\beta^{\mathrm{v}} := 0, \qquad \beta^{\mathrm{v}}(X_1^c, \dots, X_k^c) := \left[\beta(X_1, \dots, X_k)\right]^{\mathrm{v}}$$

$(X,\ X_i\ (1 \leq i \leq k)$ are vector fields on $M)$.

1.3. Definitions. (1) A function $f \in C^\infty(\mathcal{T}M)$, a vector field $X \in \mathfrak{X}(\mathcal{T}M)$, a differential form $\alpha \in \Omega(\mathcal{T}M)$ and a vector form $A \in \Psi(\mathcal{T}M)$ are called *homogeneous of degree* r (briefly r*-homogeneous*; $r \in \mathbb{Z}$) if the relations

$$Cf = rf, \quad [C, X] = (r-1)X, \quad \mathcal{L}_C\alpha = r\alpha, \quad [C, A] = (r-1)A$$

hold, respectively.

(2) A C^1 vector field $S : TM \to TTM$, smooth on $\mathcal{T}M$, is said to be a *semispray on* M if it satisfies the condition $JS = C$. A 2-homogeneous semispray is called a *spray*. A manifold endowed with a spray will be mentioned as a *spray manifold*.

(3) A differential form $\alpha \in \Omega(TM)$ is said to be *semibasic* if for any vector field X on TM, $i_{JX}\alpha = 0$. A vector form $A \in \Psi(TM)$ with this property is called semibasic if $J \circ A = 0$. The notion of semibasic tensors is analogous.

(4) Suppose that $\alpha \in \Omega(TM)$ and $A \in \Psi(TM)$ are semibasic forms of degree not less than 1. If S is any semispray on M, then $\alpha^\circ := i_S\alpha$ and $A^\circ := i_SA$ are called the *potential* of α and A, respectively.

1.4. Remarks. (1) Owing to the local basis principle 1.2, the *semibasic forms are completely determined by their action on the complete lifts.*

(2) Since the difference of two semisprays a vertical vector field, the potentials are well-defined. It is also clear that α° and A° remain semibasic.

1.5. Lemma. *If $\alpha \in \Omega^{k+1}(TM)$ is an r-homogeneous semibasic form, then its potential is $(r+1)$-homogeneous.*

Proof. By Remark 1.4(2), in forming α° we may take a spray S. Then, for any vector fields $X_1, \dots, X_k$ on M, we obtain:

$$\begin{aligned}
&\left[\mathcal{L}_C\,(i_S\alpha)\right](X_1^c, \dots, X_k^c) \overset{(5.2.1)}{=} C\left[i_S\alpha(X_1^c, \dots, X_k^c)\right] \\
&\quad - \sum_{i=1}^{k} i_S\alpha(X_1^c, \dots, [C, X_i^c], \dots, X_k^c) \overset{(5.1.5\mathrm{b})}{=} C\big[\alpha(S, X_1^c, \dots, X_k^c)\big] \\
&\quad \overset{(5.2.1)}{=} (\mathcal{L}_C\alpha)\,(S, X_1^c, \dots, X_k^c) + \alpha\big([C, S], X_1^c, \dots, X_k^c\big) = (r\alpha)(S, X_1^c, \dots, X_k^c) \\
&\quad + \alpha(S, X_1^c, \dots, X_k^c) = (r+1)\,(i_S\alpha)\,(X_1^c, \dots, X_k^c).
\end{aligned}$$

This means by 1.4(1) that $\mathcal{L}_C\alpha^\circ = (r+1)\alpha^\circ$. □

1.6. Proposition. *Let $\alpha \in \Omega^k(TM)$ be an r-homogeneous semibasic form, and suppose that $k + r \neq 0$. Then*

$$\alpha = \frac{1}{k+r}\left[(d_J\alpha)^\circ + d_J\alpha^\circ\right]. \tag{1.6.1}$$

If, in particular, α is d_J-closed, i.e. $d_J\alpha = 0$, then

$$\alpha = \frac{1}{k+r} d_J\alpha^\circ. \tag{1.6.2}$$

A proof can be found e.g. in [11].

1.7. Horizontal endomorphisms ([13], [21], [28], 5.4)**.** (1) A vector 1-form $h \in \Psi^1(TM)$, *smooth* — in general — *only on* $\mathcal{T}M$, is said to be a *horizontal endomorphism* on M if it is a projector (i.e., $h^2 = h$), and $\operatorname{Ker} h = \mathfrak{X}^{\mathrm{v}}(TM)$. If h is 1-homogeneous (in the sense of 1.3), then we speak of a *homogeneous* horizontal endomorphism. $\mathfrak{X}^h(TM) := \operatorname{Im} h$ is the module of $(h-)$ *horizontal vector fields* on TM.

(2) Let h be a horizontal endomorphism on M. The mapping

$$X \in \mathfrak{X}(M) \mapsto X^h := hX^c \in \mathfrak{X}^h(TM) \tag{1.7.1}$$

is called the *horizontal lifting* by h. If $(X_i)_{i=1}^n$ is a local basis for $\mathfrak{X}(M)$, then $(X_i^{\mathrm{v}}, X_i^h)_{i=1}^n$ is a local basis for the $C^\infty(TM)$-module $\mathfrak{X}(TM)$; this observation will also be used automatically (c.f. 1.2).

(3) $v := 1_{\mathfrak{X}(TM)} - h$ is the *vertical projector* belonging to the horizontal endomorphism h. $t := [J, h]$ is the *torsion vector 2-form* or *weak torsion* of h, $R := -\frac{1}{2}[h, h]$ is the *curvature vector 2-form* or simply *curvature* of h. A horizontal endomorphism is called *torsion-free* if its weak torsion vanishes. — Both t and R are semibasic vector forms, so they are completely determined by the maps

$$\begin{aligned} \tau : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) &\mapsto \tau(X, Y) \\ &:= t(X^c, Y^c) = [X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}} \end{aligned} \tag{1.7.2}$$

and

$$\varrho : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \varrho(X, Y) := R(X^c, Y^c) = -v[X^h, Y^h], \tag{1.7.3}$$

respectively.

(4) If h is a horizontal endomorphism, and $\widetilde{S}$ is a semispray on M, then $S := h\widetilde{S}$ is also a semispray, which is called the *semispray associated with* h.

(5) Let h be a horizontal endomorphism with associated semispray S. Then

$$F := h[S, h] - J \tag{1.7.4}$$

is an almost complex structure on TM (i.e., $F \in \Psi^1(TM)$ and $F^2 = -1_{\mathfrak{X}(TM)}$), called the *almost complex structure induced by* h.

1.8. Semisprays and horizontal endomorphisms. Our investigations considerably depend on the following fundamental discoveries, due to M. Crampin [5], [7] and J. Grifone [13].

(1) *Any semispray* $S : TM \to TTM$ *generates in a canonical way a horizontal endomorphism, given explicitly by the formula*

$$h = \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, S]\right). \tag{1.8}$$

Then h *is torsion-free, and its associated semispray is* $\frac{1}{2}(S + [C, S])$. *If, in particular,* S *is a spray, then* h *is homogeneous, and its associated semispray is just the given spray* S.

(2) *A horizontal endomorphism is generated by a semispray according to* (1.8) *if and only if it is torsion-free.*

1.9. Definition ([29]). A horizontal endomorphism is called a *Berwald endomorphism* if it is generated by a *spray.*

1.10. Remark. In view of 1.8(1) any Berwald endomorphism is homogeneous and torsion-free, but not *vice versa.*

2. Berwald-type connections

2.1. The operators D, D_J, D_h ([14], [28], [29]). (1) Any horizontal endomorphism $h \in \Psi^1(TM)$ gives rise to a rule of covariant differentiation

$$D : (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto D_X Y \in \mathfrak{X}(TM)$$

as follows:

(2.1.1a,b) $\quad D_{JX}JY := J[JX, Y], \quad D_{JX}hY := FD_{JX}JY = h[JX, Y];$

(2.1.2a,b) $\quad D_{hX}JY := v[hX, JY], \quad D_{hX}hY := FD_{hX}JY = hF[hX, JY]$

(F is given by (1.7.4)). Then (D, h) is automatically a *Finsler connection* in the sense of [28], this special Finsler connection is called the *Berwald-type connection* induced by h. If, in particular, h is a Berwald endomorphism, the we speak of a *Berwald connection.*

(2) For subsequent applications it will be useful to introduce further differential operators, namely

$$D_J, D_h : \mathcal{T}^r_s(TM) \to \mathcal{T}^r_{s+1}(TM)$$

arising from D by the rules

(2.1.3a,b) $$i_X D_J A := D_{JX} A, \quad i_X D_h A := D_{hX} A$$
$$(A \in \mathcal{T}^r_s(TM),\ X \in \mathfrak{X}(TM)).$$

2.2. Lemma (a J-Ricci formula). *Let (D, h) be a Berwald-type connection on M. If $\alpha \in \Omega^1(TM)$ is a semibasic 1-form, then for any vector fields X, Y, Z on TM*

$$D_J D_J \alpha(X, Y, Z) = D_J D_J \alpha(Y, X, Z).$$

The proof is easy and is omitted.

2.3. The splitting of the curvature. If (D, h) is a Finsler connection, then the classical curvature tensor field of D splits into three "partial curvatures", the so-called *horizontal, mixed*, and *vertical curvature*, denoted by $\mathcal{R}$, $\mathcal{P}$ and $\mathcal{Q}$, respectively ([14], [21], [28]). In case of a Berwald-type connection $\mathcal{Q}$ vanishes, $\mathcal{R}$ can be expressed with the help of the curvature vector 2-form R of h by the formula

(2.3.1) $$\mathcal{R}(X, Y)Z = [J, R(X, Y)]Z \qquad (X, Y, Z \in \mathfrak{X}(TM)),$$

while the mixed curvature is determined by

(2.3.2) $$\mathcal{P}(X, Y)Z = \big[[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}\big] \qquad (X, Y, Z \in \mathfrak{X}(M)).$$

For details we refer to [30] and [32].

2.4. Lemma. *Let h be a torsion-free horizontal endomorphism, (D,h) the Berwald-type connection induced by h, and R the curvature of h. Then*

$$\mathfrak{S}\left[D_h R(X,Y,Z)\right]=0,$$

where the symbol $\mathfrak{S}$ means cyclic permutation of the variables $X,Y,Z\in\mathfrak{X}(TM)$ and summation.

The proof is a straightforward calculation which we omit here, but see [30].

2.5. Corollary. *Under the preceding condition,*

$$\mathfrak{S}\,\mathcal{R}(X,Y)Z=0\qquad (X,Y,Z\in\mathfrak{X}(TM)). \tag{2.5}$$

Proof. Apply the algebraic Bianchi identity to the curvature tensor field of D, and use 2.4. □

2.6. Lemma (an h-Ricci formula)**.** *Let h be a torsion-free horizontal endomorphism on M, and (D,h) the corresponding Berwald-type connection. Then, for any 1-form α on TM and vector fields X, Y, Z on M, we have*

$$\begin{aligned} &D_hD_h\alpha(X^h,Y^h,Z^h)-D_hD_h\alpha(Y^h,X^h,Z^h)\\ &=-(D_J\alpha)(FR(X^h,Y^h),Z^h)-\alpha(F\mathcal{R}(X^h,Y^h)Z^h). \end{aligned} \tag{2.6}$$

Proof. Apply the classical Ricci formula (1.21) of [4]; split the curvature and the torsion of D according to 3.2 of [29], finally take into account that the vanishing of the weak torsion implies the coincidence of the torsion of D and the curvature of h (see e.g. [29], Corollary 3.6). □

2.7. Proposition. *Let h be a torsion-free horizontal endomorphism on M, and (D,h) be Berwald-type connection induced by h. If α is a semibasic 1-form on TM, then $d_h\alpha$ is also semibasic, and we have the following relations:*

(2.7.1) $\quad d_h\alpha=h^*d\alpha;$

(2.7.2) $\quad d_h\alpha(X^h,Y^h)=(D_h\alpha)(X^h,Y^h)-(D_h\alpha)(Y^h,X^h)\quad (X,Y\in\mathfrak{X}(M));$

(2.7.3a,b) $D_J\alpha=\beta,$ *where* $\beta(X,Y):=d\alpha(JX,Y)\quad (X,Y\in\mathfrak{X}(TM)).$

Proof. (1) Since α is semibasic, $d_h\alpha:=i_hd\alpha-di_h\alpha=i_hd\alpha-d\alpha$.
It can easily be checked that pairs of form $(X^{\mathrm{v}},Y^{\mathrm{v}}),\ (X^{\mathrm{v}},Y^h),\ (X^h,Y^{\mathrm{v}});$
$X,Y\in\mathfrak{X}(M)$ kill the 2-form $i_hd\alpha-d\alpha$, thus $d_h\alpha$ *is semibasic.*

(2) For any vector fields X, Y on M, $(i_hd\alpha-d\alpha)(X^h,Y^h)=i_hd\alpha(X^h,Y^h)-d\alpha(X^h,Y^h)=d\alpha(X^h,Y^h)=h^*d\alpha(X^h,Y^h)$; this and (1) imply (2.7.1).

(3) Since on the one hand $d_h\alpha(X^h,Y^h)\overset{(2.7.1)}{=}d\alpha(X^h,Y^h)=X^h\alpha(Y^h)-Y^h\alpha(X^h)-\alpha([X^h,Y^h])\overset{(5.4.5b)}{=}X^h\alpha(Y^h)-Y^h\alpha(X^h)-\alpha([X,Y]^h)$, on the other hand $D_h\alpha(X^h,Y^h)=(D_{X^h}\alpha)(Y^h)=X^h\alpha(Y^h)-\alpha\left(D_{X^h}Y^h\right)\overset{(5.5.2b)}{=}X^h\alpha(Y^h)-\alpha\big(F[X^h,Y^{\mathrm{v}}]\big)$, and h is torsion-free, we obtain: $D_h\alpha(X^h,Y^h)-D_h\alpha(Y^h,X^h)=$

$X^h\alpha(Y^h) - Y^h\alpha(X^h) - \alpha\big(F([X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}])\big) = X^h\alpha(Y^h) - Y^h\alpha(X^h) - \alpha(F[X,Y]^{\mathrm{v}}) \overset{(5.4.4c)}{=} d_h\alpha(X^h, Y^h)$, whence (2.7.2).

(4) From (2.7.3b) it is clear that the 2-tensor β is semibasic. So we have only to check that for any vector fields X, Y on M, $(D_J\alpha)(X^h, Y^h) = \beta(X^h, Y^h)$. But this is immediate:

$$(D_J\alpha)\,(X^h, Y^h) = (D_{X^{\mathrm{v}}}\alpha)\,(Y^h) = X^{\mathrm{v}}\alpha(Y^h) - \alpha\left(D_{X^{\mathrm{v}}}Y^h\right) \overset{(5.5.1b)}{=} X^{\mathrm{v}}\alpha(Y^h),$$

while $\beta(X^h, Y^h) = d\alpha(X^{\mathrm{v}}, Y^h) = X^{\mathrm{v}}\alpha(Y^h)$. □

2.8. Proposition. *Keeping the hypothesis of 2.7, suppose that α is d_J-closed. Then we have*

$$\underset{X,Y,Z\in\mathfrak{X}(M)}{\mathfrak{S}}\left[D_h d_h\alpha(X^h, Y^h, Z^h) + \beta(X^h, FR(Y^h, Z^h))\right] = 0. \tag{2.8}$$

Proof. (1) Under the condition $d_J\alpha = 0$ the tensor β is *symmetric*. Indeed, for any vector fields X, Y on M, $\beta(X^h, Y^h) - \beta(Y^h, X^h) \overset{(2.7.3a)}{=} D_J\alpha(X^h, Y^h) - D_J\alpha(Y^h, X^h) = X^{\mathrm{v}}\alpha(Y^h) - Y^{\mathrm{v}}\alpha(X^h) = d\alpha(X^{\mathrm{v}}, Y^h) + d\alpha(X^h, Y^{\mathrm{v}})$
$= i_J d\alpha(X^h, Y^h) = d_J\alpha(X^h, Y^h) = 0.$

(2) $\mathfrak{S}\left[D_h d_h\alpha(X^h, Y^h, Z^h)\right] \overset{(2.7.2)}{=} (D_hD_h\alpha)(X^h, Y^h, Z^h) -$
$(D_hD_h\alpha)(X^h, Z^h, Y^h) + (D_hD_h\alpha)(Y^h, Z^h, X^h) - (D_hD_h\alpha)(Y^h, X^h, Z^h) +$
$(D_hD_h\alpha)(Z^h, X^h, Y^h) - (D_hD_h\alpha)(Z^h, Y^h, X^h) \overset{(2.6)}{=}$
$-\,\mathfrak{S}\left[(D_J\alpha)(FR(X^h, Y^h), Z^h)\right] - \mathfrak{S}\,\alpha(F\mathcal{R}(X^h, Y^h)Z^h) \overset{(2.7.3a),\ (2.5)}{=}$
$-\,\mathfrak{S}\,\beta(X, FR(Y, Z)).$ □

3. Hilbert 1-forms and Lagrangians

3.1. Hilbert 1-forms. (1) A 1-form λ on TM is said to be a *Hilbert 1-form* (other terms: *Poincaré* or *Poincaré-Cartan 1-form*), if it is semibasic and d_J-closed, i.e., if $i_J\lambda = 0$ and $d_J\lambda = 0$.

(2) It follows immediately form 1.6 that any 0-homogeneous Hilbert 1-form λ is "d_J-exact", namely $\lambda = d_J\lambda^\circ$.

(3) The vertical lift of a 1-form is a trivial example of a Hilbert 1-form.

3.2. Lemma. *If λ is a 0-homogeneous Hilbert 1-form, then the 1-form $(d\lambda)^\circ$ is semibasic.*

Proof. Take a spray S on M and consider the Berwald endomorphism h generated by S. — For any vector field X on M, $(d\lambda)^\circ(X^{\mathrm{v}}) = (i_S d\lambda)(X^{\mathrm{v}}) = d\lambda(S, X^{\mathrm{v}}) = -X^{\mathrm{v}}\lambda(S) - \lambda([S, X^{\mathrm{v}}]) = -X^{\mathrm{v}}\lambda^\circ + \lambda(h[X^{\mathrm{v}}, S]) \overset{(5.4.7a)}{=} -X^{\mathrm{v}}\lambda^\circ + \lambda(X^h) \overset{3.1(2)}{=} -X^{\mathrm{v}}\lambda^\circ + d_J\lambda^\circ(X^h) = -X^{\mathrm{v}}\lambda^\circ + X^{\mathrm{v}}\lambda^\circ = 0.$ □

3.3. Proposition. *Suppose that (M,S) is a spray manifold, and let (D,h) be the Berwald connection determined by S. If λ is a 0-homogeneous Hilbert 1-form on TM, then*

$$d_h\lambda = D_J i_S d\lambda - D_S D_J \lambda. \tag{3.3}$$

Proof. (1) $d_h\lambda$ is semibasic by 2.7, and an easy calculation shows that the 2-forms $D_J i_S d\lambda$, $D_S D_J\lambda$ are also semibasic. Thus to verify (3.3) it is enough to check that the 2-form $D_S D_J\lambda - D_J i_S d\lambda + d_h\lambda$ kills the pairs of form (X^h, Y^h), where X, Y are vector fields on M.

(2) Using the representation $\lambda = d_J\lambda^\circ$, it follows that $\lambda(X^h) = X^{\rm v}\lambda(S)$.

(3) $D_S D_J\lambda(X^h, Y^h) \overset{(5.5.6c)}{=} S[D_J\lambda(X^h,Y^h)] - D_J\lambda(FvX^c, Y^h) - D_J\lambda(X^h, FvY^c) \overset{(5.5.1a,b)}{=} S(X^{\rm v}\lambda(Y^h)) - vX^c\lambda(Y^h) + \lambda(D_{vX^c}Y^h) - X^{\rm v}\lambda(FvY^c) + \lambda(D_{X^{\rm v}}FvY^c) \overset{(5.5.3a,b)}{=} S(X^{\rm v}\lambda(Y^h)) - X^c\lambda(Y^h) + X^h\lambda(Y^h) - X^{\rm v}\lambda(FvY^c) + \lambda([X,Y]^h) - \lambda(F[X^{\rm v}, Y^h])$.

(4) $D_J i_S d\lambda(X^h, Y^h) = (D_{X^{\rm v}} i_S d\lambda)(Y^h) \overset{(5.5.1b)}{=} X^{\rm v}[d\lambda(S, Y^h)] \overset{(2)}{=} X^{\rm v}(S\lambda(Y^h)) - [X^{\rm v}, Y^h]\lambda(S) - Y^h\lambda(X^h) - X^{\rm v}\lambda(FvY^c)$.

(5) In view of (3), (4) and (3) from the proof of 2.7 we obtain: $(D_S D_J\lambda - D_J i_S d\lambda + d_h\lambda)(X^h, Y^h) = [S, X^{\rm v}]\lambda(Y^h) - X^c\lambda(Y^h) + 2X^h\lambda(Y^h) + JF[X^{\rm v}, Y^h]\lambda(S) - \lambda(F[X^{\rm v}, Y^h]) \overset{(5.4.7a)}{=} JF[X^{\rm v}, Y^h]\lambda(S) - \lambda(F[X^{\rm v}, Y^h]) = d_J i_S\lambda(F[X^{\rm v}, Y^h]) - \lambda(F[X^{\rm v}, Y^h]) \overset{3.1(2)}{=} 0.$ □

3.4. Definitions. (1) A continuous function $L : TM \to \mathbb{R}$ will be mentioned as a *Lagrange function* or *Langrangian*, if it is smooth on $\mathcal{T}M$, and $L(a) = 0$ if and only if $a = 0$. The *energy function* of the Lagrangian L (briefly the *energy*) is the function $E := \frac{1}{2}L^2$. A Lagrangian L is called *regular*, if the 2-form $dd_J L$ is symplectic.

(2) A Lagrangian $L : TM \to \mathbb{R}$ is said to be a *Finsler-Lagrangian*, or a *Finsler structure* on M, if L is 1-homogeneous, and the *fundamental 2-form* $\omega := dd_J E = \frac{1}{2}dd_J L^2$ is symplectic.

(3) A manifold endowed with a Finsler structure is called a *Finsler manifold*. A horizontal endomorphism h on a Finsler manifold with energy E is *conservative* if $d_h E = 0$.

3.5. Remark. If L is a Lagrangian, then $d_J L$ is obviously a Hilbert 1-form.

3.6. Some basic facts. Let (M,L) be a Finsler manifold with the energy $E := \frac{1}{2}L^2$ and the fundamental 2-form $\omega := dd_J E$.

(1) *The Hilbert 1-form $d_J L$ is 0-homogeneous.* — The verification is immediate:

$$\mathcal{L}_C d_J L \overset{(5.3.10b)}{=} d_J \mathcal{L}_C L - d_{[J,C]} L \overset{(5.3.5a)}{=} d_J L - d_J L = 0.$$

(2) $i_C dd_J L = 0$. — Indeed, $i_C dd_J L = i_C dd_J L + d[d_J L(C)] = i_C dd_J L + di_C d_J L = \mathcal{L}_C(d_J L) \overset{(1)}{=} 0$. This means that C belongs to the null-space of $dd_J L$, therefore $dd_J L$ *is not symplectic* (c.f. [3], 1.41).

(3) $i_C dd_J E = d_J E$. — This is an easy consequence of (2).

(4) Nondegeneracy of the fundamental 2-form guarantees that for any 1-form $\alpha \in \Omega^1(\mathcal{T}M)$ there exists a unique vector field $\alpha^\#$ (read: α *sharp*) on $\mathcal{T}M$ such that $i_{\alpha^\#}\omega = \alpha$. Thus we obtain an isomorphism from $\Omega^1(\mathcal{T}M)$ onto $\mathfrak{X}(\mathcal{T}M)$, called the *sharp operator* with respect to ω.

(5) *Any Finsler manifold is a spray manifold* in a natural manner. — Indeed, if $S := -(dE)^\#$ over $\mathcal{T}M$ and $S(0) = 0$, then S is a spray on M, called the *canonical spray* of (M, L).

(6) *On any Finsler manifold there exists a unique conservative Berwald endomorphism* generated by the canonical spray according to (1.8). — This is the *fundamental lemma of Finsler geometry* due to J. Grifone [13]. The horizontal endomorphism in question we call the *Barthel endomorphism* of the Finsler manifold.

3.7. Lemma. *If (M, L) is a Finsler manifold and S is its canonical spray, then $i_S dd_J L = 0$.*

Proof. Take a vector field X on M, and let X^h be its horizontal lift by the Barthel endomorphism h. — On the one hand, $i_S dd_J L(X^{\mathrm{v}}) = dd_J L(S, X^{\mathrm{v}}) = -X^{\mathrm{v}} d_J L(S) - d_J L([S, X^{\mathrm{v}}]) = -X^{\mathrm{v}} L - J[S, X^{\mathrm{v}}]L \overset{(5.1.6)}{=} -X^{\mathrm{v}} L + X^{\mathrm{v}} L = 0$. On the other hand, $i_S dd_J L(X^h) = SX^{\mathrm{v}} L - X^h L - J[S, X^h]L \overset{(5.4.7b)}{=} SX^{\mathrm{v}} L - X^c L = [S, X^{\mathrm{v}}]L + X^{\mathrm{v}} SL - X^c L \overset{(5.4.7a)}{=} -2X^h L + X^{\mathrm{v}} SL = -2d_h L(X^h) + X^{\mathrm{v}}[d_h L(S)] = 0$, since h is conservative. □

3.8. Proposition (the fundamental relation)**.** *Let (M, S) be a spray manifold and $\bar{L}$ a Finsler-structure on M. Then, over $\mathcal{T}M$, the canonical spray $\bar{S}$ arising from $\bar{L}$ can be represented in the form*

$$\bar{S} = S - \frac{S\bar{L}}{\bar{L}} C - \bar{L}(i_S dd_J \bar{L})^\#, \tag{3.8}$$

where the sharp operator is taken with respect to the fundamental 2-form of $(M, \bar{L})$.

Proof. Consider the energy $\bar{E} := \frac{1}{2}\bar{L}^2$ of $(M, \bar{L})$. Then

$$\begin{aligned} i_S dd_J \bar{E} &= i_S d(\bar{L} d_J \bar{L}) = i_S(d\bar{L} \wedge d_J \bar{L} + \bar{L} dd_J \bar{L}) = (S\bar{L}) d_J \bar{L} - d\bar{L}(C\bar{L}) \\ &\quad + \bar{L} i_S dd_J \bar{L} = \frac{S\bar{L}}{\bar{L}} d_J \bar{E} - d\bar{E} + \bar{L} i_S dd_J \bar{L} \overset{3.6(3),(4),(5)}{=} \frac{S\bar{L}}{\bar{L}} i_C dd_J \bar{E} \\ &\quad + i_{\bar{S}} dd_J \bar{E} + \bar{L} i_{(i_S dd_J \bar{L})^\#} dd_J \bar{E} = i_{\bar{S} + \frac{S\bar{L}}{\bar{L}} C + \bar{L}(i_S dd_J \bar{L})^\#} dd_J \bar{E}, \end{aligned}$$

whence (3.8). □

4. Projective equivalence and Finsler-metrizability

4.1. Projective equivalence of sprays. We recall (cf. [10], [33] and the *Introduction*) that two sprays S and $\bar{S}$ on a manifold M are called *projectively equivalent* if there exists a smooth function $f : \mathcal{T}M \to \mathbb{R}$ such that $\bar{S} = S + fC$. Then f is 1-homogeneous *eo ipso*. On the other hand, if S is a spray on M and $f \in C^\infty(\mathcal{T}M)$ is a 1-homogeneous function, then $\bar{S} := S + fC$ is a spray again; the transition from S to $\bar{S}$ is mentioned as a *projective change* of the spray S. In these cases we also speak of the *projective equivalence of the spray manifolds* (M, S) and $(M, \bar{S})$.

4.2. Definitions. (1) Two *Finsler manifolds* (with common base manifold) are said to be *projectively equivalent* if they are projectively equivalent as spray manifolds.

(2) A spray manifold (M, S) is projectively equivalent to a Finsler manifold $(M, \bar{L})$ if S is projectively equivalent to the canonical spray of $(M, \bar{L})$. In this case (M, S) is called *Finsler-metrizable in a broad sense* or *projectively Finsler*.

(3) A spray manifold (M, S) is said to be *Finsler-metrizable in a natural sense* if there exists a Finsler-Lagrangian on M, whose canonical spray is the given spray S. Then we also say that S is *Finsler-variational*.

4.3. Theorem. *Let (M, S) be a spray manifold with Berwald connection (D, h) induced by S. If $\bar{L}$ is a Finsler-Lagrangian on M, then the following are equivalent:*

(1) (M, S) *is projectively equivalent to* $(M, \bar{L})$.

(2) $i_S dd_J \bar{L} = 0$.

(3) $d_h d_J \bar{L} = 0$.

(4) $D_h d_J \bar{L}(X, Y) = D_h d_J \bar{L}(Y, X) \quad (X, Y \in \mathfrak{X}(\mathcal{T}M))$.

Proof. Let $\bar{S}$ be the canonical spray of $(M, \bar{L})$.

(1) $\implies$ (2). In view of the assumption $\bar{S}$ can be written in the form $\bar{S} = S + fC$ $(f \in C^\infty(\mathcal{T}M))$. Thus $i_S dd_J \bar{L} = i_{\bar{S}} dd_J \bar{L} - f i_C dd_J \bar{L} \overset{3.7, 3.6(2)}{=} 0$.

(2) $\implies$ (1). This is clear by (3.8).

(2) $\implies$ (3). We first notice that $d_h \circ \mathcal{L}_C - \mathcal{L}_C \circ d_h = [d_h, \mathcal{L}_C] \overset{(5.3.10b)}{=} d_{[h,C]} = 0$. Thus $\mathcal{L}_C d_h d_J \bar{L} = d_h \mathcal{L}_C d_J \bar{L} \overset{3.6(1)}{=} 0$, which means that $d_h d_J \bar{L}$ is a 0-homogeneous 2-form. Hence, applying (1.6.1), it can be represented in the form

$$(*) \qquad d_h d_J \bar{L} = \frac{1}{2} \left[(d_J d_h d_J \bar{L})^\circ + d_J (d_h d_J \bar{L})^\circ \right].$$

By the vanishing of the weak torsion of h

$$d_J \circ d_h + d_h \circ d_J = [d_J, d_h] =: d_{[J,h]} = 0,$$

therefore

$$d_J d_h d_J \bar{L} = -d_h d_J^2 \bar{L} \overset{(5.3.7)}{=} 0.$$

In view of 3.6(1) and 3.2, $i_S dd_J \bar{L}$ is a semibasic 1-form. Hence

$$(d_h d_J \bar{L})^\circ = i_S d_h d_J \bar{L} \overset{(2.7.1)}{=} i_S h^* dd_J \bar{L} = i_S dd_J \bar{L} \overset{(2)}{=} 0.$$

We find that both terms on the right hand side of $(*)$ vanish. Thus $d_h d_J \bar{L} = 0$.

(3) $\implies$ (2). $i_S dd_J \bar{L} \overset{3.2, 3.6(1)}{=} i_S h^* dd_J \bar{L} \overset{(2.7.1)}{=} i_S d_h d_J \bar{L} \overset{(3)}{=} 0$.

(3) $\iff$ (4). Immediate from (2.7.2). □

4.4. Remark. We call the equivalent equations (2)–(4) in 4.3 the *fundamental* or the *Rapcsák equations of projective equivalence.*

4.5. Lemma. *Let α be a 1-form on M. If $\widetilde{\alpha} := (\alpha^{\mathrm{v}})^{\circ}$, then $d_J\widetilde{\alpha} = \alpha^{\mathrm{v}}$.*

Proof. Let (D, h) be a Berwald connection on M induced by the spray S. For any vector fields X, Y on M,

$$\begin{aligned}(D_J\alpha^{\mathrm{v}})(X^h, Y^h) &= (D_{X^{\mathrm{v}}}\alpha^{\mathrm{v}})(Y^h) = X^{\mathrm{v}}\alpha^{\mathrm{v}}(Y^h) - \alpha^{\mathrm{v}}(D_{X^{\mathrm{v}}}Y^h) \\ &\overset{(5.5.1b)}{=} X^{\mathrm{v}}\alpha^{\mathrm{v}}(Y^h) = X^{\mathrm{v}}\alpha^{\mathrm{v}}(Y^c) \overset{(1.2.2b)}{=} X^{\mathrm{v}}[\alpha(Y)]^{\mathrm{v}} \overset{(5.1.3b)}{=} 0.\end{aligned}$$

Thus we get

$$\begin{aligned}0 &= (D_J\alpha^{\mathrm{v}})(X^h, S) = (D_{X^{\mathrm{v}}}\alpha^{\mathrm{v}})(S) = X^{\mathrm{v}}\alpha^{\mathrm{v}}(S) - \alpha^{\mathrm{v}}(D_{X^{\mathrm{v}}}S) \\ &\overset{(5.5.6a)}{=} X^{\mathrm{v}}\widetilde{\alpha} - \alpha^{\mathrm{v}}(X^h) = X^{\mathrm{v}}\widetilde{\alpha} - \alpha^{\mathrm{v}}(X^c) = d_J\widetilde{\alpha}(X^c) - \alpha^{\mathrm{v}}(X^c),\end{aligned}$$

as claimed. □

4.6. An application. Let (M, L) be a Finsler manifold, α a 1-form on M, and let us consider the function $\widetilde{\alpha}$ defined in 4.5. If the function $\bar{L} := L + \widetilde{\alpha}$ is also a Finsler-Lagrangian, then we speak of a *Randers-change* of the Finsler structure L. In view of 4.3(2), $(M, \bar{L})$ is projectively equivalent to (M, L) if and only if

$$i_S(d\alpha)^{\mathrm{v}} = i_S d\alpha^{\mathrm{v}} \overset{4.5}{=} i_S dd_J\widetilde{\alpha} \overset{3.7}{=} i_S dd_J(L + \widetilde{\alpha}) = i_S dd_J\bar{L} = 0.$$

An immediate calculation shows that $(d\alpha)^{\mathrm{v}}$ is a 0-homogeneous, d_J-closed semibasic 2-from on TM. Hence, by (1.6.2), $(d\alpha)^{\mathrm{v}} = \frac{1}{2}d_J\big[i_S(d\alpha)^{\mathrm{v}}\big]$. — From these observations we infer the following result: *a Randers-change $\bar{L} := L + \widetilde{\alpha}$ yields a Finsler manifold projectively equivalent to (M, L) if and only if α is a closed 1-form on M.* By means of the classical tensor calculus, this was originally proved by M. Hashiguchi and Y. Ichijyō [17].

4.7. Corollary. *Let (M, S) be a spray manifold and h the Berwald endomorphism generated by S. (M, S) is projectively Finsler if and only if there exists a 0-homogeneous, d_h-closed Hilbert 1-form λ on TM such that λ° is nowhere zero on $\mathcal{T}M$, and the 2-form $dd_J(\lambda^{\circ})^2$ is symplectic.*

Proof. The *necessity* of the condition is obvious: if a Finsler manifold $(M, \bar{L})$ is projectively equivalent to (M, S), then by the Rapcsák equation 4.3(3) the Hilbert 1-form $d_J\bar{L}$ has all the required properties. — To verify the *sufficiency*, let $\bar{L} := \lambda^{\circ}$. Then $\bar{L}$ is a Finsler-Lagrangian, because it is 1-homogeneous by 1.5 and $dd_J\bar{L}^2$ is symplectic by assumption. It remains to check that $\bar{L}$ satisfies 4.3(3). Using our previous results, this is quite immediate:

$$i_S dd_J\bar{L} = i_S dd_J\lambda^{\circ} \overset{(1.6.2)}{=} i_S d\lambda \overset{3.2}{=} h^* i_S d\lambda = i_S h^* d\lambda \overset{(2.7.1)}{=} i_S d_h\lambda = 0,$$

since λ is d_h-closed. □

4.8. Corollary. *Let (M, S) be a spray manifold with Berwald connection (D, h) induced by S. S is Finsler-variational if and only if there exists a 0-homogeneous Hilbert 1-form λ on TM satisfying the conditions*

$$\text{(4.8a–c)}\qquad D_h\lambda = 0, \quad \lambda^\circ(a) = 0 \iff a = 0, \quad dd_J(\lambda^\circ)^2 \quad \textit{is symplectic.}$$

Proof. (1) Assume that $(M, \bar{L})$ is a Finsler manifold with canonical spray S. Then the Barthel endomorphism $\bar{h}$ arising from $\bar{L}$ is just the Berwald endomorphism h. Since $\lambda := d_J\bar{L}$ is a 0-homogeneous Hilbert 1-form satisfying (4.8c), our only task is to check (4.8a). — For any vector fields X, Y on M,

$$\begin{aligned}(D_h\lambda)(X^h, Y^h) &= (D_h d_J\bar{L})(X^h, Y^h) = (D_{X^h} d_J\bar{L})(Y^h) = X^h(Y^{\mathrm{v}}\bar{L}) \\ &- d_J\bar{L}(F[X^h, Y^{\mathrm{v}}]) = X^h(Y^{\mathrm{v}}\bar{L}) - [X^h, Y^{\mathrm{v}}]\bar{L} = Y^{\mathrm{v}}(X^h\bar{L}) \\ &= Y^{\mathrm{v}}\big(d_h\bar{L}(X^h)\big) = Y^{\mathrm{v}}\big(d_{\bar{h}}\bar{L}(X^h)\big) = 0\end{aligned}$$

since $\bar{h}$ is conservative. Thus (4.8a) holds.

(2) Conversely, let λ be a 0-homogeneous Hilbert 1-form on TM, satisfying (4.8a–c). Then, as we have already learnt, $\bar{L} := \lambda^\circ$ is a Finsler-Lagrangian. We show that the canonical spray of the Finsler manifold $(M, \bar{L})$ is just the given spray S, i.e. $i_S dd_J\bar{L}^2 = -d\bar{L}^2$, or equivalently,

$$(*)\qquad \big(i_S d\bar{L}\big)\lambda + \bar{L} i_S d\lambda = 0.$$

Here $i_S d\bar{L} = i_S d(i_S\lambda) = d(i_S\lambda)(S) = S(\lambda S) \overset{(5.5.7b)}{=} S(\lambda S) - \lambda(D_S S) = (D_S\lambda)(S) = D_h\lambda(S, S) \overset{(4.8a)}{=} 0$. The 1-form $i_S d\lambda$ in $(*)$ is semibasic by 3.2, and $i_S d\lambda = i_S d_h\lambda$ from the proof of 4.7. Since for any vector fields X, Y on$M$,

$$d_h\lambda(X^h, Y^h) \overset{(2.7.2)}{=} D_h\lambda(X^h, Y^h) - D_h\lambda(Y^h, X^h) \overset{(4.8a)}{=} 0,$$

the term $\bar{L} i_S d\lambda$ also vanishes in $(*)$. This concludes the proof. □

4.9. Theorem (the alternative theorem)**.** (i) *Let (M, S) be a spray manifold with Berwald connection induced by S. If (M, S) is projectively equivalent to a Finsler manifold $(M, \bar{L})$, then the tensor field*

$$\text{(4.9.1)}\qquad \bar{\mu} : (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto \bar{\mu}(X, Y) := dd_J\bar{L}(JX, Y)$$

has the properties

$$\text{(4.9.1a,b)}\qquad D_S\bar{\mu} = 0; \qquad \underset{X,Y,Z\in\mathfrak{X}(TM)}{\mathfrak{S}} \bar{\mu}(X, FR(Y, Z)) = 0.$$

(ii) *Conversely, let (M, L^*) be a Finsler manifold, and suppose that the tensor field μ^* formed from L^* by the rule of* (4.9.1) *satisfies* (4.9.2a,b)*. Then*

(I) *(M, S) is projectively equivalent to (M, L^*),* or

(II) *there exists, locally in general, a 1-form η on M such that — also locally — (M, S) is projectively equivalent to the Finsler manifold $(M, L^* + \widetilde{\eta})$,*

$\widetilde{\eta} := (\eta^{\mathrm{v}})^{\circ}$.

Proof. (i) Since $d_J\bar{L}$ is a 0-homogeneous Hilbert 1-form, $\bar{\mu} = D_J d_J \bar{L}$ by (2.7.3a), while (3.3) yields the relation $d_h d_J \bar{L} = D_J i_S dd_J \bar{L} - D_S\bar{\mu}$. By the Rapcsák equations 4.3(2),(3) this implies $D_S\bar{\mu} = 0$. Similarly, (4.9.2b) is an immediate consequence of (2.7.3a), (2.8) and 4.3(3).

(ii) If $i_S dd_J L^* = 0$, then we can stop by 4.3, so we have only to consider the case $i_S dd_J L^* \neq 0$. To make our reasoning more transparent, we divide it into several steps.

Step 1. We show that the 2-form $D_J i_S dd_J L^*$ is a vertical lift, namely: there exists a 2-form Λ on M, such that

(1) $$D_J i_S dd_J L^* = \Lambda^{\mathrm{v}}.$$

Notice first that (3.3) and (4.9.2a) yield

(2) $$D_J i_S dd_J L^* = d_h d_J L^*.$$

This implies that $D_J i_S dd_J L^*$ is *skew-symmetric*. Thus, taking into account 2.2, we infer

(3) *$D_J D_J i_S dd_J L^*$ is symmetric in its first two variables and skew-symmetric in its second two variables.*

Now we calculate. For any vector fields X, Y, Z on M,

$$\begin{aligned} D_J D_J i_S dd_J L^*(X^h, Y^h, Z^h) &\overset{(2)}{=} D_J d_h D_J L^*(X^h, Y^h, Z^h) \\ &\overset{(5.5.1b)}{=} X^{\mathrm{v}}\big[(d_h d_J L^*)(Y^h, Z^h)\big] \overset{(2.7.1)}{=} X^{\mathrm{v}}\big[(dd_J L^*)(Y^h, Z^h)\big] \\ &= X^{\mathrm{v}}\big(Y^h(Z^{\mathrm{v}}L^*) - Z^h(Y^{\mathrm{v}}L^*) - J[Y^h, Z^h]L^*\big) \\ &\overset{(5.4.5a)}{=} X^{\mathrm{v}}\big([Y^h, Z^{\mathrm{v}}]L^* + Z^{\mathrm{v}}(Y^h L^*) - [Z^h, Y^{\mathrm{v}}]L^* - Y^{\mathrm{v}}(Z^h L^*) - [Y, Z]^{\mathrm{v}} L^*\big) \\ &\overset{t=0}{=} X^{\mathrm{v}}\big(Z^{\mathrm{v}}(Y^h L^*)\big) - X^{\mathrm{v}}\big(Y^{\mathrm{v}}(Z^h L^*)\big). \end{aligned}$$

Using this result we obtain

(4) $$\underset{X,Y,Z\in\mathfrak{X}(M)}{\mathfrak{S}} D_J D_J i_S dd_J L^*(X^h, Y^h, Z^h) = 0.$$

(3) and (4) imply that $D_J(D_J i_S dd_J L^*) = 0$, from which (1) immediately follows.

Step 2. We claim that

(5) $$i_S\Lambda^{\mathrm{v}} = i_S dd_J L^*.$$

Starting with the obvious relation $D_J\Lambda^{\mathrm{v}} = 0$, we have, for any vector fields X, Y on M,

$$\begin{aligned} 0 = D_J\Lambda^{\mathrm{v}}(X^h, S, Y^h) &= (D_{X^{\mathrm{v}}}\Lambda^{\mathrm{v}})(S, Y^h) \overset{(5.5.6a)}{=} X^{\mathrm{v}}\Lambda^{\mathrm{v}}(S, Y^h) - \Lambda^{\mathrm{v}}(X^h, Y^h) \\ &= (D_J i_S\Lambda^{\mathrm{v}} - \Lambda^{\mathrm{v}})(X^h, Y^h). \end{aligned}$$

Thus $D_J i_S \Lambda^{\mathrm{v}} = D_J i_S dd_J L^*$, and hence

$$i_S dd_J L^* = i_S \Lambda^{\mathrm{v}} + \alpha^{\mathrm{v}}, \qquad \alpha \in \Omega^1(M).$$

But $i_S dd_J L^*$ and $i_S \Lambda^{\mathrm{v}}$ are both 1-homogeneous (see 1.5) while α^{v} is 0-homogeneous, so α^{v} must vanish.

Step 3. We show that the 2-form Λ^{v} *is closed*. — By (2.8) and (4.9.2b),

$$\begin{aligned}
0 &= \underset{X,Y,Z\in\mathfrak{X}(M)}{\mathfrak{S}} D_h d_h d_J L^*(X^h, Y^h, Z^h) \overset{(1),(2)}{=} \underset{X,Y,Z\in\mathfrak{X}(M)}{\mathfrak{S}} (D_h\Lambda^{\mathrm{v}})(X^h, Y^h, Z^h) \\
&\overset{(5.5.2b)}{=} X^h\Lambda^{\mathrm{v}}(Y^h, Z^h) - Y^h\Lambda^{\mathrm{v}}(X^h, Z^h) + Z^h\Lambda^{\mathrm{v}}(X^h, Y^h) \\
&\quad - \Lambda^{\mathrm{v}}(F([X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}]), Z^h) + \Lambda^{\mathrm{v}}(F([X^h, Z^{\mathrm{v}}] - [Z^h, X^{\mathrm{v}}]), Y^h) \\
&\quad - \Lambda^{\mathrm{v}}(F([Y^h, Z^{\mathrm{v}}] - [Z^h, Y^{\mathrm{v}}]), X^h) \overset{t=0,\ (5.4.4c)}{=} X^h\Lambda^{\mathrm{v}}(Y^h, Z^h) \\
&\quad - Y^h\Lambda^{\mathrm{v}}(X^h, Z^h) + Z^h\Lambda^{\mathrm{v}}(X^h, Y^h) - \Lambda^{\mathrm{v}}([X, Y]^h, Z^h) \\
&\quad + \Lambda^{\mathrm{v}}\big([X, Z]^h, Y^h) - \Lambda^{\mathrm{v}}([Y, Z]^h, X^h).
\end{aligned}$$

Thus, since $[X, Y]^h = [X^h, Y^h] - v[X^h, Y^h]$, *etc.*, and Λ^{v} is semibasic, we obtain

$$\begin{aligned}
0 &= X^h\Lambda^{\mathrm{v}}(Y^h, Z^h) - Y^h\Lambda^{\mathrm{v}}(X^h, Z^h) + Z^h\Lambda^{\mathrm{v}}(X^h, Y^h) - \Lambda^{\mathrm{v}}([X^h, Y^h], Z^h) \\
&\quad + \Lambda^{\mathrm{v}}([X^h, Z^h], Y^h) - \Lambda^{\mathrm{v}}([Y^h, Z^h], X^h) = (d\Lambda^{\mathrm{v}})(X^h, Y^h, Z^h)
\end{aligned}$$

whence our claim.

Step 4. Since Λ^{v} is closed and $d\Lambda^{\mathrm{v}} = (d\Lambda)^{\mathrm{v}}$, Λ is also closed. Thus there exists, at least locally, a 1-form η on M such that $d\eta = \Lambda$. Consider the function $\bar{L} := L^* - \widetilde{\eta}$, $\widetilde{\eta} := (\eta^{\mathrm{v}})^\circ$, defined on a region of TM. We show that $\bar{L}$ satisfies 4.3(2). Notice first that for any vector fields X, Y on M,

$$\begin{aligned}
dd_J\widetilde{\eta}(X^c, Y^c) &= X^c(Y^{\mathrm{v}}\widetilde{\eta}) - Y^c(X^{\mathrm{v}}\widetilde{\eta}) - J[X^c, Y^c]\widetilde{\eta} \overset{4.5,(5.1.4c),(1.2.1b)}{=} X^c(\eta(Y))^{\mathrm{v}} \\
&\quad - Y^c(\eta(X))^{\mathrm{v}} - (\eta([X, Y]))^{\mathrm{v}} = [(d\eta)(X, Y)]^{\mathrm{v}} = [\Lambda(X, Y)]^{\mathrm{v}} = \Lambda^{\mathrm{v}}(X^c, Y^c).
\end{aligned}$$

Since $dd_J\widetilde{\eta}$ is clearly semibasic, this means that $dd_J\widetilde{\eta} = \Lambda^{\mathrm{v}}$. Thus

$$i_S dd_J \bar{L} = i_S dd_J L^* - i_S dd_J\widetilde{\eta} = i_S dd_J L^* - i_S\Lambda^{\mathrm{v}} \overset{(5)}{=} i_S\Lambda^{\mathrm{v}} - i_S\Lambda^{\mathrm{v}} = 0,$$

concluding the proof of (II). □

5. Appendix: a formulary

5.1. Lifts. Suppose that f is a smooth function, X and Y are vector fields on the manifold M.

(5.1.1a,b) $f^{\mathrm{v}} := f \circ \pi$, $f^c := Sf^{\mathrm{v}}$ ($S : TM \to TTM$ is a semispray).

(5.1.2a,b) $X^c f^c := (Xf)^c$, $X^c f^{\mathrm{v}} = (Xf)^{\mathrm{v}}$.

(5.1.3a,b) $X^{\mathrm{v}} f^c = (Xf)^{\mathrm{v}}$, $X^{\mathrm{v}} f^{\mathrm{v}} = 0$.

(5.1.4a–c) $[X^{\mathrm{v}}, Y^{\mathrm{v}}] = 0$, $[X^c, Y^{\mathrm{v}}] = [X, Y]^{\mathrm{v}}$, $[X^c, Y^c] = [X, Y]^c$.

(5.1.5a,b) $[C, X^{\mathrm{v}}] = -X^{\mathrm{v}}$, $[C, X^c] = 0$.

(5.1.6) $J[X^{\mathrm{v}}, S] = X^{\mathrm{v}}$.

5.2. Derivations and vector forms.

(5.2.1) Let $A \in \mathcal{T}^r_s(N)$ and D be *tensor derivation* on M ([22], p. 43).

If θ^i $(1 \leq i \leq r)$ are 1-forms X_j $(1 \leq j \leq s)$ are vector fields on M, then

$$(DA)\ (\theta^1, \dots, \theta^r, X_1, \dots, X_s) = D[A(\theta^1, \dots, \theta^r, X_1, \dots, X_s)] - \sum_{i=1}^{r} A(\theta^1, \dots, D\theta^i, \dots, \theta^r, X_1, \dots, X_s) - \sum_{j=1}^{s} A(\theta^1, \dots, \theta^r, X_1, \dots, DX_j, \dots, X_s)$$

(the *product rule*).

(5.2.2) If D_1 and D_2 are *graded derivations* ([20], 8.1) of degrees r and s $(r, s \in \mathbb{Z})$ respectively of $\Omega(M)$, then their *graded commutator* is $[D_1, D_2] := D_1 \circ D_2 - (-1)^{rs} D_2 \circ D_1$.

(5.2.3) If $K \in \Psi^k(M)$, then i_K is a graded derivation of degree $k-1$ of $\Omega(M)$; $i_K \restriction C^\infty(M) := 0,\ i_K\alpha := \alpha \circ K \quad (\alpha \in \Omega^1(M))$.

(5.2.4) $d_K := [i_K, d] = i_K \circ d - (-1)^{k-1} d \circ i_K \quad (K \in \Psi^k(M))$.

(5.2.5) $d_K f = i_K df = df \circ K \quad (K \in \Psi^k(M),\ f \in C^\infty(M))$.

(5.2.6) $d_{[K,L]} := [d_K, d_L]$; $[K, L]$ is the *Frölicher-Nijenhuis* bracket of $K \in \Psi^k(M)$ and $L \in \Psi^\ell(M)$.

5.3. Formulas concerning vector 1-forms. Suppose that K and L are vector *1-forms*; X, Y and X_j $(1 \leq j \leq \ell)$ are vector fields on the manifold M.

(5.3.1) $i_K\alpha(X_1, \dots, X_\ell) = \sum_{j=1}^{\ell} \alpha(X_1, \dots, KX_j, \dots, X_\ell)\ (\alpha \in \Omega^\ell(M))$.

(5.3.2) $K^*\alpha(X_1, \dots, X_\ell) := \alpha(K(X_1), \dots, K(X_\ell))\ (\alpha \in \Omega^\ell(M))$.

(5.3.3.) If $\alpha \in \Omega^1(M)$, then $i_K\alpha = K^*\alpha$.

(5.3.4) $[K, Y]X = [KX, Y] - K[X, Y]$.

(5.3.5a–c) $[J, C] = J, \quad [J, X^{\mathrm{v}}] = 0, \quad [J, X^{\mathrm{c}}] = 0$.

(5.3.6) $[K, L](X, Y) = [KX, LY] + [LX, KY] + K \circ L[X, Y] + L \circ K[X, Y] - K[LX, Y] - K[X, LY] - L[KX, Y] - L[X, KY]$.

(5.3.7) $[d_J, d_J] = 2d_J^2 = 0 \implies [J, J] = 0$.

(5.3.8a,b) $[i_X, i_K] = i_{KX}, \quad [i_K, \mathcal{L}_X] = i_{[K,X]}$.

(5.3.9) $[i_K, i_L] = i_{L \circ K} - i_{K \circ L}$.

(5.3.10a,b) $[i_X, d_K] = \mathcal{L}_{KX} + i_{[K,X]}, \quad [d_K, \mathcal{L}_X] = d_{[K,X]}$.

(5.3.11) $[i_K, d_L] = d_{L \circ K} - i_{[K,L]}$.

5.4. Horizontal endomorphisms and lifts. Keeping the notations of 1.7, consider a horizontal endomorphism h. Let X and Y be vector fields on M.

(5.4.1a,b) $h \circ J = 0,\ J \circ h = J$.

(5.4.2a,b) $F \circ J = h,\ J \circ F = v$.

(5.4.3a–c) $F \circ h = -J,\ F \circ v = h \circ F,\ v \circ F = -J$.

(5.4.4a–c) $JX^h = X^{\mathrm{v}},\ FX^h = -X^{\mathrm{v}},\ FX^{\mathrm{v}} = X^h.$

(5.4.5a,b) $J[X^h, Y^h] = [X, Y]^{\mathrm{v}},\ h[X^h, Y^h] = [X, Y]^h.$

(5.4.6) $[J, vX^c] = [h, X^{\mathrm{v}}],$ if $t = 0$.

(5.4.7a,b) If h is generated by a semispray S, then

$$X^h = \frac{1}{2}\,(X^c + [X^{\mathrm{v}}, S]),\ J[X^h, S] = X^h - X^c.$$

5.5. Berwald-type connections. Let h be a horizontal endomorphism, (D, h) the corresponding Berwald-type connection, X and Y vector fields on M.

(5.5.1a,b) $D_{X^{\mathrm{v}}} Y^{\mathrm{v}} = 0,\ D_{X^{\mathrm{v}}} Y^h = 0.$

(5.5.2a,b) $D_{X^h} Y^{\mathrm{v}} = [X^h, Y^{\mathrm{v}}],\ D_{X^h} Y^h = F[X^h, Y^{\mathrm{v}}].$

(5.5.3a,b) $D_{X^{\mathrm{v}}} FvY^c = [X, Y]^h - F[X,^{\mathrm{v}}, Y^h],\ D_{vX^c} Y^h = 0.$

(5.5.4) $D_J C = J.$

(5.5.5) If h is homogeneous, then $D_h C = 0$.

(5.5.6a–c) If h is generated by the semispray S, then
$D_{X^{\mathrm{v}}} S = X^h,\ D_S X^{\mathrm{v}} = X^c - X^h,\ D_S X^h = FvX^c.$

(5.5.7a,b) If h is generated by the spray S, i.e., (D, h) is the *Berwald connection* of (M, S), then $D_{X^h} S = 0,\ D_S S = 0.$

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