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Calculus along the tangent bundle projection and projective metrizability

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Abstract. We sketch an economical framework and a simple index-free calculus for direction-dependent objects, based almost exclusively on Berwald derivative. To illustrate the efficiency of these tools, we characterize projectively Finslerian sprays, and derive Hamel's PDEs in an analytic version of Hilbert's fourth problem.

Keywords: Direction dependence, Berwald derivative, sprays, Finsler metrizability, Hilbert's fourth problem.

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1. Introduction

A part of calculus of growing importance for current research can be classified as 'calculus of direction dependent objects'. It seems to me that the most convenient and economical setting for their study is the pull-back of the tangent bundle of a base manifold over the natural projection $\mathring{\tau}$ of the slit tangent bundle. Then our direction dependent objects can be interpreted as tensors along $\mathring{\tau}$. A comprehensive calculus for this kind of objects was elaborated by E. Martínez, J. Cariñena and W. Sarlet in the early 1990's;[15,16] see also Chapter 2/E of Ref. 21. I think, for most purposes a simplified version of this calculus, with the Berwald derivative in the focus, is already sufficient. Berwald's derivative is a covariant derivative operator in our pull-back bundle, built up from a canonical vertical part and a horizontal part depending on an Ehresmann connection. In the most important applications, e.g., in the presence of a Finsler function, an Ehresmann connection can 'naturally' be constructed from the structure.

In this article I give a bold outline of a calculus based mainly on a Berwald derivative, with special emphasis on the fundamental geometric objects which may be constructed in terms of the Berwald derivative. To present some non trivial applications, I discuss the following

Metrizability problem: *Under what necessary and sufficient conditions has a spray common geodesics (as unparametrized curves) with a Finsler function?*

It will turn out that this question leads naturally to an analytic version of Hilbert's fourth problem.

2. Basic constructions and conventions

Notations. Our base manifold M will be an n-dimensional ($n \geq 2$), connected, paracompact smooth manifold. $C^\infty(M)$ stands for the ring of real-valued smooth functions on M.

T_pM is the tangent space to M at the point $p \in M$, TM is the $2n$-dimensional tangent manifold of M, and $\overset{\circ}{T}M$ is the open submanifold of the non-zero tangent vectors to M. τ and $\overset{\circ}{\tau}$ denote the natural projections of TM and $\overset{\circ}{T}M$ onto M, resp. The *tangent bundle* of M is the triplet (TM, τ, M), denoted also simply by τ or, less consequently, by TM. Similarly, $(\overset{\circ}{T}M, \overset{\circ}{\tau}, M)$ is the *slit tangent bundle* of M. The shorthand for the tangent bundle of TM and $\overset{\circ}{T}M$ will be TTM and $T\overset{\circ}{T}M$, respectively. The *vertical lift of a function* $f \in C^\infty(M)$ is $f^{\mathsf{v}} := f \circ \tau \in C^\infty(TM)$; the *complete lift* $f^{\mathsf{c}} \in C^\infty(TM)$ of f is defined by $f^{\mathsf{c}}(v) := v(f)$, $v \in TM$.

If $\varphi \colon M \to N$ is a smooth map, then φ_* will denote the smooth map of TM into TN induced by φ, the *tangent map* (or *derivative*) of φ.

Vector fields. A *rough vector field* on M is a map $X \colon M \to TM$ which satisfies $\tau \circ X = 1_M$. If, in addition, X is smooth, then it is a *vector field* on M. The $C^\infty(M)$-module of vector fields on M is denoted by $\mathfrak{X}(M)$. If $X \in \mathfrak{X}(M)$, i_X and $\mathcal{L}_X$ denote the *substitution operator* induced by X and the *Lie derivative* with respect to X, respectively. The operator of the *exterior derivative* will be denoted by d on every *manifold*.
To make the text typographically more transparent, capitals $X, Y, Z, \dots$ will stand for vector fields on the base manifold, while vector fields on TM will usually (but not exclusively) be denoted by Greek letters $\xi, \eta, \zeta, \dots$.

Given a vector field X on M, there is a unique vector field X^{c} on TM such that $X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}$ for all $f \in C^\infty(M)$. X^{c} is called the *complete lift of* X.

By a *rough second-order vector field* over M we mean a rough vector field S on TM such that $\tau_* \circ S$ is the identity on TM. S is said to be a *semispray* or a *second-order vector field* over M if it is smooth on $\mathring{T}M$ or on TM, respectively.

Basic setup. The majority of our objects will live on the pull-back of the tangent bundle of M over its projection τ or over $\mathring{\tau}$. These are vector bundles over TM and $\mathring{T}M$ whose total manifolds are the fibre products $TM \times_M TM$ and $\mathring{T}M \times_M TM$, and which will be denoted by π and $\mathring{\pi}$, respectively. For the $C^\infty(TM)$-module of sections of π we use the notation $\Gamma(\pi)$. Any section in $\Gamma(\pi)$ is of the form

$$\widetilde{X} \colon v \in TM \longmapsto \widetilde{X}(v) = (v, \underline{X}(v)) \in TM \times_M TM,$$

where $\underline{X} \colon TM \to TM$ is a smooth map such that $\tau \circ \underline{X} = \tau$. In particular, we have the *canonical section* δ given by $\delta(v) := (v, v)$, $v \in TM$.

An important class of sections of π can be obtained from vector fields on M: if $X \in \mathfrak{X}(M)$, then the map

$$\widehat{X} \colon v \in TM \longmapsto \widehat{X}(v) := (v, X(\tau(v))) \in TM \times_M TM$$

is a section of π, called the *lift of X into* $\Gamma(\pi)$ or a *basic section* of π. *If $(X^i)_{i=1}^n$ is a local basis for $\mathfrak{X}(M)$, then $(\widehat{X}^i)_{i=1}^n$ is a local basis for* $\Gamma(\pi)$; this simple observation proves to be useful in calculations.

The module $\Gamma(\pi)$ may naturally be identified with the $C^\infty(TM)$-module

$$\mathfrak{X}(\tau) := \{\underline{X} \colon TM \to TM \mid \underline{X} \textit{ is smooth, and } \tau \circ \underline{X} = \tau\}$$

of *vector fields along* τ by the map $\widetilde{X} \in \Gamma(\pi) \longmapsto \underline{X} \in \mathfrak{X}(\tau)$. We shall use this harmless identification tacitly, whenever it is convenient. Then the identity map 1_{TM} corresponds to the canonical section, while the maps $X \circ \tau$ $(X \in \mathfrak{X}(M))$ correspond to basic sections.

We may describe the $C^\infty(\mathring{T}M)$-module $\Gamma(\mathring{\pi}) \cong \mathfrak{X}(\mathring{\tau})$ analogously. The tensor algebras of the modules $\Gamma(\pi)$ and $\Gamma(\mathring{\pi})$ will be denoted by $\mathcal{T}(\pi)$ and $\mathcal{T}(\mathring{\pi})$, respectively. It is natural again to interpret the elements of these tensor algebras as 'tensors along τ or $\mathring{\tau}$', and throughout the paper, as a rule, the term 'tensor' will be used in this sense. We note finally that $\Gamma(\pi)$ will be considered as a subalgebra of $\Gamma(\mathring{\pi})$.

Coordinates. In coordinate descriptions we shall specify a chart $(U, (u^i)_{i=1}^n)$ on M and use the *induced chart* $(\tau^{-1}(U), (x^i, y^i)_{i=1}^n)$ on TM, where $x^i := (u^i)^{\mathsf{v}}$, $y^i := (u^i)^{\mathsf{c}}$. These charts lead to the local bases

$$\left(\frac{\partial}{\partial u^i}\right)_{i=1}^n, \qquad \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)_{i=1}^n \qquad \textit{and} \qquad \left(\widehat{\frac{\partial}{\partial u^i}}\right)_{i=1}^n \tag{1}$$

of $\mathfrak{X}(M)$, $\mathfrak{X}(TM)$ and $\mathfrak{X}(\mathring{\tau})$, respectively. Note that $\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial u^i}\right)^{\mathsf{c}}$.

Einstein's convention on repeated indices will be used. If $X = X^i \frac{\partial}{\partial u^i}$ is a vector field on U, then its lift into $\mathfrak{X}(\mathring{\tau})$ is locally given by

$$\widehat{X} = (X^i \circ \tau)\widehat{\frac{\partial}{\partial u^i}}.$$

The coordinate expression of the canonical section is

$$\delta \restriction \tau^{-1}(U) = y^i \widehat{\frac{\partial}{\partial u^i}}.$$

If S is a semispray over M, then in local coordinates

$$S \restriction \tau^{-1}(U) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \tag{2}$$

for some smooth functions G^i on $\mathring{\tau}^{-1}(U)$. These functions will be called the *semispray coefficients* of S with respect to the chosen chart. We note that the factor -2 in the coordinate expression of S proves to be convenient, and is more or less traditional.

3. Some vertical calculus

Canonical objects. We recall that the *fundamental exact sequence* over TM is the sequence of vector bundle maps

$$0 \longrightarrow TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \longrightarrow 0, \tag{3}$$

where $\mathbf{j}(w) := (v, \tau_*(w))$, if $w \in T_vTM$, while $\mathbf{i}$ identifies the fibre $\{v\} \times T_{\tau(v)}M$ with the tangent space $T_vT_{\tau(v)}M$ for all $v \in TM$.

The bundle maps $\mathbf{i}$ and $\mathbf{j}$ induce $C^\infty(TM)$-homomorphisms between the modules of sections, denoted by the same symbols. Thus we also have the exact sequence

$$0 \longrightarrow \mathfrak{X}(\tau) \xrightarrow{\mathbf{i}} \mathfrak{X}(TM) \xrightarrow{\mathbf{j}} \mathfrak{X}(\tau) \longrightarrow 0 \tag{4}$$

of module homomorphisms. $\mathbf{i}$ and $\mathbf{j}$ act on the local bases in (1) by

$$\mathbf{i}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = \frac{\partial}{\partial y^i}, \quad \mathbf{j}\left(\frac{\partial}{\partial x^i}\right) = \widehat{\frac{\partial}{\partial u^i}}, \quad \mathbf{j}\left(\frac{\partial}{\partial y^i}\right) = 0; \quad i \in \{1, \dots, n\}. \tag{5}$$

$\mathfrak{X}^{\mathsf{v}}(TM) := \mathbf{i}\mathfrak{X}(\tau)$ is the module of *vertical vector fields* on TM, $X^{\mathsf{v}} := \mathbf{i}(\widehat{X})$ is the *vertical lift* of the vector field X on M. We have a canonical vertical vector field on TM, the *Liouville vector field* $C := \mathbf{i}\delta$. Its bracket with any vertical lift X^{v} is given by

$$[C, X^{\mathsf{v}}] = -X^{\mathsf{v}}. \tag{6}$$

From the local form of δ we obtain by (5) that $C \restriction \tau^{-1}(U) = y^i \frac{\partial}{\partial y^i}$.

The type $\binom{1}{1}$ tensor field $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ will be called the *vertical endomorphism* of TTM. (Other terms, e.g., 'the canonical almost tangent structure on TM' are also frequently used.) Evidently $\mathbf{J}^2 = \mathbf{J} \circ \mathbf{J} = 0$. In terms of vertical and complete lifts

$$\mathbf{J}X^{\mathsf{v}} = 0, \qquad \mathbf{J}X^{\mathsf{c}} = X^{\mathsf{v}}; \qquad X \in \mathfrak{X}(M). \tag{7}$$

Thus, in particular, $\mathbf{J}\left(\frac{\partial}{\partial y^i}\right) = \mathbf{J}\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{v}} = 0$, $\mathbf{J}\left(\frac{\partial}{\partial x^i}\right) = \mathbf{J}\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{c}} = \frac{\partial}{\partial y^i}$ $(i \in \{1, \ldots, n\})$.

We shall use the *vertical differentiation* $d_{\mathbf{J}}$ which associates the semibasic 1-form $d_{\mathbf{J}}f := df \circ \mathbf{J}$ to a smooth function f on TM. In induced coordinates,

$$d_{\mathbf{J}}f \restriction \tau^{-1}(U) = \frac{\partial f}{\partial y^i} dx^i.$$

For a detailed treatment of vertical differentiation, see Ref. 8.

We define the *Frölicher-Nijenhuis bracket* $[\mathbf{J}, \xi]$ of $\mathbf{J}$ and a vector field ξ on TM by $[\mathbf{J}, \xi] := -\mathcal{L}_\xi \mathbf{J}$. Then for any vector field η on TM,

$$[\mathbf{J}, \xi]\eta = [\mathbf{J}\eta, \xi] - \mathbf{J}[\eta, \xi].$$

Observe that if S is a semispray over M, then we have

$$\mathbf{J}S = C \quad \textit{or, equivalently,} \quad \mathbf{j}S = \delta. \tag{8}$$

Conversely, condition $\tau_* \circ S = 1_{TM}$ in the definition of a semispray may be replaced by (8).

If S is a semispray over M, then

$$\mathbf{J}[\mathbf{J}\xi, S] = \mathbf{J}\xi, \quad \textit{for all} \quad \xi \in \mathfrak{X}(TM) \tag{9}$$

(*Grifone identity*). For a simple proof we refer to Ref. 22.

Vertical derivatives. Let $\widetilde{X}$ be a section in $\mathfrak{X}(\tau)$. We define an operator $\nabla^{\mathsf{v}}_{\widetilde{X}}$ specifying its action on functions in $C^\infty(TM)$ and sections in $\mathfrak{X}(\tau)$ as follows:

$$\nabla^{\mathsf{v}}_{\widetilde{X}} f := (\mathbf{i}\widetilde{X})f = (df \circ \mathbf{i})(\widetilde{X}), \qquad f \in C^\infty(TM); \tag{10}$$

$$\nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \eta]; \qquad \widetilde{Y} = \mathbf{j}(\eta) \in \mathfrak{X}(\tau), \quad \eta \in \mathfrak{X}(TM). \tag{11}$$

Then $\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}$ is well-defined: it is easy to check its independence of the choice of $\eta \in \mathfrak{X}(TM)$ satisfying $\mathbf{j}(\eta) = \widetilde{Y}$. The map ∇^{v} which assigns to each pair $(\widetilde{X}, \widetilde{Y})$ of sections the section $\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}$ obeys the same formal rules as a covariant derivative operator; the crucial derivation rule takes the form

$$\nabla^{\mathsf{v}}_{\widetilde{X}} f\widetilde{Y} = ((\mathbf{i}\widetilde{X})f)\widetilde{Y} + f\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} = (\nabla^{\mathsf{v}}_{\widetilde{X}} f)\widetilde{Y} + f\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}, \quad f \in C^\infty(TM).$$

The *(canonical) v-covariant derivative* $\nabla^{\mathsf{v}}_{\widetilde{X}}$ so obtained can be extended to operate on an arbitrary tensor, the idea is to make sure that Leibniz's rule holds. Thus, for example, if $\widetilde{A} \in \mathcal{T}^1_1(\pi)$, then for any 1-form $\widetilde{\alpha}$ and vector field $\widetilde{Y}$ along τ,

$$\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{A}(\widetilde{\alpha}, \widetilde{Y}) := (\mathbf{i}\widetilde{X})\widetilde{A}(\widetilde{\alpha}, \widetilde{Y}) - \widetilde{A}(\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{\alpha}, \widetilde{Y}) - \widetilde{A}(\widetilde{\alpha}, \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}),$$

where the 1-form $\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{\alpha}$ is given by $(\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{\alpha})(\widetilde{Z}) = \mathbf{i}\widetilde{X}(\widetilde{\alpha}(\widetilde{Z})) - \widetilde{\alpha}(\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Z})$, $\widetilde{Z} \in \mathfrak{X}(\tau)$.

Finally, we define the *(canonical) v-covariant differential* of a type $\binom{k}{l}$ tensor $\widetilde{A}$ as the type $\binom{k}{l+1}$ tensor $\nabla^{\mathsf{v}}\widetilde{A}$, which 'collects all the v-covariant derivatives of $\widetilde{A}$'. For example, if $k = l = 1$, then

$$\nabla^{\mathsf{v}}\widetilde{A}(\widetilde{\alpha}, \widetilde{X}, \widetilde{Y}) := (\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{A})(\widetilde{\alpha}, \widetilde{Y}).$$

The v-covariant differential of the canonical section is the identity operator. Indeed, using Grifone's identity (9), for any section $\widetilde{X} \in \mathfrak{X}(\tau)$ we get

$$\mathbf{i}(\nabla^{\mathsf{v}}\delta)(\widetilde{X}) = \mathbf{i}\nabla^{\mathsf{v}}_{\widetilde{X}}\delta \overset{(11)}{=} \mathbf{J}[\mathbf{i}\widetilde{X}, S] = \mathbf{i}\widetilde{X},$$

whence our claim.

For any vector field X on M we have $\nabla^{\mathsf{v}}\widehat{X} = 0$. In fact, if $Y \in \mathfrak{X}(M)$, then $\nabla^{\mathsf{v}}\widehat{X}(\widehat{Y}) = \nabla^{\mathsf{v}}_{\widehat{Y}}\widehat{X} = \mathbf{j}[Y^{\mathsf{v}}, X^{\mathsf{c}}] = 0$, since $[Y^{\mathsf{v}}, X^{\mathsf{c}}]$ is vertical. But any section of π can locally be combined from basic sections, so it follows that $\nabla^{\mathsf{v}}\widehat{X}(\widetilde{Y}) = 0$ for all $\widetilde{Y} \in \mathfrak{X}(\tau)$.

Notice that the vertical differentiation $d_{\mathbf{J}}$ and the v-covariant differential ∇^{v} are related on $C^\infty(TM)$ by $d_{\mathbf{J}} f = \nabla^{\mathsf{v}} f \circ \mathbf{j}$.

To conclude our brief overview on the 'vertical part' of the tensor calculus in $\Gamma(\pi)$, we define the *v-exterior differential* $d^{\mathsf{v}}\widetilde{\alpha}$ of a k-form $\widetilde{\alpha}$ along τ by

$$d^{\mathsf{v}}\widetilde{\alpha} := (k+1)\mathrm{Alt}\nabla^{\mathsf{v}}\widetilde{\alpha}, \tag{12}$$

where Alt is the operator of alternation. Then d^{v} has the expected property $d^{\mathsf{v}} \circ d^{\mathsf{v}} = 0$. For details, we refer to Ref. 21.

4. Horizontal extension of the vertical calculus

Ehresmann connections. In order to differentiate our tensors not only in vertical directions, we need some further structure in general. This further structure will be an Ehresmann connection in what follows. Since the term has various meanings in differential geometry, we have to fix our usage precisely. By an *Ehresmann connection* on TM we mean a map

$$\mathscr{H} \colon TM \times_M TM \to TTM$$

satisfying the following four conditions:

Ehr 1. *$\mathscr{H}$ is fibre-preserving and fibrewise linear.*

Ehr 2. $\mathbf{j} \circ \mathscr{H} = 1_{TM \times_M TM}$.

Ehr 3. *$\mathscr{H}$ is smooth over $\mathring{T}M \times_M TM$.*

Ehr 4. *If $\sigma \colon p \in M \longmapsto \sigma(p) := 0_p \in T_pM$ is the zero section of τ, then $\mathscr{H}(\sigma(p), v) := (\sigma_*)_p(v)$ for all $p \in M$, $v \in T_pM$.*

Thus, roughly speaking, *an Ehresmann connection is a right splitting of the fundamental exact sequence (3).* If we specify an Ehresmann connection $\mathscr{H}$ on TM, there is a unique left splitting of (3) 'complementary' to $\mathscr{H}$, i.e., a fibre-preserving fibrewise linear map $\mathscr{V} \colon TTM \to TM \times_M TM$, smooth on $T\mathring{T}M$, such that

$$\mathscr{V} \circ \mathbf{i} = 1_{TM \times_M TM} \qquad \text{and} \qquad \operatorname{Ker}(\mathscr{V}) = \operatorname{Im}(\mathscr{H}).$$

$\mathscr{V}$ is called the *vertical map* associated to $\mathscr{H}$. The maps

$$\mathbf{h} := \mathscr{H} \circ \mathbf{j}, \quad \mathbf{v} := \mathbf{i} \circ \mathscr{V} \quad \text{and} \quad \mathbf{F} := \mathscr{H} \circ \mathscr{V} - \mathbf{i} \circ \mathbf{j}$$

are bundle endomorphisms of TTM, called the *horizontal projector*, the *vertical projector* and the *almost complex structure* associated to $\mathscr{H}$, respectively. $\mathbf{h}$ and $\mathbf{v}$ are indeed complementary projection operators ($\mathbf{h}^2 = \mathbf{h}$, $\mathbf{v}^2 = \mathbf{v}$, $\mathbf{h} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{h} = 0$), while $\mathbf{F}^2 = -1_{TTM}$.

An Ehresmann connection, as well as its associated maps, induce module homomorphisms at the level of sections, denoted by the same letters. $\mathfrak{X}^{\mathsf{h}}(\mathring{T}M) := \mathscr{H}(\mathfrak{X}(\mathring{\tau}))$ is the module of *horizontal vector fields* on $\mathring{T}M$. The *horizontal lift* of a vector field X on M to TM is $X^{\mathsf{h}} := \mathscr{H}(\widehat{X}) = \mathbf{h}X^{\mathsf{c}}$. In particular, the horizontal lifts of the coordinate vector fields $\frac{\partial}{\partial u^j}$ can be represented in the form

$$\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}} = \frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i}, \qquad j \in \{1, \dots, n\}. \tag{13}$$

The functions N^i_j, defined on $\tau^{-1}(U)$ and smooth on $\overset{\circ}{\tau}^{-1}(U)$, are called the *Christoffel symbols* of $\mathscr{H}$ with respect to the chosen chart; the minus sign in (13) is quite traditional.

The Berwald derivative. Let an Ehresmann connection $\mathscr{H}$ be given on TM, and let $\widetilde{X}$ be a section of $\overset{\circ}{\pi}$. Following the scheme of construction of $\nabla^{\mathsf{v}}_{\widetilde{X}}$, we define a differential operator $\nabla^{\mathsf{h}}_{\widetilde{X}}$, prescribing its action

$$\begin{aligned}&\text{on } \textit{functions} \text{ by } \ \nabla^{\mathsf{h}}_{\widetilde{X}} f := (\mathscr{H}\widetilde{X})f = (df \circ \mathscr{H})(\widetilde{X}),\ f \in C^\infty(\overset{\circ}{T}M);\\ &\text{on } \textit{sections} \text{ by } \ \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := \mathscr{V}[\mathscr{H}\widetilde{X}, \mathbf{i}\widetilde{Y}],\ \widetilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}),\end{aligned} \tag{14}$$

and extending it to the whole tensor algebra $\mathcal{T}(\overset{\circ}{\pi})$ in such a way that $\nabla^{\mathsf{h}}_{\widetilde{X}}$ satisfies the derivation property. Finally, we define the ∇^{h}-*differential* of a tensor formally in the same way as in the vertical case. The operator ∇^{h} so obtained will be called the *h-Berwald derivative* determined by the Ehresmann connection $\mathscr{H}$.

Putting together the canonical v-covariant derivative and the h-Berwald derivative, we get an all-important covariant derivative operator in $\overset{\circ}{\pi}$, called the **Berwald derivative** associated to (or determined by) $\mathscr{H}$. Explicitly, for any vector field ξ on $\overset{\circ}{T}M$ and section $\widetilde{Y}$ in $\mathfrak{X}(\overset{\circ}{\tau})$,

$$\nabla_\xi \widetilde{Y} := \nabla^{\mathsf{v}}_{\mathscr{V}\xi}\widetilde{Y} + \nabla^{\mathsf{h}}_{\mathbf{j}\xi}\widetilde{Y} = \mathbf{j}[\mathbf{v}\xi, \mathscr{H}\widetilde{Y}] + \mathscr{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}]. \tag{15}$$

Then, in particular,

$$\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}, \quad \nabla_{\mathscr{H}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}; \quad \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}); \tag{16}$$

$$\nabla_{X^{\mathsf{v}}}\widehat{Y} = 0, \quad \mathbf{i}\nabla_{X^{\mathsf{h}}}\widehat{Y} = [X^{\mathsf{h}}, Y^{\mathsf{v}}]; \quad X, Y \in \mathfrak{X}(M). \tag{17}$$

From the last relation and (13) we obtain

$$\mathbf{i}\nabla_{\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}}} \widehat{\frac{\partial}{\partial u^k}} = \frac{\partial N^i_j}{\partial y^k}\frac{\partial}{\partial y^i} \quad (j, k \in \{1, \dots, n\});$$

the functions $N^i_{jk} := \frac{\partial N^i_j}{\partial y^k}$ are the *Christoffel symbols* of the Berwald derivative determined by $\mathscr{H}$.

Geometric data. By the *tension* of an Ehresmann connection we mean the h-Berwald differential of the canonical section, i.e., the type $\binom{1}{1}$ tensor $\mathbf{t} := \nabla^{\mathsf{h}}\delta$. An Ehresmann connection is said to be *homogeneous* if its tension vanishes. For any vector field X on M,

$$\mathbf{i}\mathbf{t}(\widehat{X}) = \mathbf{i}\nabla_{X^{\mathsf{h}}}\delta \overset{(14)}{=} \mathbf{i}\mathscr{V}[X^{\mathsf{h}}, C] = \mathbf{v}[X^{\mathsf{h}}, C] = [X^{\mathsf{h}}, C],$$

therefore for each $j \in \{1, \dots, n\}$,

$$\mathbf{it}\left(\widehat{\frac{\partial}{\partial u^j}}\right) = \left[\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}}, C\right] \overset{(13)}{=} \left[\frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i}, C\right] = (CN^i_j - N^i_j)\frac{\partial}{\partial y^i}.$$

It follows that *an Ehresmann connection is homogeneous, if and only if, its Christoffel symbols are positive-homogeneous of degree 1.*

The canonical surjection $\mathbf{j}$ in (3) may be interpreted as a $\mathring{\pi}$-valued 1-form on $\mathring{T}M$, so we can form its covariant exterior derivative $d^{\nabla}\mathbf{j}$ with respect to the Berwald derivative associated to an Ehresmann connection $\mathscr{H}$. By the *torsion of* $\mathscr{H}$ we mean the type $\binom{1}{2}$ tensor $\mathbf{T}$ defined by

$$\mathbf{T}(\widetilde{X}, \widetilde{Y}) := d^{\nabla}\mathbf{j}(\mathscr{H}\widetilde{X}, \mathscr{H}\widetilde{Y}); \quad \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\mathring{\tau}).$$

Evaluating at basic sections $\widehat{X}$, $\widehat{Y}$, we get the more suggestive formula

$$\mathbf{iT}(\widehat{X}, \widehat{Y}) = [X^{\mathsf{h}}, Y^{\mathsf{v}}] - [Y^{\mathsf{h}}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}}.$$

h-exterior and v-Lie derivative. Having specified an Ehresmann connection $\mathscr{H}$, we may define the operator d^{h} of the *h-exterior derivative* with respect to $\mathscr{H}$, prescribing its action

$$\begin{aligned} &\text{on } \textit{functions} \text{ by} && d^{\mathsf{h}} f(\widetilde{X}) = \nabla^{\mathsf{h}}_{\widetilde{X}} f; \quad f \in C^{\infty}(\mathring{T}M), \widetilde{X} \in \mathfrak{X}(\mathring{\tau}); \\ &\text{on } \textit{basic 1-forms} \text{ by} && d^{\mathsf{h}}(\alpha \circ \tau) := (d\alpha) \circ \tau, \quad \alpha \in \mathfrak{X}^*(M). \end{aligned} \tag{18}$$

Then d^{h} is a graded derivation of the Grassmann algebra $\Omega(\mathring{\tau})$ of differential forms along $\mathring{\tau}$. It may be shown that *if the torsion of the Ehresmann connection vanishes, then for any k-form $\widetilde{\alpha}$ along $\mathring{\tau}$ we have*

$$d^{\mathsf{h}}\widetilde{\alpha} = (k+1)\mathrm{Alt}\nabla^{\mathsf{h}}\widetilde{\alpha} \tag{19}$$

(cf. (12)), *and*

$$d^{\mathsf{h}} \circ d^{\mathsf{v}} + d^{\mathsf{v}} \circ d^{\mathsf{h}} = 0. \tag{20}$$

For a proof, see Ref. 21.

To complete our calculus summary, we define a kind of *Lie derivative* $\mathcal{L}^{\mathsf{v}}_{\xi}$ in $\mathfrak{T}(\mathring{\pi})$ ($\xi \in \mathfrak{X}(\mathring{T}M)$) by

$$\begin{aligned} \mathcal{L}^{\mathsf{v}}_{\xi} f &:= \xi(f), \quad \text{if } f \in C^{\infty}(\mathring{T}M); \\ \mathcal{L}^{\mathsf{v}}_{\xi}\widetilde{Y} &:= \mathscr{V}[\xi, \mathbf{i}\widetilde{Y}], \quad \text{if } \widetilde{Y} \in \mathfrak{X}(\mathring{\tau}) \end{aligned} \tag{21}$$

where $\mathscr{V}$ is the vertical map associated to $\mathscr{H}$.

Then, in particular, for any vector fields X, Y on M,

$$\mathbf{i}\mathcal{L}^{\mathsf{v}}_{X^{\mathsf{c}}}\widehat{Y} := \mathbf{i}\mathscr{V}[X^{\mathsf{c}}, Y^{\mathsf{v}}] = \mathbf{v}[X^{\mathsf{c}}, Y^{\mathsf{v}}] = [X^{\mathsf{c}}, Y^{\mathsf{v}}] = \mathcal{L}_{X^{\mathsf{c}}}Y^{\mathsf{v}},$$

so our generalization of the classical Lie derivative is reasonable. This is also confirmed by the fact that the operators $\mathcal{L}^{\mathsf{v}}$, d^{v} and the substitution operator $i_{\widetilde{X}}$ ($\widetilde{X} \in \mathfrak{X}(\mathring{\tau})$) are related by

$$\mathcal{L}^{\mathsf{v}}_{\mathbf{i}\widetilde{X}} = i_{\widetilde{X}} \circ d^{\mathsf{v}} + d^{\mathsf{v}} \circ i_{\widetilde{X}}, \tag{22}$$

which is a just a mutation of H. Cartan's magic formula.

Homogeneous tensors. Let k be an integer. A covariant or a $\mathring{\pi}$-valued covariant tensor $\widetilde{\alpha}$ along $\mathring{\tau}$ is said to be *homogeneous of degree* k, briefly k*-homogeneous*, if $\nabla_C\widetilde{\alpha} = k\alpha$.

Lemma 4.1. *If $\widetilde{\alpha}$ is a 0-homogeneous 2-form along $\mathring{\tau}$, then we have*

$$\widetilde{\alpha} = \frac{1}{2}(i_\delta d^{\mathsf{v}}\widetilde{\alpha} + d^{\mathsf{v}} i_\delta \widetilde{\alpha}).$$

Proof. By 'Cartan's formula' (22),

$$i_\delta d^{\mathsf{v}}\widetilde{\alpha} + d^{\mathsf{v}} i_\delta\widetilde{\alpha} = \mathcal{L}^{\mathsf{v}}_{\mathbf{i}\delta}\widetilde{\alpha} = \mathcal{L}^{\mathsf{v}}_C\widetilde{\alpha}.$$

For any vector fields X, Y on M,

$$\begin{aligned}
\mathcal{L}^{\mathsf{v}}_C\widetilde{\alpha}(\widehat{X}, \widehat{Y}) &= C\widetilde{\alpha}(\widehat{X}, \widehat{Y}) - \widetilde{\alpha}(\mathcal{L}^{\mathsf{v}}_C\widehat{X}, \widehat{Y}) - \widetilde{\alpha}(\widehat{X}, \mathcal{L}^{\mathsf{v}}_C\widehat{Y}) \overset{(21)}{=} \\
&= C\widetilde{\alpha}(\widehat{X}, \widehat{Y}) - \widetilde{\alpha}(\mathscr{V}[C, X^{\mathsf{v}}], \widehat{Y}) - \widetilde{\alpha}(\widehat{X}, \mathscr{V}[C, Y^{\mathsf{v}}]) \overset{(6)}{=} \\
&= C\widetilde{\alpha}(\widehat{X}, \widehat{Y}) + 2\widetilde{\alpha}(\widehat{X}, \widehat{Y}) = \\
&= (\nabla_C\widetilde{\alpha})(\widehat{X}, \widehat{Y}) + 2\widetilde{\alpha}(\widehat{X}, \widehat{Y}) = 2\widetilde{\alpha}(\widehat{X}, \widehat{Y}).
\end{aligned}$$

This proves our assertion. □

5. Curvatures of an Ehresmann connection

Throughout this section we assume that an Ehresmann connection $\mathscr{H}$ is specified on TM. ∇ is the Berwald derivative determined by $\mathscr{H}$, d^∇ is the corresponding covariant exterior derivative. We denote by R^∇ the curvature of ∇.

Affine curvatures. The vertical map $\mathscr{V}$ associated to $\mathscr{H}$ may also be interpreted as a $\mathring{\pi}$-valued 1-form on $\mathring{T}M$, so it makes sense to speak of its covariant exterior derivative. By the *fundamental affine curvature of* $\mathscr{H}$ we mean the type $\binom{1}{2}$ tensor field $\mathbf{R}$ along $\mathring{\tau}$ defined by

$$\mathbf{R}(\widetilde{X},\widetilde{Y}) := d^{\nabla}\mathscr{V}(\mathscr{H}\widetilde{X},\mathscr{H}\widetilde{Y}); \qquad \widetilde{X},\widetilde{Y}\in\mathfrak{X}(\mathring{\tau}). \tag{23}$$

Then for vector fields X, Y on M we get

$$\mathbf{iR}(\widehat{X},\widehat{Y}) = -\mathbf{v}[X^{\mathsf{h}},Y^{\mathsf{h}}], \tag{24}$$

so $\mathbf{R}$ is just the obstruction tensor to integrability of the horizontal distribution $\mathrm{Im}(\mathscr{H})\subset T\mathring{T}M$.

We define the *affine curvature tensor* $\mathbf{H}$ of the Ehresmann connection by

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathscr{H}\widetilde{X},\mathscr{H}\widetilde{Y})\widetilde{Z}. \tag{25}$$

It is related to the fundamental affine curvature by

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\widetilde{Z} = (\nabla^{\mathsf{v}}_{\widetilde{Z}}\mathbf{R})(\widetilde{X},\widetilde{Y}). \tag{26}$$

We remark that the terms 'fundamental affine curvature' and 'affine curvature' are borrowed from Berwald's paper,[5] whose terminology we try to adopt as far as possible.

The Berwald curvature. An immediate calculation shows that for any sections $\widetilde{X},\widetilde{Y},\widetilde{Z}$ along $\mathring{\tau}$, $R^{\nabla}(\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y})\widetilde{Z}=0$. However, the type $\binom{1}{3}$ tensor $\mathbf{B}$ along $\mathring{\tau}$ defined by

$$\mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathbf{i}\widetilde{X},\mathscr{H}\widetilde{Y})\widetilde{Z}, \tag{27}$$

and called the *Berwald curvature* of the Ehresmann connection, comprises important new information. Notice first that for any vector fields X, Y, Z on M,

$$\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} = [[X^{\mathsf{v}},Y^{\mathsf{h}}],Z^{\mathsf{v}}]. \tag{28}$$

From this relation, using the Jacobi identity and the fact that the Lie brackets of vertically lifted vector fields vanish, we infer: *the Berwald curvature is symmetric in its first and third variable. If,* in addition, *the Ehresmann connection is homogeneous or torsion-free, then the Berwald curvature is totally symmetric.*[21]

To clarify the meaning of the Berwald curvature, we recall that any covariant derivative operator D on M gives rise to an Ehresmann connection $\mathscr{H}_D$ on TM by the rule

$$(v,w)\in TM\times_M TM \longmapsto \mathscr{H}_D(v,w) := Y_*(w) - \mathbf{i}(v, D_wY)\in TTM,$$

where $Y \in \mathfrak{X}(M)$ is any vector field such that $Y(\tau(v)) = v$. Then $\mathscr{H}_D$ *is homogeneous and has vanishing Berwald curvature.* The torsions T^D of D and $\mathbf{T}$ of $\mathscr{H}_D$ are related by

$$(T^D(X,Y))^{\mathsf{v}} = \mathbf{i}\mathbf{T}(\widehat{X},\widehat{Y}); \qquad X,Y \in \mathfrak{X}(M), \tag{29}$$

while the relation between the curvature R^D of D and the affine curvature $\mathbf{H}$ of $\mathscr{H}_D$ is given by

$$(R^D(X,Y)Z)^{\mathsf{v}} = \mathbf{i}\mathbf{H}(\widehat{X},\widehat{Y})\widehat{Z}; \qquad X,Y,Z \in \mathfrak{X}(M). \tag{30}$$

Conversely, if $\mathscr{H}$ is a *homogeneous* Ehresmann connection *of class* C^1 on $TM \times_M TM$, then there exists a (necessarily unique) covariant derivative D on M such that for any vector fields X, Y on M we have

$$(D_X Y)^{\mathsf{v}} = \mathbf{i}\nabla^{\mathsf{h}}_{\widehat{X}}\widehat{Y} = [X^{\mathsf{h}}, Y^{\mathsf{v}}]. \tag{31}$$

The Ehresmann connection arising from D is just the given connection $\mathscr{H}$, therefore *the Berwald curvature of* $\mathscr{H}$ *vanishes.*

6. Semisprays and Ehresmann connections

A semispray S over M is said to be a *spray* if it is of class C^1 on TM and positive-homogeneous of degree 2 (briefly, 2^+-homogeneous) in the sense that $[C, S] = S$. By an *affine spray* we mean a spray which is of class C^2 (and hence is smooth) on its whole domain TM. Equivalently, an affine spray is a 2^+-homogeneous second-order vector field.

There is an important relation between affine sprays and covariant derivatives, formulated clearly and explicitly by Ambrose, Palais and Singer[1] first. In our terms, *if* $S \in \mathfrak{X}(TM)$ *is an affine spray, then there exists a covariant derivative* D *on* M *such that* $S = \mathscr{H}_D \circ \delta$. Conversely, if D is a covariant derivative on M and $\mathscr{H}_D$ is the Ehresmann connection determined by D, then $S := \mathscr{H}_D \circ \delta$ is an affine spray. This construction was generalized to a great extent by M. Crampin and J. Grifone in the early 1970s, independently (see Ref. 6 and Ref. 9).

Theorem 6.1 (M. Crampin and J. Grifone). *If* S *is a semispray over* M*, then there exists a unique Ehresmann connection* $\mathscr{H}_S$ *on* TM *such that the horizontal lift of a vector field on* M *with respect to* $\mathscr{H}_S$ *is given by*

$$X^{\mathsf{h}} := \mathscr{H}_S(\widehat{X}) = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]) \qquad \text{(Crampin's formula)}, \tag{32}$$

or, equivalently, the horizontal projector associated to $\mathscr{H}_S$ *is*

$$\mathbf{h}_S = \frac{1}{2}(1_{\mathfrak{X}(TM)} + [\mathbf{J}, S]) \qquad \text{(Grifone's formula)}. \tag{33}$$

We list some basic properties of $\mathscr{H}_S$.

CrGr 1. *The torsion of $\mathscr{H}_S$ vanishes.*

This can be checked by a direct calculation.

CrGr 2. *If $\mathscr{H}$ is an Ehresmann connection with vanishing torsion, then there is a semispray S such that $\mathscr{H} = \mathscr{H}_S$.*

This result of M. Crampin[7] is far from being trivial, see also E. K. Ayassou's *Thèse de doctorat.*[2]

CrGr 3. *$\mathscr{H}_S$ is homogeneous, if and only if, there is a 2^+-homogeneous semispray $\bar{S}$ and a vector field X on M such that $S = \bar{S} + X^{\mathsf{v}}$.*

For a proof we refer to Ref. 21.

CrGr 4. *The semispray $\mathscr{H}_S \circ \delta$ coincides with S, if and only if, $[C, S] = S$.*

Applying, e.g., Grifone's formula (33) and Grifone's identity (9), the proof of this claim is routine.

By the (affine and Berwald) *curvatures of a semispray S* we shall mean the (corresponding) curvatures of $\mathscr{H}_S$. When there is no risk of confusion, instead of $\mathscr{H}_S$ we shall simply write $\mathscr{H}$.

The *affine deviation tensor* (L. Berwald[5]) or the *Jacobi endomorphism* (W. Sarlet et al.[15]) of a semispray S is the type $\binom{1}{1}$ tensor $\mathbf{K}$ defined by

$$\mathbf{K}(\widetilde{X}) := \mathscr{V}[S, \mathscr{H}\widetilde{X}]; \qquad \widetilde{X} \in \mathfrak{X}(\mathring{\tau}), \quad \mathscr{H} := \mathscr{H}_S. \tag{34}$$

The Jacobi endomorphism and the fundamental affine curvature $\mathbf{R}$ of S are related by

$$\mathbf{R}(\widetilde{X}, \widetilde{Y}) = \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{K}(\widetilde{Y}, \widetilde{X}) - \nabla^{\mathsf{v}}\mathbf{K}(\widetilde{X}, \widetilde{Y})); \quad \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\mathring{\tau}), \tag{35}$$

for a proof see Ref. 21.

If S is a *spray*, then the tensors $\mathbf{K}$, $\mathbf{R}$ and $\mathbf{H}$ carry the same information: each of them can be expressed from another one. Adopting Z. Shen's terminology,[20] we say that a spray is *R-flat*, if one (hence every) of its affine curvatures vanishes. By a *flat spray* we mean an affine R-flat spray. Property **CrGr 3** and our remarks concerning the Berwald curvature at the end of the preceding section imply immediately the following

Characterization of affine sprays. *A spray is affine, if and only if, its Berwald curvature vanishes.*

The next important observation goes back to the 1920s, i.e., to the first golden age of the 'geometry of paths'.

Characterization of flat sprays. *A spray S is flat, if and only if, there is a coordinate system $(u^i)_{i=1}^n$ for M at any point $p \in M$ such that the* *coordinate expression of S in the induced coordinate system $(x^i, y^i)_{i=1}^n$ is* $S = y^i \frac{\partial}{\partial x^i}$.

Then (x^i, y^i) will be called a *rectilinear coordinate system* on TM.

We sketch here the main steps of the proof, cf. Ref. 20: 8.1.6. The sufficiency of the condition is obvious. To prove the converse, suppose that S is flat. Then $\mathbf{B} = 0$ and $\mathbf{H} = 0$. Let the coordinate expression of S in an induced coordinate system $(x_0^i, y_0^i)_{i=1}^n$ $(x_0^i := u_0^i \circ \tau)$ be

$$S = y_0^i \frac{\partial}{\partial x_0^i} - 2\overset{\circ}{G}{}^i \frac{\partial}{\partial y_0^i}.$$

Condition $\mathbf{B} = 0$ implies the existence of a covariant derivative D on M satisfying (31). If the Christoffel symbols of D with respect to $(u_0^i)_{i=1}^n$ are the functions $\overset{\circ}{\Gamma}{}^i_{jk}$, then

$$\overset{\circ}{\Gamma}{}^i_{jk} \circ \tau = \overset{\circ}{G}{}^i_{jk} := \frac{\partial^2 \overset{\circ}{G}{}^i}{\partial y_0^j \partial y_0^k}; \qquad i, j, k \in \{1, \dots, n\}. \tag{36}$$

Our assumption $\mathbf{H} = 0$ implies by (30) that $R^D = 0$. From this we infer that there is a chart $(U, (u^i)_{i=1}^n)$ on M such that in the u^i*-coordinates the Christoffel symbols of D vanish.* If the spray coefficients of S in the induced coordinate system $(x^i, y^i)_{i=1}^n$ are the functions G^i, then $0 = (\Gamma^k_{ij} \circ \tau) y^k \overset{(36)}{=} G^i_{jk} y^k = G^i_j$ and $0 = G^i_j y^j = 2G^i$ by the 2^+-homogeneity of the G^is, therefore $S \restriction \overset{\circ}{\tau}{}^{-1}(U) = y^i \frac{\partial}{\partial x^i}$.

We recall that two sprays S and $\bar{S}$ over M are said to be *projectively related* if there is a function P, smooth on $\overset{\circ}{T}M$, of class C^1 on TM such that $\bar{S} = S - 2PC$. Then the *projective factor* P is necessarily positive-homogeneous of degree 1. We shall find useful the following simple

Observation. *A spray S is projectively related to a given spray $\bar{S}$, if and only if, S is contained by the $C^\infty(\overset{\circ}{T}M)$-submodule of $\mathfrak{X}(\overset{\circ}{T}M)$ generated by $\bar{S}$ and C.*

Indeed, the necessity of the condition is obvious. Conversely, if $S = f\bar{S} + hC$ $(f, h \in C^\infty(\overset{\circ}{T}M))$, then using relation (8) we get $C = fC$. Hence f is the constant function with the value 1, and $S = \bar{S} + hC$, as we claimed.

7. Basic facts on Finsler functions

By a *Finsler function* we mean a function $F \colon TM \to \mathbb{R}$ satisfying the following conditions:

Fins 1. *F is smooth on $\overset{\circ}{T}M$.*
Fins 2. *$CF = F$, i.e., F is positive-homogeneous of degree 1.*
Fins 3. *The metric tensor $g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2$ is (fibrewise) non-degenerate.*

A pair (M, F) consisting of a manifold and a Finsler function on its tangent manifold is called a *Finsler manifold*. A Finsler manifold (M, F) is *positive definite* if the condition

Fins 4. *$F(v) > 0$ for all $v \in \overset{\circ}{T}M$ and $F(0) = 0$*

is also satisfied. It may be shown (see Ref. 14) that the nomenclature is correct: **Fins 4** implies that the metric tensor is positive definite. It is immediately verified that a positive definite Finsler manifold (M, F) reduces to a Riemannian manifold (M, γ) in the sense that $g = \gamma \circ \tau$, if and only if, $\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 = 0$.

If F is a Finsler function, then $\theta := d_{\mathbf{J}}F$ is a 1-form, $\omega := dd_{\mathbf{J}}F^2$ is a 2-form on $\overset{\circ}{T}M$, called the *Hilbert 1-form* and the *fundamental 2-form* of (M, F), respectively. **Fins 3** implies that ω is non-degenerate, and the converse is also true. The 1-forms θ and $d_{\mathbf{J}}F^2$, and the 2-form ω have the following properties:

$$\mathcal{L}_C\theta = 0, \quad \mathcal{L}_C d_{\mathbf{J}}F^2 = d_{\mathbf{J}}F^2, \quad \mathcal{L}_C\omega = \omega; \tag{37}$$

$$i_C\omega = d_{\mathbf{J}}F^2; \tag{38}$$

$$\omega(\mathbf{J}\xi, \eta) + \omega(\xi, \mathbf{J}\eta) = 0 \quad (\xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M)), \text{ i.e.,} \tag{39}$$
the vertical endomorphism is skew-symmetric with respect to the fundamental 2-form.

The next result is the miracle of Finsler geometry.

Proposition 7.1 (the fundamental lemma of Finsler geometry) and definition. *If (M, F) is a Finsler manifold, then there exists a unique spray S such that $i_S\omega = -dF^2$. The Ehresmann connection $\mathscr{H}$ determined by S according to Theorem 6.1 is homogeneous and* conservative *in the sense that $d^{\mathsf{h}}F = 0$. $\mathscr{H}$ is said to be the* canonical connection *of (M, F); this is the only torsion-free, homogeneous, conservative Ehresmann connection on TM.*

The first intrinsic formulation and index-free proof of the fundamental lemma is due to J. Grifone.[9] For our formulation and recent proofs we refer to Refs. 21,23.

Lemma 7.1. *If S is spray over M, then it is related to the canonical spray $\bar{S}$ of a Finsler manifold $(M, \bar{F})$ over $\overset{\circ}{T}M$ by*

$$\bar{S} = S - \frac{S\bar{F}}{\bar{F}}C - \bar{F}(i_S dd_{\mathbf{J}}\bar{F})^{\sharp} \tag{40}$$

where the sharp operator is taken with respect to the fundamental 2-form of $(M, \bar{F})$.

A proof of this useful observation can be found in Ref. 24.

8. Projectively Finslerian sprays

We say that a spray is *Finsler metrizable in a broad sense*, or *projectively Finslerian* if there is a Finsler function whose canonical spray is *projectively related* to the given spray. From relation (40) it follows at once that a spray S is projectively related to the canonical spray of a Finsler manifold $(M, \bar{F})$, if and only if, $i_S dd_{\mathbf{J}}\bar{F} = 0$. In this section we give a more illuminating derivation of this important observation, and deal with several equivalent characterizations of Finsler metrizability in a broad sense.

We need some preparatory results.

Lemma 8.1. *A function $\bar{F}\colon TM \to \mathbb{R}$ satisfying **Fins 1** and **Fins 2** is a Finsler function, if and only if, the 2-form $dd_{\mathbf{J}}\bar{F}$ on $\overset{\circ}{T}M$ is of rank $2n-2$.*

The proof is quite immediate.

Lemma 8.2. *If $\bar{F}$ is a Finsler function, then the nullspace of the 2-form $dd_{\mathbf{J}}\bar{F}$ is generated by the canonical spray of $\bar{F}$ and the Liouville vector field.*

By the preceding Lemma it is enough to check that the canonical spray and the Liouville vector field indeed belong to the nullspace, and this is a routine verification.

Lemma 8.3. *Let $\bar{F}\colon TM \to \mathbb{R}$ be a function satisfying **Fins 1** and **Fins 2**. Let S be a spray over M. If $\nabla = (\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ is the Berwald derivative determined by S, then relations*

$$i_S dd_{\mathbf{J}}\bar{F} = 0 \qquad \text{and} \qquad \nabla_S \nabla^{\mathsf{v}}\bar{F} = \nabla^{\mathsf{h}}\bar{F}$$

are both equivalent to the 'Euler–Lagrange equation'

$$S(X^{\mathsf{v}}\bar{F}) - X^{\mathsf{c}}\bar{F} = 0, \qquad X \in \mathfrak{X}(M). \tag{41}$$

The proof is just a calculation again: it is enough to evaluate the 1-form $i_S dd_{\mathbf{J}}\bar{F}$ on $\overset{\circ}{T}M$ at a complete lift and the Finsler 1-form $\nabla_S\nabla^{\mathsf{v}}\bar{F} - \nabla^{\mathsf{h}}\bar{F}$ at a basic vector field.

Theorem 8.1. *Let S be a spray over M, and let $\nabla = (\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ be the Berwald derivative determined by S. If $\bar{F}$ is a Finsler function, then the following four statements are equivalent:*

Proj. *S and the canonical spray of $(M, \bar{F})$ are projectively related.*
Rap 1. $i_S dd_{\mathbf{J}}\bar{F} = 0$.
Rap 2. $\nabla_S\nabla^{\mathsf{v}}\bar{F} = \nabla^{\mathsf{h}}\bar{F}$.
Rap 3. $d^{\mathsf{h}}d^{\mathsf{v}}\bar{F} = 0$.

If in an induced coordinate system $(x^i, y^i)_{i=1}^n$ on TM the spray coefficients of S are the functions G^i, and $G^i_j := \frac{\partial G^i}{\partial y^j}$, then the common local expression of **Rap 1** *and* **Rap 2** *is*

$$y^i \frac{\partial^2 \bar{F}}{\partial x^i \partial y^j} - 2G^i \frac{\partial^2 \bar{F}}{\partial y^i \partial y^j} = \frac{\partial \bar{F}}{\partial x^j} \qquad (j \in \{1, \dots, n\}), \tag{42}$$

while the local expression of **Rap 3** *is*

$$\frac{\partial^2 \bar{F}}{\partial x^j \partial y^k} - G^i_j \frac{\partial^2 \bar{F}}{\partial y^i \partial y^k} = \frac{\partial^2 \bar{F}}{\partial x^k \partial y^j} - G^i_k \frac{\partial^2 \bar{F}}{\partial y^i \partial y^j} \qquad (j, k \in \{1, \dots, n\}) \tag{43}$$

Proof. Let $\bar{S}$ be the canonical spray of $(M, \bar{F})$. By the Observation in the end of section 6, S and $\bar{S}$ are projectively related, if and only if, S is in the submodule generated by $\bar{S}$ and C. This submodule is just the nullspace of $dd_{\mathbf{J}}\bar{F}$ by Lemma 8.2, so we obtain the equivalence of **Proj** and **Rap 1**. Since for any vector field X on M we have

$$d^{\mathsf{h}}d^{\mathsf{v}}\bar{F}(\delta, \widehat{X}) = S(X^{\mathsf{v}}\bar{F}) - X^{\mathsf{c}}\bar{F},$$

applying Lemma 8.3 we conclude that **Rap 3** implies **Rap 1**. To show the converse, observe that the 2-form $d^{\mathsf{h}}d^{\mathsf{v}}\bar{F}$ is 0^+-homogeneous, i.e., $\nabla_C d^{\mathsf{h}}d^{\mathsf{v}}\bar{F} = 0$, and hence it may be represented in the form

$$d^{\mathsf{h}}d^{\mathsf{v}}\bar{F} = \frac{1}{2}(i_\delta d^{\mathsf{v}}d^{\mathsf{h}}d^{\mathsf{v}}\bar{F} + d^{\mathsf{v}}i_\delta d^{\mathsf{h}}d^{\mathsf{v}}\bar{F}) \tag{44}$$

by Lemma 4.1. Using (20) and relation $d^{\mathsf{v}} \circ d^{\mathsf{v}} = 0$, it follows that the first term in the right-hand side of (44) vanishes. As to the second term, taking into account Lemma 8.3 again, we get for any vector field X on M

$$i_\delta d^{\mathsf{h}}d^{\mathsf{v}}\bar{F}(\widehat{X}) = d^{\mathsf{h}}d^{\mathsf{v}}\bar{F}(\delta, \widehat{X}) = S(X^{\mathsf{v}}\bar{F}) - X^{\mathsf{c}}\bar{F} = i_S dd_{\mathbf{J}}\bar{F}(\widehat{X}),$$

hence assuming **Rap 1**, we conclude that $d^{\mathsf{h}} d^{\mathsf{v}} \bar{F} = 0$.
Our statements concerning the coordinate expressions can immediately be checked. □

Relations (42) and (43), as criteria for Finsler metrizability of a spray in a broad sense are due to A. Rapcsák.[18] The index-free form **Rap 1** and an equivalent of **Rap 3** were first published in Ref. 24. Later I realized that the equivalence of **Proj** and **Rap 1** has already been discovered by J. Klein and A. Voutier.[13] There is nothing new under the sun...

Now I am in a position to make some remarks about the following

Analytic version of Hilbert's 4th problem: *find the Finsler functions whose canonical spray is projectively related to a given* **flat spray**.

This reformulation of Hilbert's problem 4 differs very much from Hilbert's original formulation,[12] in which he immediately connected the problem to his actual research on the axiomatic foundations of geometry. However, the first thorough approach to the problem was analytic in G. Hamel's thesis.[10] Hamel's work was supervised by Hilbert, immediately after his famous lecture in Paris in 1900. This indicates that the reformulation of the problem in differential geometric terms alien to classical geometry was found relevant by Hilbert himself. The next corollary of Theorem 8.1 makes it clear that our formulation is just an index-free expression of Hamel's analytic interpretation of Hilbert's 4th problem.

Corollary 8.1. *A flat spray is (locally) projectively related to the canonical spray of a Finsler function $\bar{F}$, if and only if, one of the following relations holds in any rectilinear coordinate system:*

$$\textbf{Ham 1}. \qquad y^i \frac{\partial^2 \bar{F}}{\partial x^i \partial y^j} = \frac{\partial \bar{F}}{\partial x^j}, \qquad j \in \{1, \dots, n\}.$$

$$\textbf{Ham 2}. \qquad \frac{\partial^2 \bar{F}}{\partial x^i \partial y^k} = \frac{\partial^2 \bar{F}}{\partial x^k \partial y^j}, \qquad j, k \in \{1, \dots, n\}.$$

In Hamel's paper[11] equation **Ham 2** was derived and solved in 3 dimensions. In the 2-dimensional case an extremely elegant solution of **Ham 2** was given by A. V. Pogorelov;[17] see also Álvarez Paiva's delightful account.[3]

The next observation makes it possible to transform the problem into the quest of a suitable 0^+-homogeneous function instead of a 1^+-homogeneous Finsler function.

Proposition 8.1. *A flat spray S is (locally) projectively Finslerian, if and only if, there is a 0^+-homogeneous function f on TM, smooth on $\mathring{T}M$, such that $\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}(Sf)^2$ is a non-degenerate 2-form along $\mathring{\tau}$, and in any rectilinear coordinate system $(x^i, y^i)_{i=1}^n$ on TM we have*

$$\frac{\partial f}{\partial y^j \partial x^k} y^k = 0, \qquad j \in \{1, \ldots, n\}. \tag{45}$$

A sketchy solution of (45) can be found in Rapcsák's paper,[19] but his solution is probably incomplete. On the other side, an interesting new strategy has been shown to solve Hilbert's fourth problem by Álvarez Paiva in Ref. 4. Notice that the Liouville vector field and the given spray S generate an integrable distribution, and we can (at least locally) take the quotient by its leaves. The result is a manifold of dimension $2n-2$, the so-called *path space*, whose points represent the unparametrized geodesics of S. It follows from Lemma 8.2 that $dd_{\mathbf{J}}\bar{F}$ defines a symplectic 2-form on the path space. The idea of Álvarez Paiva is to specify the properties of the symplectic form on the path space. It would be illuminating to establish an exact relation between the solutions of (45) and the general form of Álvarez Paiva's 'admissible symplectic forms'.

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References

1. W. Ambrose, R.S. Palais and I.M. Singer, Sprays, *An. Acad. Bras. Ciênc.* **32** (1960) 163–178.
2. E. K. Ayassou, Cohomologies définies pour une 1-forme vectorielle plate, Thèse de Doctorat, Université de Grenoble, 1985.
3. J.C. Álvarez Paiva, Hilbert's fourth problem in two dimensions, *In: Mass Selecta: teaching and learning advanced undergraduate mathematics* (Amer. Math. Soc., Providence, RI, 2003) 165–183.
4. J.C. Álvarez Paiva, Symplectic geometry and Hilbert's fourth problem, *J. Diff. Geom.* **69** (2005) 353–378.
5. L. Berwald, Ueber Finslersche und Cartansche Geometrie IV, *Ann. Math.* **48** (1947) 755–781.

6. M. Crampin, On horizontal distributions on the tangent bundle of a differentiable manifold, *J. London Math. Soc. (2)* **3** (1971) 178–182.
7. M. Crampin, Generalized Bianchi identities for horizontal distributions, *Math. Proc. Camb. Phil. Soc.* **94** (1983) 125–132.
8. M. de León and P.R. Rodrigues, *Methods of differential geometry in analytical mechanics* (North-Holland, Amsterdam, 1989).
9. J. Grifone, Structure presque tangente et connexions, I, *Ann. Inst. Fourier, Grenoble* **22** (1) (1972) 287–334.
10. G. Hamel, Über die Geometrien, in denen die Geraden die Kürzesten sind, Dissertation, Göttingen, 1901, pp. 90.
11. G. Hamel, Über die Geometrien, in denen die Geraden die Kürzesten sind, *Math. Ann.* **57** (1903) 231–264.
12. D. Hilbert, Mathematical Problems, *Bull. Am. Math. Soc.* **37** (4) (2000) 407–436.
13. J. Klein et A. Voutier, Formes extérieures génératrices de sprays, *Ann. Inst. Fourier, Grenoble* **18** (1) (1968) 241–260.
14. R.L. Lovas, A note on Finsler-Minkowski norms, *Houston J. Math.* **33** (2007) 701–707.
15. E. Martínez, J.F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Diff. Geometry and its Applications* **2** (1992) 17–43.
16. E. Martínez, J.F. Cariñena and W. Sarlet, *Derivations of differential forms along the tangent bundle projection, II*, *Ibid.* **3** (1993) 1–29.
17. A.V. Pogorelov, *Hilbert's fourth problem* (Scripta Series in Mathematics, Winston and Sons, New York, 1979).
18. A. Rapcsák, Über die bahntreuen Abbildungen metrischer Räume, *Publ. Math. Debrecen* **8** (1961) 285–290.
19. A. Rapcsák, Die Bestimmung der Grundfunktionen projektiv-ebener metrischer Räume, *Publ. Math. Debrecen* **9** (1962) 164–167.
20. Z. Shen, *Differential Geometry of Spray and Finsler Spaces* (Kluwer Academic Publishers, Dordrecht, 2001).
21. J. Szilasi, A Setting for Spray and Finsler Geometry, *In: Handbook of Finsler Geometry* (Kluwer Academic Publishers, Dordrecht, 2003) 1183–1426.
22. J. Szilasi and Á. Győry, A generalization of a theorem of H. Weyl, *Rep. Math. Phys.* **53** (2004) 261–273.
23. J. Szilasi and R.L. Lovas, Some aspects of differential theories, *In: Handbook of Global Analysis* (Elsevier, 2007) 1071–1116.
24. J. Szilasi and Sz. Vattamány, On the Finsler-metrizabilities of spray manifolds, *Periodica Mathematica Hungarica* **44** (2002) 81–100.