Good metric derivatives for a class of Moór – Vanstone metrics

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Abstract

By a Moór – Vanstone metric we mean a pseudo-Riemannian metric which is defined in the pull-back bundle of a tangent bundle over itself and which satisfies some additional conditions. In this context, we say that a metric covariant derivative is 'good' if it is associated to an Ehresmann connection determined by the metric alone. Every Ehresmann connection determines a metric covariant derivative, the so-called Miron's derivative, in a natural manner. The main result of this paper is the following: an Ehresmann connection of a certain type is associated to Miron's derivative arising from it if and only if it satisfies a mixed system of a partial differential equation and two algebraic equations. We also determine all the solutions of that system.

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1 Introduction

A *(generalized) metric* on a manifold is a symmetric, non-degenerate, type (0,2) tensor in the pull-back bundle of the tangent bundle over itself. This type of metrics has obviously a velocity-dependent character. For technical reasons, we shall assume that they are defined only on the open submanifold of non-zero tangent vectors rather than on the whole tangent manifold. A *Moór – Vanstone metric* is a homogeneous metric whose absolute energy is a (pseudo-)Finslerian energy function. This special class of generalized metrics was introduced by the Hungarian Finslerist A. Moór and the theoretical physicist J. I. Horváth in 1955 [4]. The next year A. Moór, and a few years later, in 1962, the excellent Canadian geometer J. R. Vanstone enriched the subject with further important contributions [10, 13]. We note that in [4] the new class of metrics also obtained an interesting application to Yukawa's bilocal field theory – an aspect which deserves serious attention. Generalized metrics proved to be useful also in relativistic optics [9].

A covariant derivative operator in our pull-back bundle is *metric* if the covariant differential of the metric tensor with respect to it vanishes. In 1987, M. Matsumoto, the leading Finslerist of the last decades, drew up the following important heuristic principle: *'Through the author's several experiences, he became convinced that there should exist the* best *Finsler connection for every theory of Finsler spaces'* [6]. In this paper, we try to extend Matsumoto's principle to the world of Moór – Vanstone metrics, and to find the 'best' metric derivative, or, at least, some 'good' metric derivatives for one of their special classes.

If an Ehresmann connection is specified over the manifold, there is a unique metric covariant derivative in the pull-back bundle whose vertical torsion vanishes and whose horizontal torsion coincides with the torsion of the given Ehresmann connection. We call this covariant derivative *Miron's derivative* arising from the Ehresmann connection [8, 11]. A covariant derivative is *associated* to an Ehresmann connection if the kernel of its deflection is the horizontal subspace of the Ehresmann connection at each point of the tangent manifold. All the notable covariant derivatives in Finsler geometry enjoy this property, even the non-metric ones. Thus, by the 'goodness' of a metric derivative we shall informally mean that it is associated to an Ehresmann connection *determined by the metric alone.* In the case of Moór – Vanstone metrics, we have a natural Ehresmann connection: the Barthel connection $\mathcal{H}_E$ arising from the absolute energy E of the metric. However, Miron's derivative constructed from $\mathcal{H}_E$ is not 'good' in the sense above. Thus, to obtain 'good' metric derivatives for this class of metrics, we have to search more suitable Ehresmann connections than $\mathcal{H}_E$.

The new Ehresmann connection $\mathcal{H}$ is completely determined by its difference tensor P from $\mathcal{H}_E$, therefore we shall look for P rather than directly for $\mathcal{H}$. We shall also assume that the absolute energy E is a first integral of the $\mathcal{H}$-horizontal vector fields and that the semispray associated to $\mathcal{H}$ coincides with the canonical spray belonging to the Finsler energy E. All these (geometrically natural) conditions, together with the 'goodness' of Miron's derivative arising from $\mathcal{H}$, will yield a mixed system of a partial differential equation and two algebraic equations for P. This system will be linear, thus we may determine its solutions as the sum of the general solution of the homogeneous part and a particular solution. We shall find the general solution of the homogeneous part by decomposing it into the sum of a symmetric and a skew-symmetric term. In establishing our partial differential equation, the unpublished manuscript [5] was as inspiring as helpful for us, in spite of the fact that it contains some mistakes.

2 Basic setup

2.1. Throughout this paper M denotes an n-dimensional smooth manifold whose topology is Hausdorff, 2nd countable and connected. The $\mathbb{R}$-algebra of smooth real-valued functions on M is denoted by $C^\infty(M)$. The $2n$-dimensional tangent manifold of M is TM, and $\overset{\circ}{T}M$ stands for the open submanifold of non-zero tangent vectors to M.

2.2. The tangent bundle of M is $\tau : TM \to M$, while $\overset{\circ}{\tau} : \overset{\circ}{T}M \to M$ is the deleted bundle for τ. The tangent bundle of the latter is $\tau_{\overset{\circ}{T}M} : T\overset{\circ}{T}M \to \overset{\circ}{T}M$. $\overset{\circ}{\tau}{}^*\tau$ is the *pull-back of* τ *over* $\overset{\circ}{\tau}$. The total space of this vector bundle is the fibre product $\overset{\circ}{T}M \times_M TM$, the base manifold is $\overset{\circ}{T}M$, and the fibre over $v \in \overset{\circ}{T}M$ is $\{v\} \times T_{\overset{\circ}{\tau}(v)}M \cong T_{\overset{\circ}{\tau}(v)}M$. $\mathfrak{X}(\overset{\circ}{\tau})$ denotes the $C^\infty\big(\overset{\circ}{T}M\big)$-module of (smooth) sections of $\overset{\circ}{\tau}{}^*\tau$, and its typical elements are denoted by $\tilde{X}, \tilde{Y} \dots$. These sections are also mentioned as *vector fields along* $\overset{\circ}{\tau}$. The *canonical section* δ of $\overset{\circ}{\tau}{}^*\tau$ duplicates the elements of $\overset{\circ}{T}M$, i.e.,

$$\delta : v \in \overset{\circ}{T}M \mapsto \delta(v) := (v, v) \in \overset{\circ}{T}M \times_M TM.$$

Every vector field X on M induces a vector field $\hat{X}$ along $\overset{\circ}{\tau}$ by 'parallel propagation', i.e., by the rule

$$v \in \overset{\circ}{T}M \mapsto \hat{X}(v) := \Big(v, X\big(\overset{\circ}{\tau}(v)\big)\Big) \in \overset{\circ}{T}M \times_M TM.$$

$\hat{X}$ is called a *basic vector field* along $\overset{\circ}{\tau}$. The set of basic vector fields generates the module $\mathfrak{X}(\overset{\circ}{\tau})$.

2.3. The *basic short exact sequence* is the sequence

$$0 \to \overset{\circ}{T}M \times_M TM \xrightarrow{\mathbf{i}} T\overset{\circ}{T}M \xrightarrow{\mathbf{j}} \overset{\circ}{T}M \times_M TM \to 0,$$

where $\mathbf{i}$ identifies the fibre $\{v\} \times T_{\overset{\circ}{\tau}(v)}M$ of $\overset{\circ}{\tau}{}^*\tau$ with the tangent space $T_vT_{\overset{\circ}{\tau}(v)}M$ for all $v \in \overset{\circ}{T}M$, and $\mathbf{j} = \Big(\tau_{\overset{\circ}{T}M}, \overset{\circ}{\tau}_*\Big)$ ($\overset{\circ}{\tau}_*$ denotes the differential of $\overset{\circ}{\tau}$). The (strong) bundle maps $\mathbf{i}, \mathbf{j}$ induce the $C^\infty\left(\overset{\circ}{T}M\right)$-homomorphisms

$$\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \mathbf{i}\tilde{X} := \mathbf{i} \circ \tilde{X} \in \mathfrak{X}\left(\overset{\circ}{T}M\right), \ \xi \in \mathfrak{X}\left(\overset{\circ}{T}M\right) \mapsto \mathbf{j}\xi := \mathbf{j} \circ \xi \in \mathfrak{X}(\overset{\circ}{\tau}),$$

which will also be denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively. For any vector field X on M, $X^v := \mathbf{i}\hat{X}$ is the *vertical lift* of X. With the help of $\mathbf{i}$ and $\mathbf{j}$ it is possible to define two additional canonical objects, the *Liouville vector field* $C := \mathbf{i}\delta$ and the *vertical endomorphism* $J := \mathbf{i} \circ \mathbf{j}$. A *second-order vector field* or a *semispray* over M is a vector field χ on $\overset{\circ}{T}M$ such that $J\chi = C$ (or equivalently, $\mathbf{j}\chi = \delta$). If, in addition, $[C, \chi] = \chi$, i.e. χ is positively homogeneous of degree 2, then χ is called a *spray*.

2.4. For a given vector field $\tilde{X}$ along $\overset{\circ}{\tau}$ a 'covariant-like' differential operator, the *(canonical) v-covariant derivative* $\nabla^v_{\tilde{X}}$ can be constructed as follows. Let

$$\begin{cases} \nabla^v_{\tilde{X}} F := \left(\mathbf{i}\tilde{X}\right) F, & \text{if } F \in C^\infty\left(\overset{\circ}{T}M\right); \\ \nabla^v_{\tilde{X}} \tilde{Y} := \mathbf{j}\left[\mathbf{i}\tilde{X}, \eta\right], & \text{if } \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}), \text{ and } \eta \in \mathfrak{X}\left(\overset{\circ}{T}M\right) \text{ such that } \mathbf{j}\eta = \tilde{Y}. \end{cases}$$

Then $\nabla^v_{\tilde{X}} \tilde{Y}$ is well-defined, and $\nabla^v_{\tilde{X}}$ may be uniquely extended to be a tensor derivation of the tensor algebra of $\mathfrak{X}(\overset{\circ}{\tau})$. For any tensor A along $\overset{\circ}{\tau}$ the v-covariant *differential* $\nabla^v A$ collects in the usual manner all the v-covariant derivatives of A. In particular, if $F : \overset{\circ}{T}M \to \mathbb{R}$ is a smooth function, then the second v-covariant differential $g_F := \nabla^v \nabla^v F$ is a symmetric, type (0,2) tensor along $\overset{\circ}{\tau}$, called the *Hessian* of F.

2.5. By an *Ehresmann connection* (or a *nonlinear connection*) $\mathcal{H}$ on $\overset{\circ}{T}M$ we mean a right splitting of the basic short exact sequence, i.e. a strong bundle map $\mathcal{H} : \overset{\circ}{T}M \times_M TM \to T\overset{\circ}{T}M$ such that $\mathbf{j} \circ \mathcal{H} = 1_{\overset{\circ}{T}M \times_M TM}$. The type (1,1) tensor field $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ on $\overset{\circ}{T}M$ is the *horizontal projector* belonging to $\mathcal{H}$, and $\mathbf{v} := 1_{T\overset{\circ}{T}M} - \mathbf{h}$ is the complementary *vertical projector*. If $\mathcal{V} : T\overset{\circ}{T}M \to \overset{\circ}{T}M \times_M TM$ is a left splitting of the basic exact sequence such that $\operatorname{Ker} \mathcal{V} = \operatorname{Im} \mathcal{H}$, then $\mathcal{V}$ is called the *vertical map* associated to $\mathcal{H}$. The bundle maps $\mathcal{H}$ and $\mathcal{V}$ induce $C^\infty\left(\overset{\circ}{T}M\right)$-homomorphisms at the level of sections by the rules

$$\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \mathcal{H}\tilde{X} := \mathcal{H} \circ \tilde{X} \in \mathfrak{X}\left(\overset{\circ}{T}M\right), \ \xi \in \mathfrak{X}\left(\overset{\circ}{T}M\right) \mapsto \mathcal{V}\xi := \mathcal{V} \circ \xi \in \mathfrak{X}(\overset{\circ}{\tau}).$$

For any vector field X on M, $X^h := \mathcal{H}\hat{X}$ is the *horizontal lift* of X (with respect to $\mathcal{H}$). If χ is a semispray over M, then there is a unique Ehresmann connection $\mathcal{H}_\chi : \overset{\circ}{T}M \times_M TM \to T\overset{\circ}{T}M$ such that

$$\mathcal{H}_\chi \hat{X} = \frac{1}{2}(X^c + [X^v, \chi]) \quad \text{for all } X \in \mathfrak{X}(M),$$

where X^c is the complete lift of X. This Ehresmann connection is said to be the *Crampin – Grifone connection* arising from χ [1, 3].

2.6. Suppose

$$D : \mathfrak{X}\left(\overset{\circ}{T}M\right) \times \mathfrak{X}(\overset{\circ}{\tau}) \to \mathfrak{X}(\overset{\circ}{\tau}),\ \left(\xi, \tilde{Y}\right) \mapsto D_\xi \tilde{Y}$$

is a covariant derivative operator in the vector bundle $\overset{\circ}{\tau}{}^*\tau$. By the *vertical torsion* of D we mean the tensor $\mathcal{Q}$ along $\overset{\circ}{\tau}$ given by

$$\mathcal{Q}\left(\tilde{X}, \tilde{Y}\right) := D_{\mathbf{i}\tilde{X}}\tilde{Y} - D_{\mathbf{i}\tilde{Y}}\tilde{X} - \mathbf{i}^{-1}\left[\mathbf{i}\tilde{X}, \mathbf{i}\tilde{Y}\right].$$

In the presence of an Ehresmann connection $\mathcal{H}$, the *horizontal torsion* $\mathcal{T}$ of D is the type (1,2) tensor along $\overset{\circ}{\tau}$ such that

$$\mathcal{T}\left(\tilde{X}, \tilde{Y}\right) := D_{\mathcal{H}\tilde{X}}\tilde{Y} - D_{\mathcal{H}\tilde{Y}}\tilde{X} - \mathbf{j}\left[\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}\right].$$

By the *deflection* of D we mean the covariant differential $\mu := D\delta$ of the canonical section. The type (1,1) tensor field

$$\tilde{\mu} := \mu \circ \mathbf{i} : \tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \tilde{\mu}\tilde{X} = D_{\mathbf{i}\tilde{X}}\delta \in \mathfrak{X}(\overset{\circ}{\tau})$$

along $\overset{\circ}{\tau}$ is the *v-deflection* of D. The covariant derivative D is called *regular* if $\tilde{\mu}$ is (pointwise) invertible, and *strongly regular* if $\tilde{\mu} = 1_{\mathfrak{X}(\overset{\circ}{\tau})}$. If an Ehresmann connection $\mathcal{H}$ is also specified, we can introduce the *h-deflection*

$$\mu^h := \mu \circ \mathcal{H} : \tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \mu^h\tilde{X} = D_{\mathcal{H}\tilde{X}}\delta \in \mathfrak{X}(\overset{\circ}{\tau}) =: \left(D^h\delta\right)\tilde{X}.$$

The covariant derivative D is said to be *associated to* $\mathcal{H}$ if $\operatorname{Im}\mathcal{H} = \operatorname{Ker}\mu$, and *strongly associated to* $\mathcal{H}$ if its deflection is just the vertical map $\mathcal{V}$ determined by $\mathcal{H}$, i.e. $\mu = \mathcal{V}$. It follows immediately that D is associated to $\mathcal{H}$ if and only if it is regular and its h-deflection vanishes, and it is strongly associated to $\mathcal{H}$ if and only if it is strongly regular and its h-deflection vanishes. If $\mathcal{H} : \overset{\circ}{T}M \times_M TM \to T\overset{\circ}{T}M$ is an Ehresmann connection, then the rules

$$\nabla_{\mathcal{H}\tilde{X}}\tilde{Y} := \mathcal{V}\left[\mathcal{H}\tilde{X}, \mathbf{i}\tilde{Y}\right], \quad \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla^v_{\tilde{X}}\tilde{Y}; \quad \tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$$

define a covariant derivative operator in $\overset{\circ}{\tau}{}^*\tau$, called *Berwald's derivative* induced by $\mathcal{H}$. The h-deflection $\nabla^h\delta =: \mathbf{t}$ of Berwald's derivative is called the *tension* of $\mathcal{H}$. By the *torsion* of $\mathcal{H}$ we mean the horizontal torsion of ∇.

3 Generalized metrics and Miron's construction

3.1. By a *generalized metric* (also called a *generalized Finsler* or a *generalized Lagrange* metric), briefly a *metric* we mean a pseudo-Riemannian metric in the pull-back bundle $\overset{\circ}{\tau}{}^*\tau$, i.e., a smooth assignment of non-degenerate scalar products g_v to the fibres $T_{\overset{\circ}{\tau}(v)}M$, $v \in \overset{\circ}{T}M$. The 1-form

$$\vartheta_g : \tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \vartheta_g \tilde{X} := g\left(\tilde{X}, \delta\right) \in C^\infty\left(\overset{\circ}{T}M\right)$$

along $\overset{\circ}{\tau}$ is the *Lagrange 1-form* associated to g. The smooth function $E := \frac{1}{2}g(\delta,\delta)$ on $\overset{\circ}{T}M$ is called the *absolute energy* of g. If the Hessian $g_E = \nabla^v\nabla^v E$ is non-degenerate, we say that the metric g is *energy-regular*. If $\nabla^v_\delta g = 0$, then g is called *homogeneous*. If a metric is energy-regular and homogeneous, then we speak of a *Moór – Vanstone metric*.

3.2. By a *Finsler energy* on $\overset{\circ}{T}M$ we mean a smooth function $E : \overset{\circ}{T}M \to \mathbb{R}$ which is positively homogeneous of degree 2 and has the property that $g_E = \nabla^v\nabla^v E$ is non-degenerate, i.e. it is a metric in the sense above. Then we say that g_E is a *Finsler metric* on $\overset{\circ}{T}M$. (Our concept of a Finsler energy is more general than the usual one since the positiveness of E is not required. It is a nice problem to prove that if E is positive everywhere, then the metric tensor g_E is positive definite.) The absolute energy of g_E is just the given Finsler energy E, and g_E is homogeneous. Thus any Finsler metric is a Moór – Vanstone metric. The converse is, of course, definitely false in general. The *canonical spray* χ for the Finsler energy E is determined via the Lagrange equation

$$i_\chi d(dE \circ J) = -dE.$$

By the Crampin – Grifone construction, χ induces an Ehresmann connection $\mathcal{H}_E$, called the *Barthel connection* of the Finsler manifold (M, E). The Berwald derivative arising from $\mathcal{H}_E$ will be denoted by $\overset{E}{\nabla}$.

3.3. The *(covariant) Cartan tensor* of g is the v-covariant differential $\mathcal{C}_\flat := \nabla^v g$ of the metric. The type (1,2) tensor $\mathcal{C}$ along $\overset{\circ}{\tau}$ determined by

$$g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \tilde{Z}\right) = \mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right); \quad \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$$

i.e., by a raising operation, is also called the *Cartan tensor* of g. If g is a Finsler metric, then $\mathcal{C}_\flat$ is totally symmetric. In general, it is symmetric only in its last two variables, and this is one of the main sources of the difficulties one has to face studying generalized metrics. Notice that g is homogeneous if and only if $\mathcal{C}\left(\delta, \tilde{X}\right) = 0$ for all $\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})$.

3.4 Definition. *Let g be a metric in $\overset{\circ}{\tau}{}^*\tau$. We say that g is* weakly normal *if $\mathcal{C}_\flat\left(\tilde{X}, \delta, \delta\right) = 0$ for all $\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})$.*

This class can be characterized as follows [7].

3.5 Proposition. *A metric g is weakly normal if and only if $\vartheta_g = \nabla^v E$. The absolute energy of a weakly normal metric is positive-homogeneous of degree 2.*

3.6. Let a metric g in $\overset{\circ}{\tau}{}^*\tau$ and an Ehresmann connection $\mathcal{H}$ over M be given. There is a unique metric covariant derivative D in $\overset{\circ}{\tau}{}^*\tau$ with vanishing vertical torsion such that the horizontal torsion of D coincides with the torsion of $\mathcal{H}$. With the help of Berwald's derivative ∇ induced by $\mathcal{H}$, D may be constructed as follows:

(1) We introduce the h-Cartan tensor $\mathcal{C}^h$ of g (with respect to $\mathcal{H}$) by means of the relation $g\left(\mathcal{C}^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) := (\nabla_{\mathcal{H}\tilde{X}}g)\left(\tilde{Y},\tilde{Z}\right)$.

(2) Using Christoffel's trick, we define two new tensors $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ along $\overset{\circ}{\tau}$:

$$g\left(\overset{\circ}{\mathcal{C}}\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) := g\left(\mathcal{C}\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) + g\left(\mathcal{C}\left(\tilde{Y},\tilde{Z}\right),\tilde{X}\right) - g\left(\mathcal{C}\left(\tilde{Z},\tilde{X}\right),\tilde{Y}\right),$$
$$g\left(\overset{\circ}{\mathcal{C}}{}^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) := g\left(\mathcal{C}^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) + g\left(\mathcal{C}^h\left(\tilde{Y},\tilde{Z}\right),\tilde{X}\right) - g\left(\mathcal{C}^h\left(\tilde{Z},\tilde{X}\right),\tilde{Y}\right).$$

(3) If

$$D_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\tilde{X},\tilde{Y}\right),\ D_{\mathcal{H}\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\tilde{X},\tilde{Y}\right),$$

then D is the desired covariant derivative.

We call the metric derivative so obtained *Miron's derivative* arising from $\mathcal{H}$. In the case of a Finsler metric and its Barthel connection this construction leads to Cartan's covariant derivative [11, 12]. In the general case, however, our initial data g and $\mathcal{H}$ are not related at all, thus it can hardly be expected that D is associated to $\mathcal{H}$.

4 The main results

4.1. In this section, g will be a *weakly normal Moór – Vanstone metric*, χ the canonical spray of the Finsler energy E, $\mathcal{H}_E$ the Barthel connection, and $\overset{E}{\nabla}$ Berwald's covariant derivative arising from $\mathcal{H}_E$. Other data of $\mathcal{H}_E$ will be distinguished from those of an arbitrary Ehresmann connection by a subscript E. The symbol $\sharp$ will denote the sharp operator with respect to g.

4.2 Proposition. *Given a type (1,1) tensor field P along $\overset{\circ}{\tau}$, suppose that $\mathcal{H} := \mathcal{H}_E - \mathbf{i}\circ P$ is an Ehresmann connection such that $\mathcal{H}\delta = \chi$ and $\operatorname{Im}\mathcal{H} \subset \operatorname{Ker} dE$. Then Miron's derivative D arising from $\mathcal{H}$ is strongly associated to $\mathcal{H}$ if and only if*

$$g\left((\nabla^v_\delta P)\left(\tilde{X}\right),\tilde{Y}\right) + g\left(\tilde{X},(\nabla^v_\delta P)\left(\tilde{Y}\right)\right) = -\left(\overset{E}{\nabla}_\chi g\right)\left(\tilde{X},\tilde{Y}\right)$$

for any $\tilde{X},\tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$.

Proof. Let ∇ be Berwald's derivative arising from $\mathcal{H}$. In 2.6 we noted that D is associated to $\mathcal{H}$ if and only if it is strongly regular and its h-deflection vanishes. In the first step, we show that D is automatically strongly regular. If $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$, we have

$$\tilde{\mu}\tilde{X} = D_{\mathbf{i}\tilde{X}}\delta = \nabla_{\mathbf{i}\tilde{X}}\delta + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\tilde{X}, \delta\right),$$

$$g\left(\overset{\circ}{\mathcal{C}}\left(\tilde{X}, \delta\right), \tilde{Y}\right) = g\left(\mathcal{C}\left(\tilde{X}, \delta\right), \tilde{Y}\right) + g\left(\mathcal{C}\left(\delta, \tilde{Y}\right), \tilde{X}\right) - g\left(\mathcal{C}\left(\tilde{Y}, \tilde{X}\right), \delta\right)$$
$$= g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \delta\right) + (\nabla^v_\delta g)\left(\tilde{Y}, \tilde{X}\right) - g\left(\mathcal{C}\left(\tilde{X}, \tilde{Y}\right), \delta\right) = 0,$$

thus $\tilde{\mu}$ is indeed the identity map. In the second step we show that the h-deflection of D with respect to $\mathcal{H}$ vanishes if and only if the equation in the proposition holds. Since g is non-degenerate, it is enough to consider the expression $g\left(D_{X^h}\delta, \hat{Y}\right)$:

$$2g\left(D_{X^h}\delta, \hat{Y}\right) = 2g\left(\nabla_{X^h}\delta, \hat{Y}\right) + g\left(\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X}, \delta\right), \hat{Y}\right) = 2g\left(\mathbf{t}\hat{X}, \hat{Y}\right)$$
$$+ g\left(\mathcal{C}^h\left(\hat{X}, \delta\right), \hat{Y}\right) + g\left(\mathcal{C}^h\left(\delta, \hat{Y}\right), \hat{X}\right) - g\left(\mathcal{C}^h\left(\hat{Y}, \hat{X}\right), \delta\right)$$
$$= 2g\left(\mathbf{t}\hat{X}, \hat{Y}\right) + (\nabla_{X^h}g)\left(\hat{Y}, \delta\right) + (\nabla_\chi g)\left(\hat{X}, \hat{Y}\right) - (\nabla_{Y^h}g)\left(\hat{X}, \delta\right)$$
$$= 2g\left(\mathbf{t}\hat{X}, \hat{Y}\right) + X^h g\left(\hat{Y}, \delta\right) - g\left(\nabla_{X^h}\hat{Y}, \delta\right) - g\left(\hat{Y}, \nabla_{X^h}\delta\right)$$
$$+ \chi g\left(\hat{X}, \hat{Y}\right) - g\left(\nabla_\chi\hat{X}, \hat{Y}\right) - g\left(\hat{X}, \nabla_\chi\hat{Y}\right)$$
$$- Y^h g\left(\hat{X}, \delta\right) + g\left(\nabla_{Y^h}\hat{X}, \delta\right) + g\left(\hat{X}, \nabla_{Y^h}\delta\right).$$

Since g is weakly normal, by 3.5 we have $\vartheta_g = \nabla^v E$. This implies that $g\left(\hat{X}, \delta\right) = \vartheta_g\hat{X} = X^v E$ for all $X \in \mathfrak{X}(M)$, thus we get

$$2g\left(D_{X^h}\delta, \hat{Y}\right) = 2g\left(\mathbf{t}\hat{X}, \hat{Y}\right) + X^h Y^v E - \mathbf{i}\left(\nabla_{X^h}\hat{Y}\right)E - g\left(\hat{Y}, \mathbf{t}\hat{X}\right)$$
$$+ \left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X}, \hat{Y}\right) + g\left(\overset{E}{\nabla}_\chi\hat{X}, \hat{Y}\right) + g\left(\hat{X}, \overset{E}{\nabla}_\chi\hat{Y}\right) - g\left(\nabla_\chi\hat{X}, \hat{Y}\right)$$
$$- g\left(\hat{X}, \nabla_\chi\hat{Y}\right) - Y^h X^v E + \mathbf{i}\left(\nabla_{Y^h}\hat{X}\right)E + g\left(\hat{X}, \mathbf{t}\hat{Y}\right)$$
$$= g\left(\mathbf{t}\hat{X}, \hat{Y}\right) + g\left(\hat{X}, \mathbf{t}\hat{Y}\right) + X^h Y^v E - \left[X^h, Y^v\right]E + \left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X}, \hat{Y}\right)$$
$$+ g\left(\overset{E}{\nabla}_\chi\hat{X} - \nabla_\chi\hat{X}, \hat{Y}\right) + g\left(\hat{X}, \overset{E}{\nabla}_\chi\hat{Y} - \nabla_\chi\hat{Y}\right) - Y^h X^v E + \left[Y^h, X^v\right]E,$$

where we also used the definition of the Berwald derivative ∇. Finally, taking into account that $\chi = \mathcal{H}_E\delta = \mathcal{H}\delta$, that the relation $\mathcal{H}_E - \mathcal{H} = \mathbf{i}\circ P$ is equivalent to $\mathcal{V} - \mathcal{V}_E = P \circ \mathbf{j}$, and that $\mathbf{j}[\chi, X^v] = \hat{X}$, $\mathbf{j}[\chi, Y^v] = \hat{Y}$ (see e.g. [11, p. 1349]), we obtain

$$
\begin{aligned}
2g\left(D_{X^h}\delta,\hat{Y}\right) &= g\left(\mathbf{t}\hat{X},\hat{Y}\right)+g\left(\hat{X},\mathbf{t}\hat{Y}\right)+Y^vX^hE \\
&\quad + g\left(\mathcal{V}_E[\chi,X^v]-\mathcal{V}[\chi,X^v],\hat{Y}\right)+g\left(\hat{X},\mathcal{V}_E[\chi,Y^v]-\mathcal{V}[\chi,Y^v]\right) \\
&\quad + \left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X},\hat{Y}\right)-X^vY^hE \\
&= g\left(\mathbf{t}\hat{X},\hat{Y}\right)+g\left(\hat{X},\mathbf{t}\hat{Y}\right) \\
&\quad - g\left(P\mathbf{j}[\chi,X^v],\hat{Y}\right)-g\left(\hat{X},P\mathbf{j}[\chi,Y^v]\right)+\left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X},\hat{Y}\right) \\
&= g\left(P\hat{X},\hat{Y}\right)+g\left(\mathbf{t}\hat{X},\hat{Y}\right)+g\left(\hat{X},P\hat{Y}\right)+g\left(\hat{X},\mathbf{t}\hat{Y}\right)+\left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X},\hat{Y}\right).
\end{aligned}
$$

Hence it remains only to show that $\nabla^v_\delta P = P+\mathbf{t}$. Since $\mathcal{L}_C J=-J$, we have

$$
\begin{aligned}
\mathbf{i}P\hat{X}+\mathbf{i}\mathbf{t}\hat{X} &= J\mathcal{H}P\hat{X}-\left[C,X^h\right]=-(\mathcal{L}_CJ)\mathcal{H}P\hat{X}-\left[C,X^{h_E}-\mathbf{i}P\hat{X}\right] \\
&= -\left[C,\mathbf{i}P\hat{X}\right]+J\left[C,\mathcal{H}P\hat{X}\right]+\left[C,\mathbf{i}P\hat{X}\right]=\mathbf{i}\nabla^v_\delta\left(P\hat{X}\right)=\mathbf{i}\left(\nabla^v_\delta P\right)\left(\hat{X}\right),
\end{aligned}
$$

thus proving our proposition.

4.3 Theorem. *Define a type (1,1) tensor field P along $\overset{\circ}{\tau}$ by*

(1) $P\tilde{X} := -\frac{1}{2}\left(i_{\tilde{X}}\overset{E}{\nabla}_\chi g\right)^\sharp + P_s\tilde{X}+P_a\tilde{X},\ \tilde{X}\in\mathfrak{X}(\overset{\circ}{\tau}),$

where

(2) *P_s is a symmetric, P_a is a skew-symmetric, type (1,1) tensor along $\overset{\circ}{\tau}$;*

(3) *P_s is homogeneous of degree 0, i.e. $\nabla^v_\delta P_s = 0$;*

(4) *$\operatorname{Im} P_s$ and $\operatorname{Im} P_a$ are contained in the g-orthogonal complement of the canonical section.*

Then the Ehresmann connection $\mathcal{H} := \mathcal{H}_E - \mathbf{i}\circ P$ has the properties

(5) *$\mathcal{H}\delta = \chi$ (and hence $\mathcal{H}$ and χ have common geodesics);*

(6) *E is a first integral of the $\mathcal{H}$-horizontal vector fields ($\operatorname{Im}\mathcal{H}\subset\operatorname{Ker} dE$);*

and Miron's derivative D arising from $\mathcal{H}$ is strongly associated to $\mathcal{H}$, i.e.,

(7) *$D\delta = \mathcal{V}$.*

Conversely, if $\mathcal{H} := \mathcal{H}_E - \mathbf{i}\circ P$ is an Ehresmann connection with the properties (5) and (6), and the Miron derivative arising from $\mathcal{H}$ is strongly associated to $\mathcal{H}$, then P is of the form (1), satisfying relations (2), (3) and (4).

Proof. By the previous proposition, taking into account that $\mathcal{H}_E$ also satisfies (5) and (6), we have to solve the following mixed system of algebraic equations and a partial differential equation for P:

(i) $$P\delta = 0$$

(ii) $$\left(\mathbf{i}P\tilde{X}\right)E = 0 \ \left(\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})\right)$$

(iii) $$g\left((\nabla^v_\delta P)\left(\tilde{X}\right), \tilde{Y}\right) + g\left(\tilde{X}, (\nabla^v_\delta P)\left(\tilde{Y}\right)\right) = -\left(\overset{E}{\nabla}_\chi g\right)\left(\tilde{X}, \tilde{Y}\right) \quad \left(\tilde{X}, \tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})\right).$$

As the system is linear, we may search its general solution as the sum of the general solution of the homogeneous part and a particular solution. First we show that the tensor P defined by

$$\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto P\tilde{X} := -\frac{1}{2}\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp \in \mathfrak{X}(\overset{\circ}{\tau})$$

(roughly speaking, the first term of (1)) is a particular solution, i.e., it satisfies (i)–(iii). To verify relation (i), it is enough to check that $g\left(P\delta, \hat{X}\right) = 0$ for all $X \in \mathfrak{X}(M)$:

$$\begin{aligned}
-2g\left(P\delta, \hat{X}\right) &= g\left(\left(i_\delta\overset{E}{\nabla}_\chi g\right)^\sharp, \hat{X}\right) = \left(i_\delta\overset{E}{\nabla}_\chi g\right)\left(\hat{X}\right) = \left(\overset{E}{\nabla}_\chi g\right)\left(\delta, \hat{X}\right) \\
&= \chi g\left(\delta, \hat{X}\right) - g\left(\overset{E}{\nabla}_\chi \delta, \hat{X}\right) - g\left(\delta, \overset{E}{\nabla}_\chi \hat{X}\right) \\
&= \chi\left(\vartheta_g \hat{X}\right) - g\left(\mathcal{V}_E[\chi, C], \hat{X}\right) - \vartheta_g\left(\overset{E}{\nabla}_\chi \hat{X}\right) = \chi X^v E - \left(\mathbf{i}\overset{E}{\nabla}_\chi \hat{X}\right)E \\
&= [\chi, X^v]E - (\mathbf{v}_E[\chi, X^v])E = (\mathbf{h}_E[\chi, X^v])E = 0,
\end{aligned}$$

using in the last step that E is a first integral of the $\mathcal{H}_E$-horizontal vector fields. Thus relation (i) is indeed satisfied. On the other hand,

$$\mathbf{i}\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp E = g\left(\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp, \delta\right) = \left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)(\delta) = \left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X}, \delta\right) = 0,$$

as before; this proves (ii). To verify that (iii) is satisfied, first we show that $\nabla^v_\delta\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp = \left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp$, i.e., the vector field $\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp$ is positively homogeneous of degree 1. Using the definition $\nabla^v_\delta g = 0$ of the homogeneity of g, we obtain

$$\begin{aligned}
g\left(\nabla^v_\delta\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp, \hat{Y}\right) &= (\mathbf{i}\delta)g\left(\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp, \hat{Y}\right) - g\left(\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp, \nabla^v_\delta \hat{Y}\right) \\
&= C\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)\left(\hat{Y}\right) = C\left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X}, \hat{Y}\right)
\end{aligned}$$

$$= C\chi g\left(\hat{X},\hat{Y}\right) - Cg\left(\overset{E}{\nabla}_\chi \hat{X},\hat{Y}\right) - Cg\left(\hat{X},\overset{E}{\nabla}_\chi \hat{Y}\right).$$

Now we treat the three terms appearing on the right-hand side separately:

$$C\chi g\left(\hat{X},\hat{Y}\right) = [C,\chi]g\left(\hat{X},\hat{Y}\right) + \chi C g\left(\hat{X},\hat{Y}\right) = [C,\chi]g\left(\hat{X},\hat{Y}\right) = \chi g\left(\hat{X},\hat{Y}\right),$$

$$\begin{aligned}
Cg\left(\overset{E}{\nabla}_\chi \hat{X},\hat{Y}\right) &= g\left(\nabla^v_\delta \overset{E}{\nabla}_\chi \hat{X},\hat{Y}\right) + g\left(\overset{E}{\nabla}_\chi \hat{X},\nabla^v_\delta \hat{Y}\right)\\
&= g\left(\mathbf{j}[C,\mathcal{H}_E\mathcal{V}_E[\chi,X^v]],\hat{Y}\right) = g\left(\mathbf{j}[C,\mathcal{H}_E\mathcal{V}_E X^c]],\hat{Y}\right)\\
&= g\left(\mathbf{i}^{-1}([C,\mathbf{v}_E X^c] - (\mathcal{L}_C J)\mathcal{H}_E\mathcal{V}_E X^c),\hat{Y}\right)\\
&= g\left(\mathbf{i}^{-1}\left([C,X^c] - [C,X^{h_E}]\right) + \mathcal{V}_E X^c,\hat{Y}\right)\\
&= g\left(\mathcal{V}_E[\chi,X^v],\hat{Y}\right) = g\left(\overset{E}{\nabla}_\chi \hat{X},\hat{Y}\right),
\end{aligned}$$

and similarly,

$$Cg\left(\hat{X},\overset{E}{\nabla}_\chi \hat{Y}\right) = g\left(\hat{X},\overset{E}{\nabla}_\chi \hat{Y}\right).$$

Thus

$$\begin{aligned}
g\left(\nabla^v_\delta\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp,\hat{Y}\right) &= \chi g\left(\hat{X},\hat{Y}\right) - g\left(\overset{E}{\nabla}_\chi \hat{X},\hat{Y}\right) - g\left(\hat{X},\overset{E}{\nabla}_\chi \hat{Y}\right)\\
&= \left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X},\hat{Y}\right) = \left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)\left(\hat{Y}\right) = g\left(\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp,\hat{Y}\right),
\end{aligned}$$

which proves that $\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp$ is indeed positively homogeneous of degree 1. Finally, substituting it into the left-hand side of (iii), we obtain

$$\begin{aligned}
&-\frac{1}{2}g\left(\nabla^v_\delta\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp,\hat{Y}\right) - \frac{1}{2}g\left(\hat{X},\nabla^v_\delta\left(i_{\hat{Y}}\overset{E}{\nabla}_\chi g\right)^\sharp\right)\\
&= -\frac{1}{2}g\left(\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)^\sharp,\hat{Y}\right) - \frac{1}{2}g\left(\hat{X},\left(i_{\hat{Y}}\overset{E}{\nabla}_\chi g\right)^\sharp\right)\\
&= -\frac{1}{2}\left(i_{\hat{X}}\overset{E}{\nabla}_\chi g\right)\left(\hat{Y}\right) - \frac{1}{2}\left(i_{\hat{Y}}\overset{E}{\nabla}_\chi g\right)\left(\hat{X}\right)\\
&= -\frac{1}{2}\left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X},\hat{Y}\right) - \frac{1}{2}\left(\overset{E}{\nabla}_\chi g\right)\left(\hat{Y},\hat{X}\right) = -\left(\overset{E}{\nabla}_\chi g\right)\left(\hat{X},\hat{Y}\right),
\end{aligned}$$

thus (iii) is satisfied as well. Now we turn to the solution of the homogeneous part of our system. To make a clear distinction we denote the unknown tensor field by P_h rather than P. Then the system takes the form

(i) $$P_h\delta = 0$$

(ii) $$\left(\mathbf{i}P_h\tilde{X}\right)E = 0 \quad \left(\tilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})\right)$$

(iii') $$g\left(\left(\nabla^v_\delta P_h\right)\left(\tilde{X}\right),\tilde{Y}\right) + g\left(\tilde{X},\left(\nabla^v_\delta P_h\right)\left(\tilde{Y}\right)\right) = 0 \quad \left(\tilde{X},\tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})\right).$$

Decomposing P_h into the sum of a symmetric part P_s and a skew-symmetric part P_a, we obtain

$$\begin{aligned}
0 &= g\left((\nabla^v_\delta P_s)\hat{X} + (\nabla^v_\delta P_a)\hat{X}, \hat{Y}\right) + g\left(\hat{X}, (\nabla^v_\delta P_s)\hat{Y} + (\nabla^v_\delta P_a)\hat{Y}\right) \\
&= g\left(\nabla^v_\delta\left(P_s\hat{X}\right) + \nabla^v_\delta\left(P_a\hat{X}\right), \hat{Y}\right) + g\left(\hat{X}, \nabla^v_\delta\left(P_s\hat{Y}\right) + \nabla^v_\delta\left(P_a\hat{Y}\right)\right) \\
&= Cg\Big(P_s\hat{X}, \hat{Y}\Big) + Cg\Big(\hat{X}, P_s\hat{Y}\Big) + Cg\Big(P_a\hat{X}, \hat{Y}\Big) + Cg\Big(\hat{X}, P_a\hat{Y}\Big) \\
&= 2Cg\Big(P_s\hat{X}, \hat{Y}\Big) = 2g\left((\nabla^v_\delta P_s)\hat{X}, \hat{Y}\right),
\end{aligned}$$

thus (iii') holds if and only if P_s is homogeneous of degree 0. It remains to find the conditions imposed by (i) and (ii) on P_s and P_a. Equation (i) implies

$$\begin{aligned}
g\left(P_h\delta, \tilde{X}\right) &= g\left(P_s\delta + P_h\delta, \tilde{X}\right) = g\left(\delta, P_s\tilde{X}\right) - \left(\delta, P_h\tilde{X}\right) \\
&= g\left(\delta, (P_s - P_h)\left(\tilde{X}\right)\right) = 0,
\end{aligned}$$

whereas equation (ii) is equivalent to

$$\begin{aligned}
&\left(\mathbf{i}P_s\tilde{X}\right)E + \left(\mathbf{i}P_a\tilde{X}\right)E = \vartheta_g\left(P_s\tilde{X}\right) + \vartheta_g\left(P_a\tilde{X}\right) \\
&\quad = g\left(P_s\tilde{X}, \delta\right) + g\left(P_a\tilde{X}, \delta\right) = g\left((P_s + P_a)\left(\tilde{X}\right), \delta\right) = 0,
\end{aligned}$$

thus (i) and (ii) holds if and only if both the sum and the difference of P_s and P_a are contained in the orthogonal complement of δ. This concludes the proof of the theorem.

We note that our result provides, in fact, a family of good metric derivatives, since one has a considerable freedom in the choice of P_s and P_a in (1).

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