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PSEUDOCONNECTIONS AND FINSLER-TYPE CONNECTIONS

J. SZILASI - Z. KOVÁCS

1. INTRODUCTION

Koszul's concept of a linear connection is one of the basic definitions of elementary differential geometry. As is well-known, in Koszul's sense a linear connection ∇ on a manifold M is a correspondence $\nabla: (X,Y) \in \mathfrak{X}(M)\times\mathfrak{X}(M) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ which is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in Y and has the property

$$(*)\qquad \nabla_X fY = (Xf)Y + f\nabla_X Y \quad (f \in C^\infty(M)).$$

For the applications different generalizations of this concept are also important. A weakening of the Koszul-axioms yields e.g. the so-called homogeneous connections, see [7]. In 1969 an interesting modification of the defining conditions was treated by C. di Comite [2]. He considered a mapping $\nabla: \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ and an endomorphism A of the tangent bundle of M as well. Di Comite called the pair (∇, A) a pseudoconnection on M, if - instead of $(*)$ - ∇ satisfies the rule

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$$\nabla_X fY = (A\circ X) fY + f\nabla_X Y$$

and the further Koszul-axioms remain (formally) unaltered. This notion was extended and generalized to arbitrary vector bundles (of finite rank) by I. Candela in 1982 [1]. In the present paper we propose and discuss a further extension and generalization of Candela's concept of pseudoconnections, motivated by the demands of the theory of Finsler-type connections. We shall see, indeed, that this new type of generalized connections arises in a very natural manner in the construction of Finsler-type connections.

For the reader's convenience, in section 2 a minimum of the necessary preparatory material is collected. Pseudoconnections (in our sense) are introduced and illustrated in section 3. Section 4 is the core of the paper, here a "representation theorem" of Finsler-type connections is proved. The concluding section is devoted to a brief discussion of Berwald-connections. Further Finsler-geometric applications of pseudoconnections will be investigated in forthcoming papers.

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2. PRELIMINARIES

Our basic works of reference are [3] and [4], we adopt their conventions as closely as feasible. Accordingly, we work in the category of DIFF-manifolds (so all data will be assumed to be smooth in this paper) and in the Abelian category VB of vector bundles. In the sequel $\xi = (E,\pi,B,F) \in VB$ will denote a fixed vector bundle of rank r over the n-dimensional base manifold B with typical fiber F, Sec ξ is the $C^{\infty}(B)$-module of sections of ξ. In particular, the tangent bundle of an m-dimensional manifold M is the quadruple $\tau_M = (TM, \pi_M, M, \mathbb{R}^m)$. In this case we write $\mathfrak{X}(M)$ instead of Sec τ_M, as usual.

We need the simple

LEMMA 1. *If* $\tilde{\xi} = (\tilde{E}, \tilde{\pi}, \tilde{B}, \tilde{F}) \in VB$,

$$(g,f): \quad \xi \to \tilde{\xi}, \qquad \begin{array}{ccc} E & \xrightarrow{f} & \tilde{E} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ B & \xrightarrow{g} & \tilde{B} \end{array}$$

is a VB-morphism and f is fiberwise a linear isomorphism, then for each section $\tilde{\sigma} \in$ Sec $\tilde{\xi}$, *the mapping*

$$f^{\#}\sigma : B \to E, \quad x \mapsto f^{\#}\sigma(x) := f_x^{-1}[\tilde{\sigma}(g(x))]$$

is a section of ξ.

$V\xi = (VE, \pi_V, E, \mathbb{R}^r)$ denotes the *vertical subbundle* of τ_E and $\mathfrak{X}_V E :=$ Sec $V\xi$ is the module of vertical vector fields of E. As is well-known we have the following

LEMMA 2. *There exists a natural morphism*

$$(\pi,\alpha): \quad V\xi \to \xi, \qquad \begin{array}{ccc} VE & \xrightarrow{\alpha} & E \\ \pi_V \downarrow & & \downarrow \pi \\ E & \xrightarrow{\pi} & B \end{array} ;$$

where the so-called canonical map α is fiberwise an isomorphism.

Now let us consider the pullback bundle

$$\pi^*(\tau_B) = (E \times_B TB, pr_1, E, \mathbb{R}^n)$$

and the mapping

$$j: TE \to E\times_B TB,\ v \in T_zE \mapsto (z, T\pi(v))$$

(T is the tangent functor). Then j determines an E-morphism $j: \tau_E \to \pi^*(\tau_B)$ and we get the canonical short exact sequence

$$\text{(SEQ)} \qquad 0 \to V\xi \overset{i}{\to} \tau_E \overset{j}{\to} \pi^*(\tau_B) \to 0$$

(i is inclusion). A (right) splitting $\mathscr{H}: \pi^*(\tau_B) \to \tau_E$ of (SEQ) will be called a *horizontal map*. (Recall that $\mathscr{H}$ is characterized by $j\circ\mathscr{H} = 1_{\pi^*(\tau_B)}$.) $H\xi := \mathrm{Im}\mathscr{H}$ is the *horizontal subbundle*, $\mathfrak{X}_H E :=$ Sec $H\xi$ is the module of *horizontal vector fields* of E. It is an elementary but fundamental fact that $\tau_E = V\xi \oplus H\xi$ (Whitney sum). In the sequel, a *horizontal map* $\mathscr{H}$ *will be fixed.* determines the *horizontal projection* $h := \mathscr{H}\circ j : TE \to HE$ and the *vertical projection* $v = 1_{TE} - h: TE \to VE$, they will be important tools in our considerations.

In local calculations we choose and fix a trivializing map $\psi: V\times F \to \pi^{-1}(V)$ for ξ, a chart $(U;u)$ $U \subset V$, $u = (u^1,\dots,u^n)$ for B and a basis $(a_\kappa)_{\kappa=1}^r$ for F. We use the summation convention, with Latin indices running from 1 to n and Greek indices running from 1 to r. If $\rho: F \to \mathbb{R}^r$, $\xi^\kappa a_\kappa \mapsto (\xi^\kappa)$, then one gets the *fibered chart* $(\pi^{-1}(U), (u\times\rho)\circ\psi^{-1})$ for E. The corresponding coordinate-functions are

$$x^i := u^i\circ\pi \quad (1 \leqq i \leqq n)$$

and

$$y^\kappa := l^\kappa\circ p_2\circ\psi^{-1} \quad (1 \leqq \kappa \leqq r)$$

where (l^κ) is the dual of the basis (a_κ) and p_2 is the second projection of the product $U\times F$. We note that the mappings

$$e_\kappa : V \to E,\ x \mapsto e_\kappa(x) := \psi(x, a_\kappa) \quad (1 \leqq \kappa \leqq r)$$

constitute a *framing* ([4], Vol. 2, p. 352) for ξ. We also have the framings $\left(\frac{\partial}{\partial x^i}\right)_{i=1}^{n}$, $\left(\frac{\partial}{\partial y^\kappa}\right)_{\kappa=1}^{r}$ and $\left(\frac{\partial}{\partial y^\kappa}\right)_{\kappa=1}^{r}$ for τ_E and $V\xi$, resp. (actually they are constituted by vector fields defined on $\pi^{-1}(U)$). If the horizontal map $\mathscr{H}$ is described locally by the functions $N_i^\kappa : \pi^{-1}(U) \to \mathbb{R}$, then the vector fields

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^\kappa \frac{\partial}{\partial y^\kappa} : \pi^{-1}(U) \to TE \quad (1 \leqq i \leqq n)$$

provide a framing for $H\xi$ and we get the "adapted frame" $\left(\frac{\delta}{\delta x^i}\right)$, $\left(\frac{\partial}{\partial y^\kappa}\right)$ on τ_E over U.

With these notations one obtains the following statements:

LEMMA 3.

(i) $$\alpha^{\#} e_\beta = \frac{\partial}{\partial y^\beta} \quad (1 \leqq \beta \leqq r),$$

$$\alpha^{\#}\sigma = (\sigma^\beta \circ \pi)\frac{\partial}{\partial y^\beta} \quad (\sigma = \sigma^\beta e_\beta).$$

(ii) *Consider the VB-morphism*

$$(\pi, pr_2) : \pi^*(\tau_B) \to \tau_B, \quad pr_1 \downarrow \begin{array}{ccc} E \times_B TB & \xrightarrow{pr_2} & TB \\ \downarrow & & \downarrow \pi_B . \\ E & \xrightarrow{\pi} & B \end{array}$$

Then

$$pr_2^{\#} \frac{\partial}{\partial u^i} = j \circ \frac{\delta}{\delta x^i} \quad (1 \leq i \leq n),$$

consequently the sections $pr_2^{\#} \frac{\partial}{\partial u^i}$ *constitute a framing for* $\pi^*(\tau_B)$. *Moreover,*

$$pr_2^{\#} X = (X^i \circ \pi_B) pr_2^{\#} \frac{\partial}{\partial u^i} \quad \left(X = X^i \frac{\partial}{\partial u^i} \right).$$

PROOF. (i) is obvious from the definitions. To prove (ii), assume that $z \in F_x := \pi^{-1}(x)$, $x \in U$. Then

$$pr_1 \left(pr_2^{\#} \frac{\partial}{\partial u^i}(z) \right) = z, \text{ since } pr_2^{\#} \frac{\partial}{\partial u^i} \in \text{Sec } \pi^*(\tau_B).$$

On the other hand

$$pr_2 \left(pr_2^{\#} \frac{\partial}{\partial u^i}(z) \right) = pr_2 \left[(pr_2)_z^{-1} \left(\frac{\partial}{\partial u^i} \right)_x \right] = \left(\frac{\partial}{\partial u^i} \right)_x,$$

while

$$j \left(\frac{\delta}{\delta x^i} \right)_z = j \left[\left(\frac{\partial}{\partial x^i} \right)_z - N_i^\kappa(z) \left(\frac{\partial}{\partial y^\kappa} \right)_z \right] =$$

$$= \left(z, T\pi \left(\frac{\partial}{\partial x^i} \right)_z - N_i^\kappa(z) \, T\pi \left(\frac{\partial}{\partial y^\kappa} \right)_z \right) =$$

$$= \left(z, \left(\frac{\partial}{\partial u^i} \right)_x \right) \quad (1 \leq i \leq n),$$

thus (ii) also follows.

3. PSEUDOCONNECTIONS

A linear connection in ξ is a mapping ∇: Sec $\tau_B \times$ $\times$ Sec $\xi \to$ Sec ξ characterized on the analogy of the case $\xi = \tau_B$. Replacing now τ_B by an arbitrary vector bundle over B, we start from the following

DEFINITION. Let ξ and also $\tilde{\xi} = (\tilde{E}, \tilde{\pi}, B, \tilde{F}) \in VB$ be given. A *pseudoconnection* in the vector bundle ξ (with respect to $\tilde{\xi}$) is a pair (∇, A), where $A: \tilde{\xi} \to \tau_B$ is a B-morphism and ∇: Sec $\tilde{\xi} \times$ Sec $\xi \to$ Sec ξ is a mapping satisfying the following conditions:

(I) $\qquad \nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y;$

(II) $\qquad \nabla_X (Y_1+Y_2) = \nabla_X Y_1 + \nabla_X Y_2;$

(III) $\qquad \nabla_{fX} Y = f\nabla_X Y;$

(IV) $\qquad \nabla_X fY = (A \circ X) fY + f\nabla_X Y$

($X, X_1, X_2 \in$ Sec $\tilde{\xi}$; $Y, Y_1, Y_2 \in$ Sec ξ; $f \in C^\infty(B)$).

Thus the requirements (I)-(III) formally coincide with the corresponding Koszul-axioms, the only but important new feature is the property (IV). If $\tilde{\xi} = \tau_B$ and $A = 1_{\tau_B}$, a pseudoconnection becomes a linear connection, while in case of $\tilde{\xi} = \xi$ one can recover Candela's concept of pseudoconnections. Both as illustrations and for our further purposes, it will be useful to consider the following very simple but not incurious examples.

EXAMPLE 1. Let ∇ be a linear connection in τ_E (that is - by the usual "abuse of language" - on the total space

E of ξ) or in the vector bundle $H\xi$. Then the correspondence

$$(X,Y) \in \text{Sec } \pi^*(\tau_B) \times \text{Sec } \pi^*(\tau_B) \mapsto j \circ \nabla_{\mathscr{H} \circ X} \circ Y \in \text{Sec } \pi^*(\tau_B)$$

together with the E-morphism is a pseudoconnection in $\pi^*(\tau_B)$.

EXAMPLE 2. If ∇ is a linear connection on E or in $V\xi$, then the mapping

$$\text{Sec } \pi^*(\tau_B) \times \mathfrak{X}_V E \to \mathfrak{X}_V E, \quad (X,Y) \mapsto v\nabla_{\mathscr{H} \circ X} Y$$

and the horizontal map $\mathscr{H}$ constitute a pseudoconnection in $V\xi$ with respect to $\pi^*(\tau_B)$.

EXAMPLE 3. The mapping

$$\mathfrak{X}_V E \times \mathfrak{X}_V E \to \mathfrak{X}_V E, \quad (X,Y) \mapsto v\nabla_X Y$$

together with the obvious E-morphism i is a pseudoconnection in $V\xi$, whenever ∇ is a linear connection on E.

EXAMPLE 4. Let ∇ be again a linear connection on E or in $H\xi$. The assignment

$$(X,Y) \in \mathfrak{X}_V E \times \text{Sec } \pi^*(\tau_B) \mapsto j \circ \nabla_X \mathscr{H} \circ Y \in \text{Sec } \pi^*(\tau_B)$$

and the E-morphism i provide a pseudoconnection in $\pi^*(\tau_B)$ w.r.t. $V\xi$.

The next results illustrate, how one can construct pseudoconnections with the help of "pullback".

LEMMA 4. (i) *Let ∇ be a linear connection on the manifold B. There exists a unique pseudoconnection*

$$(\nabla^{\#}, \mathscr{H}),\ \nabla^{\#}: \mathrm{Sec}\ \pi^*(\tau_B) \times_{\bullet} \mathrm{Sec}\ \pi^*(\tau_B) \to \mathrm{Sec}\ \pi^*(\tau_B)$$

in $\pi^*(\tau_B)$ satisfying the condition

$$\nabla^{\#}_{pr_2^{\#}X} pr_2^{\#}Y = pr_2^{\#}(\nabla_X Y) \quad (X, Y \in \mathfrak{X}(B)).$$

(ii) *If ∇ is a linear connection in ξ, then there is a unique pseudoconnection*

$$(\tilde{\nabla}^{\#}, \mathscr{H}),\ \tilde{\nabla}^{\#}: \mathrm{Sec}\ \pi^*(\tau_B) \times \mathfrak{X}_V E \to \mathfrak{X}_V E$$

in $V\xi$ such that

$$\forall\ X \in \mathfrak{X}(B),\ \sigma \in \mathrm{Sec}\ \xi:\ \tilde{\nabla}^{\#}_{pr_2^{\#}X} \alpha^{\#}\sigma = \alpha^{\#}(\nabla_X \sigma).$$

PROOF. The unicity statements are clear. As for the existence, we work in the chart $(U;(u^i))$.

(i) As in the case of a linear connection, the mapping $\nabla^{\#}$ is determined by its values over a framing, therefore we have to define it by

$$\nabla^{\#}_{pr_2^{\#}\frac{\partial}{\partial u^i}} pr_2^{\#} \frac{\partial}{\partial u^j} = pr_2^{\#}\left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^i}\right), \quad (1 \leqq i, j \leqq n).$$

We show that this defining property is independent of the choice of the chart $(U,(u^i))$, that is, for another chart $(\tilde{U},(\tilde{u}^i))$,

$$\nabla^{\#}_{pr_2^{\#}\frac{\partial}{\partial \tilde{u}^i}} pr_2^{\#} \frac{\partial}{\partial \tilde{u}^j} = pr_2^{\#}\left(\nabla_{\frac{\partial}{\partial \tilde{u}^i}} \frac{\partial}{\partial \tilde{u}^j}\right) \quad (1 \leqq i, j \leqq n)$$

again holds. - In the neighbourhood $\tilde{U} \cap U$ we can write

$$\frac{\partial}{\partial \tilde{u}^i} = A_i^l \frac{\partial}{\partial u^l}, \quad A_i^l : \tilde{U} \cap U \to \mathbb{R} \quad (1 \leqq i \leqq n).$$

Using axioms (I)-(IV), part (ii) of Lemma 3, the obvious relation $\mathscr{H} \circ j \circ \frac{\delta}{\delta x^i} = \frac{\delta}{\delta x^i}$ $(1 \leqq i \leqq n)$ and calculating the partial derivative $\frac{\delta}{\delta x^l}(A_j^k \circ \pi)$ by the generalized chain rule, we have:

$$\nabla^{\#}_{pr_2^{\#} \frac{\partial}{\partial \tilde{u}^i}} \; pr_2^{\#} \frac{\partial}{\partial \tilde{u}^j} = \nabla^{\#}_{(A_i^l \circ \pi) pr_2^{\#} \frac{\partial}{\partial u^l}} \; (A_j^k \circ \pi) pr_2^{\#} \frac{\partial}{\partial u^k} =$$

$$= (A_i^l \circ \pi) \left[\left(\mathscr{H} \circ j \circ \frac{\delta}{\delta x^l} \right) (A_j^k \circ \pi) \, pr_2^{\#} \frac{\partial}{\partial u^k} + \right.$$

$$\left. + (A_j^k \circ \pi) \nabla^{\#}_{pr_2^{\#} \frac{\partial}{\partial u^l}} \; pr_2^{\#} \frac{\partial}{\partial u^k} \right] =$$

$$= (A_i^l \circ \pi) \left[\left(\frac{\partial A_j^k}{\partial u^l} \circ \pi \right) pr_2^{\#} \frac{\partial}{\partial u^k} + (A_j^k \circ \pi) pr_2^{\#} \left(\nabla_{\frac{\partial}{\partial u^l}} \frac{\partial}{\partial u^k} \right) \right] =$$

$$= pr_2^{\#} \left[A_i^l \left(\frac{\partial A_j^k}{\partial u^l} \frac{\partial}{\partial u^k} + A_j^k \nabla_{\frac{\partial}{\partial u^l}} \frac{\partial}{\partial u^k} \right) \right] =$$

$$= pr_2^{\#} \left(\nabla_{A_i^l \frac{\partial}{\partial u^l}} A_j^k \frac{\partial}{\partial u^k} \right) =$$

$$= pr_2^{\#} \left(\nabla_{\frac{\partial}{\partial \tilde{u}^i}} \frac{\partial}{\partial \tilde{u}^j} \right).$$

Likewise, we can easily check that for any vector field $X = X^i \frac{\partial}{\partial u^i}$, $Y = Y^i \frac{\partial}{\partial u^i} : U \to TB$ the requirement $\nabla^{\#}_{pr_2^{\#} X} pr_2^{\#} Y = pr_2^{\#}(\nabla_X Y)$ holds.

(ii) We define $\tilde{\nabla}^{\#}$ by

$$\tilde{\nabla}^{\#}_{pr_2^{\#} \frac{\partial}{\partial u^i}} \frac{\partial}{\partial y^\kappa} := \alpha^{\#}\left(\nabla_{\frac{\partial}{\partial u^i}} e_\kappa\right) \quad (1 \leqq i \leqq n,\ 1 \leqq \kappa \leqq r),$$

the argument is similar to the previous one.

An analogous reasoning gives the following

LEMMA 5. (i) *There exists a unique pseudoconnection*

$$(\mathring{\nabla}, i);\quad \mathring{\nabla}: \mathfrak{X}_V E \times \mathrm{Sec}\ \pi^*(\tau_B) \to \mathrm{Sec}\ \pi^*(\tau_B)$$

in $\pi^*(\tau_B)$ *with respect to* $V\xi$ *such that*

$$\forall\ X \in \mathfrak{X}_V E,\ Y \in \mathfrak{X}(B):\ \mathring{\nabla}_X pr_2^{\#} Y = 0.$$

(ii) *There is exactly one pseudoconnection*

$$(\mathring{D}, i);\ \mathring{D}: \mathfrak{X}_V E \times \mathfrak{X}_V E \to \mathfrak{X}_V E$$

in $V\xi$ *satisfying the condition*

$$\mathring{D}_X \alpha^{\#} \sigma = 0;\ X \in \mathfrak{X}_V E,\ \sigma \in \mathrm{Sec}\ \xi.$$

REMARK. The last two pseudoconnections are nontrivial: it is *not* true that each section of $\pi^*(\tau_B)$ and $V\xi$ has vanishing pseudocovariant derivatives by $\mathring{\nabla}$ and $\mathring{D}$ respectively.

4. FINSLER-TYPE CONNECTIONS

First we recall Miron's concept of a Finsler-type connection. (A detailed treatment can be found in [5].)

DEFINITION. A linear connection $\nabla: \mathfrak{X}(E)\times\mathfrak{X}(E) \to \mathfrak{X}(E)$ on the total space of ξ is called *Finsler-type* (with respect to) if for an arbitrary vector field $X: E \to TE$ we have

(F1) $\quad Y \in \mathfrak{X}_V E \Rightarrow \nabla_X Y \in \mathfrak{X}_V E;$

(F2) $\quad Y \in \mathfrak{X}_H E \Rightarrow \nabla_X Y \in \mathfrak{X}_H E.$

Using the introduced terms we can make the following basic observation.

THEOREM. *Suppose that*

$$(\nabla^h, \mathscr{H});\ \nabla^h: \text{Sec}\ \pi^*(\tau_B) \times \text{Sec}\ \pi^*(\tau_B) \to \text{Sec}\ \pi^*(\tau_B),$$

$$(\tilde{\nabla}^h, \mathscr{H});\ \tilde{\nabla}^h: \text{Sec}\ \pi^*(\tau_B) \times \mathfrak{X}_V E \to \mathfrak{X}_V E,$$

$$(\nabla^v, i);\ \nabla^v: \mathfrak{X}_V E \times \mathfrak{X}_V E \to \mathfrak{X}_V E,$$

$$(\tilde{\nabla}^v, i);\ \tilde{\nabla}^v: \mathfrak{X}_V E \times \text{Sec}\ \pi^*(\tau_B) \to \text{Sec}\ \pi^*(\tau_B)$$

are pseudoconnections in $\pi^*(\tau_B)$, $V\xi$, $V\xi$, $\pi^*(\tau_B)$ *respectively. Then*

$$\nabla: \mathfrak{X}(E)\times\mathfrak{X}(E) \to \mathfrak{X}(E),$$

(1)

$$(X,Y) \mapsto \nabla_X Y := \mathscr{H}\circ\nabla^h_{j\circ X}\, j\circ Y + \nabla^v_{vX}\, vY + \mathscr{H}\circ\tilde{\nabla}^v_{vX}\, j\circ Y + \tilde{\nabla}^h_{j\circ X}\, vY$$

is a Finsler-type connection in E. Conversely, any Finsler-type connection can be uniquely represented in the form (1).

PROOF. (a) *∇ is a linear connection.* We check only the last Koszul-axiom

$$\nabla_X fY = (Xf)Y + f\nabla_X Y \quad (X,Y \in \mathfrak{X}(E),\ f \in C^\infty(E)),$$

the others hold trivially. Using the "pointwise principle" and the properties (I)-(IV), we obtain from the definition:

$$\nabla_X fY = \mathcal{H}\circ\overset{h}{\nabla}_{j\circ X} j\circ fY + \overset{v}{\nabla}_{vX} v(fY) + \mathcal{H}\circ\tilde{\overset{v}{\nabla}}_{vX} j\circ fY + \tilde{\overset{h}{\nabla}}_{j\circ X} v(fY) =$$

$$= \mathcal{H}\circ\overset{h}{\nabla}_{j\circ X} f(j\circ Y) + \mathcal{H}\circ\tilde{\overset{v}{\nabla}}_{vX} f(j\circ Y) + \overset{v}{\nabla}_{vX} f(vY) + \tilde{\overset{h}{\nabla}}_{j\circ X} f(vY) =$$

$$= \mathcal{H}\circ[(\mathcal{H}\circ j\circ X)f(j\circ Y) + f\overset{h}{\nabla}_{j\circ X} j\circ Y + (vX)f(j\circ Y) + f\tilde{\overset{v}{\nabla}}_{vX} j\circ Y] +$$

$$+ (vX)f(vY) + f\overset{v}{\nabla}_{vX} vY + (\mathcal{H}\circ j\circ X)f(vY) + f\tilde{\overset{h}{\nabla}}_{j\circ X} vY =$$

$$= \mathcal{H}\circ[(hX+vX)f(j\circ Y) + f(\overset{h}{\nabla}_{j\circ X} j\circ Y + \tilde{\overset{v}{\nabla}}_{vX} j\circ Y)] +$$

$$+ (vX + hX)f(vY) + f(\overset{v}{\nabla}_{vX} vY + \tilde{\overset{h}{\nabla}}_{j\circ X} vY) =$$

$$= (Xf)Y + f\nabla_X Y,$$

as desired.

(b) *∇ is a Finsler-type connection.* Assume that $X \in \mathfrak{X}(E)$. If $Y \in \mathfrak{X}_V E$ then $vY = Y$, $j\circ Y = 0$, hence

$$\nabla_X Y = \overset{v}{\nabla}_{vX} Y + \tilde{\overset{h}{\nabla}}_{j\circ X} Y \in \mathfrak{X}_V E,$$

thus $(F1)$ is proved. Likewise, if $Y \in \mathfrak{X}_H E$ then $vY = 0$ and therefore

$$\nabla_X Y = \mathcal{H} \circ (\nabla^h_{j \circ X} j \circ Y + \tilde{\nabla}^v_{vX} j \circ Y) \in \mathfrak{X}_H E,$$

as was to be shown.

(c) Conversely, assume that ∇ is a Finsler-type connection on E. To prove that ∇ can be represented in the form (1), consider those pseudoconnections

$$(\nabla^h, \mathcal{H}), \ (\tilde{\nabla}^h, \mathcal{H}), \ (\nabla^v, i), \ (\tilde{\nabla}^v, i)$$

in the vector bundle $\pi^*(\tau_B)$, $\pi^*(\tau_B)$, $V\xi$, $V\xi$ respectively, which are constructed from ∇ according to Example 1, 2, 3, 4 respectively. Then $\forall\ X, Y \in \mathfrak{X}(E)$:

$$\mathcal{H} \circ \nabla^h_{j \circ X} j \circ Y + \nabla^v_{vX} vY + \ \circ \tilde{\nabla}^v_{vX} j \circ Y + \tilde{\nabla}^h_{j \circ X} vY =$$

$$= \mathcal{H} \circ j \circ \nabla_{\mathcal{H} \circ j \circ X} \mathcal{H} \circ j \circ Y + v\nabla_{vX} vY + \mathcal{H} \circ j \circ \nabla_{vX} \mathcal{H} \circ j \circ Y + v\nabla_{\mathcal{H} \circ j \circ X} vY =$$

$$= h\nabla_{hX} hY + v\nabla_{vX} vY + h\nabla_{vX} hY + v\nabla_{hX} vY =$$

$$= \nabla_{hX} hY + \nabla_{vX} hY + \nabla_{hX} vY + \nabla_{vX} vY = \nabla_X Y$$

as we claimed. (In the penultimate step we used the fact that ∇ is Finsler-type.) It remains to show that the representation (1) of a Finsler-type connection is unique. Suppose first that $X, Y \in \mathfrak{X}_V E$. Then $j \circ X = j \circ Y = 0$, hence from (1) $\nabla_X Y = \nabla_{vX} vY = \nabla^v_X Y$. On the other hand $\nabla_X Y = v\nabla_X Y$ (because ∇ is Finsler-type), therefore

$$\nabla^v_X Y = v\nabla_X Y.$$

This means that the pseudoconnection (∇^{v}, i) is the same as that of Example 3. If $X, Y \in \mathrm{Sec}\,\pi^{*}(\tau_B)$, then $\mathcal{H}\circ X, \mathcal{H}\circ Y \in \mathfrak{X}_H E$ and by (1)

$$\nabla_{\mathcal{H}\circ X}\mathcal{H}\circ Y = \mathcal{H}\circ\nabla^{h}_{j\circ\mathcal{H}\circ X} j\circ\mathcal{H}\circ Y = \mathcal{H}\circ\nabla^{h}_{X} Y.$$

Applying the mapping j, one gets

$$\nabla^{h}_{X} Y = j\circ\nabla_{\mathcal{H}\circ X}\mathcal{H}\circ Y,$$

therefore the pseudoconnection $(\nabla^{h}, \mathcal{H})$ coincides with the pseudoconnection of Example 1. The unicity of the other two terms can be seen analogously.

REMARK. The Theorem was motivated by a result of Z. I. Szabó [6]. The idea of such a representation of Finsler-type connections in τ_B originated with P. Dombrowski.

Keeping the notations of the Theorem, we have the following consequences:

COROLLARY 1. *The mappings*

$$(2)\qquad (X, Y) \in \mathfrak{X}(E)\times\mathfrak{X}_V E \mapsto \nabla^{v}_{vX} Y + \tilde{\nabla}^{h}_{j\circ X} Y,$$

$$(3)\qquad (X, Y) \in \mathfrak{X}(E)\times\mathfrak{X}_H E \mapsto \mathcal{H}\circ\nabla^{h}_{j\circ X} j\circ Y + \mathcal{H}\circ\tilde{\nabla}^{v}_{vX} j\circ Y$$

are linear connections in $V\xi$ and $H\xi$ respectively. Conversely, any linear connection in $V\xi$ and $H\xi$ can be uniquely represented in the form (2) *and* (3) *respectively.*

PROOF. These assertions can be gathered from the proof of the Theorem.

As special cases one obtains the following results.

COROLLARY 2. *Let* ∇_B *be a linear connection in the base manifold of* ξ. *Consider the "pullback" pseudoconnection* $(\nabla^{\#},\mathscr{H})$ *and the pseudoconnection* $(\overset{\circ}{\nabla},i)$ *in* $\pi^*(\tau_B)$ *constructed in Lemmas 4 and 5. Then the mapping*

$$\mathfrak{X}(E)\times\mathfrak{X}_H E \to \mathfrak{X}_H E,\quad (X,Y)\mapsto \mathscr{H}\circ\nabla^{\#}_{j\circ X}j\circ Y+\mathscr{H}\circ\overset{\circ}{\nabla}_{vX}j\circ Y \tag{4}$$

is a linear connection in $H\xi$ *which is called induced by* ∇_B.

COROLLARY 3. *Let* ∇_ξ *be a linear connection in* ξ. *The pullback connection* $(\tilde{\nabla}^{\#},\mathscr{H})$ *(Lemma 4,* (ii)*) and the pseudoconnection* $(\overset{\circ}{D},i)$ *(Lemma 5,* (ii)*) determine a linear connection in* $V\xi$ *by the correspondence*

$$(X,Y)\ \in\ \mathfrak{X}(E)\times\mathfrak{X}_V E\mapsto\ \overset{\circ}{D}_{vX}Y+\tilde{\nabla}^{\#}_{j\circ X}Y.$$

It is called induced by ∇_ξ.

REMARK. The Finsler-type connection

$$(X,Y)\in\mathfrak{X}(E)\times\mathfrak{X}(E)\ \mapsto \mathscr{H}\circ\nabla^{\#}_{j\circ X}j\circ Y+$$

$$+\overset{\circ}{D}_{vX}vY+\mathscr{H}\circ\overset{\circ}{\nabla}_{vX}j\circ Y+\tilde{\nabla}^{\#}_{j\circ X}vY\in\mathfrak{X}(E)$$

corresponding to the latter two linear connections may be called *induced* by ∇_B and ∇_ξ. A different (rather geometric) construction of this is essentially contained in [9].

5. BERWALD CONNECTION

In order to give an illustrative application of the ideas just presented we conclude the paper with a particularly simple description of the *Berwald connection*. (This terminology is a little ambiguous, we use it in Vilms' sense [10]. For another construction see [9].) We need a preliminary observation.

LEMMA 6. *The pair*

$$(\overset{B_\circ}{\nabla},\mathscr{H}),\ \overset{B_\circ}{\nabla}:\mathrm{Sec}\,\pi^*(\tau_B)\times \mathfrak{X}_V E\to \mathfrak{X}_V E,\ (X,Y)\mapsto v[\mathscr{H}\circ X,Y]$$

is a pseudoconnection in $V\xi$ *with respect to* $\pi^*(\tau_B)$.

PROOF. Evidently, $\overset{B_\circ}{\nabla}$ is additive in X and Y. Let $f\in C^\infty(E)$. Then on the one hand (applying the "pointwise principle")

$$\overset{B_\circ}{\nabla}_{fX}Y = v[\mathscr{H}\circ fX,Y] = v[f(\mathscr{H}\circ X),Y] =$$

$$= v(f[\mathscr{H}\circ X,Y]-(Yf)\mathscr{H}\circ X) = fv[\mathscr{H}\circ X,Y] = f\overset{B_\circ}{\nabla}_X Y,$$

on the other hand

$$\overset{B_\circ}{\nabla}_X fY = v[\mathscr{H}\circ X,fY] = v(f[\mathscr{H}\circ X,Y]+(\mathscr{H}\circ X)fY) =$$

$$= fv[\mathscr{H}\circ X,Y]+(\mathscr{H}\circ X)fY = (\mathscr{H}\circ X)fY+f\overset{B_\circ}{\nabla}_X fY.$$

COROLLARY 4. *The mapping*

$$\overset{B}{\nabla}:\mathfrak{X}(E)\times\mathfrak{X}_V E\to\mathfrak{X}_V E,\ (X,Y)\mapsto \overset{B}{\nabla}_X Y := \overset{\circ}{D}_{vX}Y+\overset{B_\circ}{\nabla}_{j\circ X}Y$$

(where $\overset{\circ}{D}$ *was constructed in Lemma 5* (ii)*) is a linear*

connection in $V\xi$ which is called Berwald-connection belonging to $\mathcal{H}$.

As for the importance of Berwald-connections in the general theory of connections, we refer to the papers [9], [10]. However, it will be instructive to show how one can derive Jaak Vilms' next clever observation in our approach.

PROPOSITION. *The curvature tensor field of $\mathcal{H}$ is equal to the exterior covariant derivative of the vertical projection v with respect to the $\overset{B}{\nabla}$ belonging to $\mathcal{H}$.*

PROOF. The curvature tensor field of $\mathcal{H}$ is by definition $R = -\frac{1}{2}[h,h]$, where the bracket means Nijenhuis-torsion [8]. We also recall from [8] the useful computational formula

$$R(X,Y) = -v[hX,hY] \quad (X,Y \in \mathfrak{X}(E)).$$

By the rule of exterior covariant differentiation,

$$\overset{B}{\nabla} v(X,Y) = \overset{B}{\nabla}_X vY - \overset{B}{\nabla}_Y vX - v[X,Y] = \overset{\circ}{D}_{vX} vY + v[hX,vY] -$$

$$- \overset{\circ}{D}_{vY} vX - v[hY,vX] - v[X,Y] =$$

$$= \overset{\circ}{D}_{vX} vY - \overset{\circ}{D}_{vY} vX - v[hX,hY] - [vX,vY].$$

Here $\overset{\circ}{D}_{vX} vY - \overset{\circ}{D}_{vY} vX - [vX,vY] = 0$ because locally (assuming that $X = X^i \frac{\delta}{\delta x^i} + \tilde{X}^\kappa \frac{\partial}{\partial y^\kappa}$, $Y = Y^i \frac{\delta}{\delta x^i} + \tilde{Y}^\kappa \frac{\partial}{\partial y^\kappa}$)

$$\overset{\circ}{D}_{vX}vY-\overset{\circ}{D}_{vY}vX-[vX,vY] =$$

$$= \overset{\circ}{D}_{\tilde{X}^{\kappa}\frac{\partial}{\partial y^{\kappa}}}\tilde{Y}^{\lambda}\frac{\partial}{\partial y^{\lambda}} - \overset{\circ}{D}_{\tilde{Y}^{\kappa}\frac{\partial}{\partial y^{\kappa}}}\tilde{X}^{\lambda}\frac{\partial}{\partial y^{\lambda}} - \left[\tilde{X}^{\kappa}\frac{\partial}{\partial y^{\kappa}},\tilde{Y}^{\lambda}\frac{\partial}{\partial y^{\lambda}}\right] =$$

$$= \tilde{X}^{\kappa}\frac{\partial\tilde{Y}^{\lambda}}{\partial y^{\kappa}}\frac{\partial}{\partial y^{\lambda}} - \tilde{Y}^{\kappa}\frac{\partial\tilde{X}^{\lambda}}{\partial y^{\kappa}}\frac{\partial}{\partial y^{\lambda}} - \left[\tilde{X}^{\kappa}\frac{\partial}{\partial y^{\kappa}},\tilde{Y}^{\lambda}\frac{\partial}{\partial y^{\lambda}}\right] = 0.$$

This implies $\overset{B}{\nabla} v(X,Y) = -v[hX,hY] = R(X,Y)$.

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J. Szilasi, Z. Kovács
Department of Mathematics
University of Debrecen
H-4010 Debrecen Pf. 12
Hungary