REMARKS ON FINSLER-TYPE CONNECTIONS

by

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In this note inspired by Miron's paper [9] our aim is to sketch a somewhat more general setting for the theory of Finsler-type connections from a vector bundle viewpoint and in terms of such geometric data as vertical projection, horizontal subbundles etc. In sections 1 and 2 preparatory material is presented mainly from the folklore for the sake of completeness and readers convenience. Our program is realized in section 3 together with some examples by way of illustration.

1. Vector bundles and lifts ([1], [4])

We work in the category of DIFF-manifolds ([2], p.17), more precisely in the Abelian category VB of vector bundles over a fixed base manifold from DIFF. /The underlying topological structures are always assumed to be Hausdorff and 2nd countable, as usual. / The "standard representative" from a VB will be denoted by $\xi = (E, \pi, B, F)$. Here B is the n-dimensional base manifold, F is the r-dimensional typical fibre, E is the total space and $\pi : E \to B$ is the projection. The fibre over a point $x \in B$ is written as F_x or $(E)_x$. $\mathrm{Sec}\,\xi$ and $A^p(B;\xi)$ $(0 \leq p \in \mathbb{N})$ denote the $C^\infty(B)$-modules of cross-sections and ξ-valued p-forms on B, respectively. /In case of $p=0$, these modules coincide./ For local calculation an open subset U of B is selected as a trivializing neighbourhood, making the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{\ \psi\ } & \pi^{-1}U \\ & {\scriptstyle pr_1} \searrow \quad \swarrow {\scriptstyle \pi|\pi^{-1}(U)} & \\ & U & \end{array}$$

commutative /the diffeomorphism ψ is the so-called trivializing map/. If $(u^1,\ldots,u^n)$ is a local coordinate system on U then the functions $x^i := u^i \circ \pi$, $y^\alpha := \ell^\alpha \circ pr_2 \circ \psi^{-1}$ / (ℓ^α) is a fixed basis of the conjugate space F^*; Latin indices run over $1,\ldots,n$, Greek indices run over $1,\ldots,r$, if not otherwise stated/ constitute a local coordinate system for ξ over $\pi^{-1}(U)$. The tangent bundle of the total space of ξ /and of an arbitrary DIFF-manifold, in general/ is written in the form $\tau_E = (TE,\pi_E,E,\mathbb{R}^{n+r})$. As for $\mathrm{Sec}\,\tau_E$, we use the standard notation and terminology: $\mathrm{Sec}\,\tau_E \equiv \mathfrak{X}(E)$ is the module of vector fields on E. It is a well-known basic fact that the manifold TE carries another VB-structure $T\xi = (TE,d\pi,TE,\mathbb{R}^{2r})$, this is the tangent fibration of ξ. For better clarity, the total space of $T\xi$ will sometimes be distinguished by the notation T^tE. If $\sigma \in \mathrm{Sec}\,\xi$, then the dervative map $d\sigma : TB \to TE$ is an element of $\mathrm{Sec}T\xi$. It is also called the <u>complete lift</u> of σ /cf. [14] / and for this reason $d\sigma$ will also be written as σ^c .

We shall use frequently the so-called canonical involution $S: TTE \to TTE$. Some of its basic properties are

(S1) <u>S^2=identity</u> /that is, S is indeed an involution;/

(S2) the diagram

$$\begin{array}{ccc} TTE & \overset{S}{\underset{S}{\rightleftarrows}} & TTE \\ & {\scriptstyle \pi_{TE}}\searrow \quad \swarrow{\scriptstyle d\pi_E} & \\ & TE & \end{array}$$

commutes;

(S3) if $\varphi : E \to M$ is a map in the DIFF-category, then <u>$S \circ d^2\varphi = d^2\varphi \circ S$</u> /where the same S denotes different involutions, of course/;

(S4) if X is a section $B \to TB$, that is a vector field on B, then <u>$X^c := S \circ dX$</u> is an element of <u>$\mathrm{Sec}T\tau_B$</u>; it is called the <u>complete lift</u> of $X \in \mathcal{X}(B)$ to $\mathrm{Sec}T\tau_B$.

/An equivalent construction of X^c can be found in [16], §5./

<u>$V\xi = (VE, \pi_V, E, \mathbb{R}^r)$</u> is the <u>vertical subbundle</u> of τ_E or, by abuse of language, of ξ . $\mathrm{Sec}V\xi$ is called the module of vertical vector fields on E and it is denoted by $\mathcal{X}_V E$. Really, $\mathcal{X}_V E$ is a submodule of $\mathcal{X}(E)$, moreover $\mathcal{X}_V E$ is a direct summand of $\mathcal{X}(E)$. It is known that there exists a canonical VB-morphism $\alpha : V\xi \to \xi$ which is a linear isomorphism on the fibres V_zE $(z \in E)$ such that the diagram

$$\begin{array}{ccc} VE & \xrightarrow{\alpha} & E \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi} & B \end{array}$$

commutes.

α is called the canonical map for $V\xi$. We also need the canonical maps $\alpha^t : VT^tE \to T^tE$ and $\alpha^v : VVE \to VE$. There is an interesting relation between S, α and α^t:

(35) $$d\alpha = \alpha^t \circ S .$$

The canonical map $\alpha : V\xi \to \xi$ has infinitely many "right inverses": for each $\sigma \in \mathrm{Sec}\,\xi$ there exists a morphism $\beta_\sigma : \xi \to V\xi$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \beta_\sigma\ } & VE \\ \pi \downarrow & & \downarrow \pi_V \\ B & \xrightarrow{\ \sigma\ } & E \end{array}$$

commutes and $\alpha \circ \beta_\sigma$ = identity. Obviously, $\beta_\sigma \in \mathcal{X}_V E$, so it is also called the vertical lift of σ. /For a local construction of β_σ, see [9], p.153 ./ A more important vertical vector field in the Finsler-theory is the intrinsic vector field /a term of M.Matsumoto, cf. [8], p.3o/

$$C : E \to TE,\ z \mapsto C(z) := \alpha_z^{-1}(z).$$

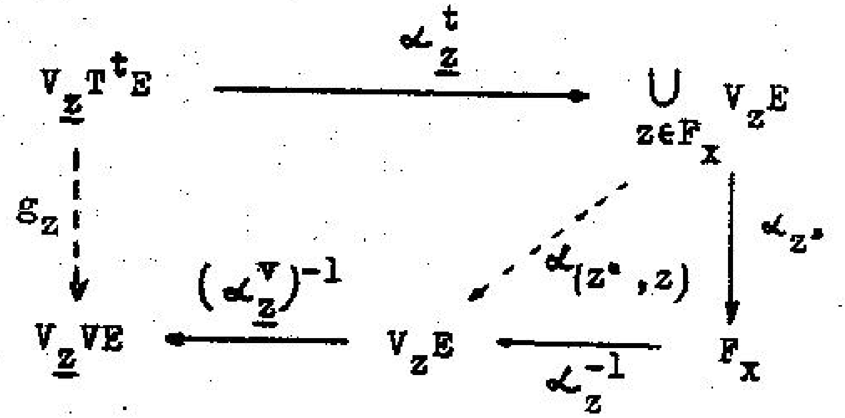
Locally, $C = y^\alpha \frac{\partial}{\partial y^\alpha}$.

This first part of the preparations is finished with a useful

Lemma: There exists a canonical epimorphism $g : VT\xi \,|\, V\xi \to VV\xi$, where the restriction means that g acts only on the fibres $V_{\underline{z}}T^tE$, $\underline{z} \in VE$.

Proof: If $\underline{z} \in V_zE$, $\pi(z) = x \in B$, then the image space $\alpha^t_{\underline{z}}(V_{\underline{z}}T^tE)$ is the $T\xi$-fibre $(T^tE)_{\underline{o}(x)}$, where $\underline{o}(x) = d\pi(\underline{z})$ is the zero element of the vector space T_xB. But $(T^tE)_{\underline{o}(x)}$ is the union $\bigcup_{z \in F_x} V_zE$, so the image of an element $\underline{a}$ of

$V_{\underline{z}}T^tE$ under $\alpha^t_{\underline{z}}$ belongs to a vertical subspace $V_{z^*}E$, where $z^* \neq z$ in general. After this comment, the result follows by the next diagram:



/in the second step the suitable partial map acts/. □

A different reasoning can be found in [15], p.1127.

2. Connections ([3], [4], [12])

Consider the canonical short exact sequence

$$\text{(SEQ)} \qquad 0 \longrightarrow V\xi \xrightarrow{\;i\;} \tau_E \xrightarrow{\;\widetilde{d\pi}\;} \pi^*(\tau_B) \longrightarrow 0 \;,$$

where i is the inclusion map, $\widetilde{d\pi}\,|\,T_zE := (d\pi)_z$ and $*$ denotes pull-back. A splitting $H:\pi^*(\tau_B)\to\tau_E$ of (SEQ) is called a <u>horizontal map</u> or a horizontal map <u>over ξ</u>. It induces a Whitney-decomposition $\tau_E = V\xi \oplus H\xi$, where $H\xi := \mathrm{Im}H = (HE, \pi_H, E, \mathbb{R}^n)$ is called a <u>horizontal subbundle</u> of τ_E. Locally, in the chart $(\pi^{-1}(U);(x^i, y^\alpha))$ H can be described with some /unique/ functions $N^\alpha_i : \pi^{-1}(U)\to\mathbb{R}$, which are called <u>connection parameters</u>. The basic geometric data constructed starting from a horizontal map H are the following:

① <u>h-lift</u> It is the E-isomorphism /in the sense of [1], p.78/

$\varrho^h := (\widetilde{d\pi} \mid \mathrm{Im}H)^{-1}$. If $X \in \mathfrak{X}(B)$, its horizontal lift is $X^h := \varrho^h \circ X \circ \pi \in \mathfrak{X}_H E := \mathrm{Sec}\ \mathrm{Im}H$.

② $h := H \circ \widetilde{d\pi}$, $v := 1 - h$, $K := \mathcal{A} \circ v$ are the horizontal projection, vertical projection, and Dombrowski-map belonging to H , respectively. /For a nice recent introduction also to the idea of Dombrowski-map see [6]./

③ $\nabla : \mathrm{Sec}\,\xi \to A^1(B;\xi)$, $\sigma \mapsto \nabla\sigma := K \circ d\sigma = K \circ \sigma^c$, is the general connection induced by H. If $X \in \mathfrak{X}(B)$ and $i(X)$ denotes the substitution operator $A^1(B;\xi) \to \mathrm{Sec}\,\xi$,

$\Phi \mapsto i(X)\Phi$, $(i(X)\Phi)(x) := \Phi(x)(X(x))$, then

$\nabla_X := i(X) \circ \nabla : \mathrm{Sec}\,\xi \to \mathrm{Sec}\,\xi$ is the covariant derivative by X . We speak about a linear connection if H satisfies the homogeneity condition

(HC) $\forall\, t \in \mathbb{R} : h \circ d\mu_t = d\mu_t \circ h$ $/\mu_t : E \to E$, $z \mapsto tz$ /.

In this case the covariant derivatives have the usual Koszul-properties.

④ $R := -\frac{1}{2}[h,h]$ is the curvature tensor field of H / the bracket means Nijenhuis-torsion /.

The following observations are very useful:

TEST 1: A bundle endomorphism $v : \tau_E \to \tau_E$ is a vertical projection for a horizontal map iff $\mathrm{Im}\, v \subset V\xi$ and $v \mid V\xi$ entity . □

/ In this case v is also called a vertical projection for ξ ./

Corollary / Duc's lemma / : If $\varphi : \xi \to \xi'$ is a morphism which is an isomorphism on the fibres and v' is a vertical projection for ξ' , then $d\varphi \mid VE : VE \to VE'$ is an isomorphism and $v = (d\varphi \mid VE)^{-1} \circ v' \circ d\varphi$ is a vertical projection for ξ . □

<u>TEST 2:</u> A bundle homomorphism $K: \tau_E \to \xi$ is a Dombrowski-map for a horizontal map iff $\forall \sigma \in \mathrm{Sec}\,\xi$:

$$\underline{K \circ \beta_* = \text{identity}} .$$

<u>TEST 3:</u> H satisfies (HC) iff

/a/ $\forall\ t \in \mathbb{R}$: $K \circ d\mu_t = \mu_t \circ K$ or

/b/ $\forall\ X \in \mathcal{X}(B)$: $\underline{\mathcal{T}(X^h) = o}$, where $\mathcal{T} : \mathcal{X}(E) \to \mathcal{X}(E)$, $Z \mapsto \mathcal{T}(Z) := [C, hZ]$ is the <u>tension tensor field</u> of H /cf. [5]/.

/For further equivalents and a weekening of (HC), see [12]/

A very important observation is formulated in

<u>Proposition 1 /Yano-Kobayachi-Vilms/</u> : Let $K: TE \to E$ be the Dombrowski-map belonging to H. Then

/a/ $\underline{K^c := dK \circ S}$ is a Dombrowski-map for $T\xi$.

/b/ If ∇^c and ∇ are the corresponding general connections, then $\forall\ X \in \mathcal{X}(B)$, $\sigma \in \mathrm{Sec}\,\xi$:

$$\underline{\nabla^c_{X^c} \sigma^c = (\nabla_X \sigma)^c} .$$

<u>Proof</u>: Using TEST 2 the verification of part /a/ becomes an easy but somewhat lengthy calculation. To see /b/ is also easy and elegant as well:

$$(\nabla_X \sigma)^c := d(\nabla_X \sigma) = d(K \circ d\sigma \circ X) = dK \circ d^2\sigma \circ dX \overset{(S1)}{=} dK \circ S \circ S \circ d^2\sigma \circ dX \overset{/a/ (S3)}{=} K^c \circ d^2\sigma \circ S \circ dX \overset{(S4)}{=} K^c \circ d\sigma^c \circ X^c :=$$

$$= \nabla^c_{X^c} \sigma^c . \ \square$$

Okay, now we are in a position to come to the point.

3. <u>Finsler-type connections</u>

The underlying differential geometric structure in Matsumoto's theory of Finsler connections is a principal bundle, the pull-back of the associated <u>frame bundle</u> of τ_B

via the projection map π_B. This bundle is called a <u>Finsler bundle</u> and it is denoted by $\mathcal{F}(B)$. A <u>Finsler connection</u> is e.g. a pair $(\Gamma, \mathcal{H}_B)$, where Γ is a principal connection on $\mathcal{F}(B)$ and $\mathcal{H}_B$ is a horizontal map over τ_B /or in the standard jargon a "nonlinear connection on τ_B"/. It is at first sight clear that the associated <u>vector bundle</u> of $\mathcal{F}(B)$ is exactly $V\tau_B$, so from VB - viewpoint the equivalent of the "principal part" of a Finsler-pair is a linear connection on the vertical subbundle. As for the "transplantation" of Matsumoto'ideas into a vector bundle context see e.g. the nice papers [7], [10]. Probably V.Oproiu was the first who extended a linear connection from $V\tau_B$ to the whole tangent bundle of TB in [11]. / A further extension is constructed and investigated in [13]./ The connections of this type will be distinguished by the attribute "<u>Finsler-type</u>". Miron's paper [9] is a logical final step in a sense and also a starting point of the recent advance of the Finsler-theory in this direction. - Now we reformulate the "abstract essence" of Miron's approach in the spirit hinted at in the introductory lines.

<u>Definitions</u>:

/a/ Let ξ be a Whitney sum of its subbundles ξ^1 and ξ^2 : $\xi = \xi^1 \oplus \xi^2$. /For the next following we suppose that rank $\xi^1 = r_1$, rank $\xi^2 = r_2$; $\alpha_i = 1, \ldots, r_i$, $i = 1, 2$; Einstein's summation convention is applied accordingly./ A splitting H of (SEQ) is called <u>invariant</u> with respect to the decomposition $\xi^1 \oplus \xi^2$ if

(INV1) $\qquad v \cdot dp^1 \cdot h = o$

(INV2) $\qquad v \cdot dp^2 \cdot h = o$

/ $p^i : \xi \to \xi^i$; $i = 1,2$ are the projection morphisms / are satisfied.

/b/ In particular, a splitting H^F of the (SEQ) constructed from the tangent bundle of the total space of ξ is called Finsler-type if it is invariant with respect to the Whitney-decomposition $\tau_E = V\xi \oplus H\xi$ induced by a horizontal map H over ξ that is if

$$\text{(F1)} \qquad \underline{v^F \circ dv \circ h^F = 0} \quad \text{and}$$

$$\text{(F2)} \qquad \underline{v^F \circ dh \circ h^F = 0}$$

hold, where, of course, v^F and h^F denote the vertical and horizontal projections belonging to H^F, respectively. /The corresponding Dombrowski-map will be denoted by K^F./ The pair (H^F, H) is called a Finsler-pair and the general connection $\nabla^F : \mathcal{X}(E) \to A^1(E, \tau_E) \cong \mathcal{X}^1_1(E)$ belonging to (H^F, H) is called a Finsler-type /general/ connection. The mapping $\overset{h}{\nabla}{}^F : \mathcal{X}(E) \to \mathcal{X}^1_1(E)$, $Z \mapsto \overset{h}{\nabla}{}^F Z := K^F \circ dZ \circ h$ is called an h-connection, the notion of v-connection $\overset{v}{\nabla}{}^F$ is analogous. The deflection tensor field D and the Cartan-tensor field CAR of ∇^F are given by $D := \overset{h}{\nabla}{}^F C$ and $CAR : Z \in \mathcal{X}_V E \mapsto CAR(Z) := \nabla^F_Z C - Z \in \mathcal{X}_V E$ respectively.

Of course, we cannot make any detailed analysis of the introduced concepts here, so e.g. the study of the important tensor field CAR /compare with Matsumoto's theory/ will be totally omitted. We have only a look at the general situation outlined in part /a/ of the definition and after this we turn to some applications and illustrations of our observations and ideas.

Proposition 2: If H is $\xi^1 \oplus \xi^2$ invariant then

/a/ the Dombrowski-map belonging to H has the form

$\underline{K = p^1 \circ K \circ dp^1 + p^2 \circ K \circ dp^2}$;

/b/ $\forall z_i \in E^i : \underline{H_{z_i} E \subset T_{z_i} E^i}$ /i=1,2/

Proof: By a local calculation we have:

$$(\text{INV1}) \text{ and } (\text{INV2}) \Longleftrightarrow \begin{array}{l} N_i^{\alpha_1} \circ p^1 = N_i^{\alpha_1} \\ N_i^{\alpha_1} \circ p^2 = 0 \\ N_i^{\alpha_2} \circ p^1 = 0 \\ N_i^{\alpha_2} \circ p^2 = N_i^{\alpha_2} \end{array} ;$$

at the same time

/a/ $\Longleftrightarrow N_i^{\alpha_1} \circ p^1 = N_i^{\alpha_1}$ and $N_i^{\alpha_2} \circ p^2 = N_i^{\alpha_2}$,

consequently (INV1) and (INV2) $\Longrightarrow$ /a/ /but not conversely!/

The verification of /b/ is similar. □

Proposition 3: Let ξ be a Whitney-sum $\xi^1 \oplus \xi^2$ of its subbundles and suppose that v^i /i=1,2/ are vertical projections for ξ^i. Then

$$\underline{v = (dp^1|VE^1)^{-1} \circ v^1 \circ dp^1 + (dp^2|VE^2)^{-1} \circ v^2 \circ dp^2}$$

is a vertical projection for a $\xi^1 \oplus \xi^2$-invariant horizontal map.

Proof: Apply TEST 1 and check on (INV1) and (INV2) directly. □

Proposition 4: Let H be a splitting of (SEQ) satisfying (HG). In this case the following conditions are equivalent:

/1/ H is $\xi^1 \oplus \xi^2$ invariant.

/2/ $K = p^1 \circ K \circ dp^1 + p^2 \circ K \circ dp^2$.

/3/ $\forall z_i \in E^i : H_{z_i} E \subset T_{z_i} E^i$ /i=1,2/.

/4/ $\forall \sigma_i \in \mathrm{Sec}\, \xi^i : \nabla \sigma_i \in A^1(B; \xi^i)$ /i=1,2/.

/5/ $\forall \sigma_i \in \mathrm{Sec}\, \xi^i,\ X \in \mathcal{X}(B) : \nabla_X \sigma_i \in \mathrm{Sec}\, \xi^i$ /i=1,2/.

/6/ $N_i^{\alpha 1} \circ p^1 = N_i^{\alpha 1}$ and $N_i^{\alpha 2} \circ p^2 = N_i^{\alpha 2}$.

<u>Proof</u>: Compute, compute. □

The equivalence /1/⟺/5/ in Proposition 4 shows that our definition of Finsler-type connection coincides with that of Miron in the linear case. Proposition 3 provides an obvious answer to the question: how to construct a Finsler-type connection, or more precisely, a Finsler-type vertical projection. The only task is to give a vertical projection for $H\xi$ and $V\xi$.

<u>Example 1</u>: Let $v_b : TTB \to VTB$ be a vertical projection for τ_B. By Duc's lemma and the peculiarities of the h-lift, the diagram

$$\begin{array}{ccc} THE & \xrightarrow{(d\ell^h)^{-1}} & TTB \\ \downarrow & & \downarrow v_b \\ VHE & \xleftarrow{d\ell^h | VTB} & VTB \end{array}$$

defines the vertical projection $\underline{v_h = d\ell^h | VTB \circ v_b \circ (d\ell^h)^{-1}}$ for $H\xi$.

<u>Example 2</u>: Again by Duc's lemma /or immediately/ it is clear that if $v : TE \to VE$ is a vertical projection for ξ , then $(d\alpha | VVE)^{-1} \circ v \circ d\alpha$ is a vertical projection for $V\xi$. The corresponding connection is not too interesting for its own sake since the only nonvanishing coefficients are the connection parameters of the starting splitting,

composed by α .

Example 3: If $v^t : TT^tE \to VT^tE$ is an arbitrary vertical projection for $T\xi$ then $g\circ(v^t|TVE)$ is a vertical projection for the vertical subbundle $V\xi$.
/ Remember, g is the epimorphism constructed in the Lemma. For brevity, we shall write simply $g\circ v^t$ instead of $g\circ(v^t|TVE)$./ This assertion can also be verified by TEST 1. At this point there is an all-important special case when v^t is derived from a vertical projection $v : TE \to VE$. To see this, consider the Dombrowski-map $K^c = dK\circ S$ from Proposition 1. By virtue of

$$\alpha^t\circ(S\circ dv\circ S) = (\alpha^t\circ S)\circ dv\circ S \overset{(S5)}{=} d\alpha\circ dv\circ S = d(\alpha\circ v)\circ S = dK\circ S ,$$

K^c belongs to the vertical projection $v^t = S\circ dv\circ S$.
Hence in view of the Lemma we have the vertical projection

$$g\circ v^t = (\alpha^v)^{-1}\circ \alpha_{(\cdot,\cdot)}\circ \alpha^t\circ S\circ dv\circ S = (\alpha^v)^{-1}\circ \alpha_{(\cdot,\cdot)}\circ dK\circ S$$

for $V\xi$ which is denoted by v^B and the corresponding connection is called /following [15]/ Berwald-connection. Its Dombrowski-map is $K^B = \alpha^v\circ v^B = \alpha_{(\cdot,\cdot)}\circ dK\circ S$. - We conclude our sketch with some nice properties of Berwald-connections.

Proposition 5 /cf. [15]/: Let ∇^B be the Berwald-connection constructed starting from a horizontal map H over ξ . Then

/B1/ ∇^B is always a linear connection.

/B2/ For the curvature tensor field of H we have the relation $R = \nabla^B v$ where the vertical projection is interpreted as an element of $A^1(E;V\xi)$ and ∇^B as a covariant exterior derivative on the module of $\nabla\xi$-valued forms on E.

/B3/ If D^B is the deflection tensor field of ∇^B, then $\forall X \in \mathcal{X}(B): \underline{D^B(X^h) = \mathcal{T}(X^h)}$, consequently by TEST 3b H satisfies (HC) iff ∇^B has no deflection.

Proof: The verification of (B1) is a fine local calculation to check TEST 3a. If the connection parameters of H are the functions $N_i^\alpha : \pi^{-1}(U) \to \mathbb{R}$, then we have

$$\nabla^B_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^\alpha} = \frac{\partial N_i^\beta}{\partial y^\alpha}\frac{\partial}{\partial y^\beta} \quad , \quad \nabla^B_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha} = 0 \quad ,$$

consequently $\forall \; Z = Z^i\left(\frac{\partial}{\partial x^i} - N_i^\alpha \frac{\partial}{\partial y^\alpha}\right) \in \mathcal{X}_H E$:

$$D^B Z = \nabla^B_{Z^i\left(\frac{\partial}{\partial x^i} - N_i^\alpha \frac{\partial}{\partial y^\alpha}\right)} C = Z^i \, \nabla^B_{\frac{\partial}{\partial x^i}}\left(y^\beta \frac{\partial}{\partial y^\beta}\right) - Z^i N_i^\alpha \, \nabla^B_{\frac{\partial}{\partial y^\alpha}} \, y^\beta \frac{\partial}{\partial y^\beta} =$$

$$= Z^i\left(y^\beta \frac{\partial N_i^\alpha}{\partial y^\beta} - N_i^\alpha\right)\frac{\partial}{\partial y^\alpha} \; ,$$ that is /B3/ is okay.

J. Vilms' clever observation /B2/ also can be derived in local coordinates by brute force /details are horrible/. □

Our concluding remark: Berwald-connection informs us on the homogeneity of the starting horizontal map as well as on its curvature.

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