A Setting for Spray and Finsler Geometry

by

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Introduction

The aim of the present study is to give a systematic and fairly comprehensive account of the fundamentals of a manifold endowed with a second-order differential equation or, in particular, a Finsler metric. 'Fairly comprehensive' does not mean 'complete' or 'encyclopaedic' as the size of the Handbook limits the length.

Philosophy

Any author who undertakes to work out a conceptual and calculative apparatus for Finsler geometry should first make a choice between the possible geometric frameworks under discussion. At the start two approaches offer themselves: that of principal fibre bundles and that of vector bundles. The theory of Finsler connections based on principal bundles was constructed in a masterly way and in full detail by M. Matsumoto in the nineteen-sixties [52], [53]. The theory of Matsumoto is a self-contained entity, nothing can be taken away from it, and nothing essential can be added to it without impairing the whole construction. It is a cornerstone of the edifice of modern Finsler geometry.

Within the framework of vector bundles the road branches off in at least three directions; among these, however, there are crosswalks.

(1) The fundamental structure is τ_{TM}, the tangent bundle of the tangent manifold of the base manifold M.

(2) The geometric framework is provided by $V\tau_M$, the vertical bundle to the vector bundle τ_M.

(3) The theory is developed in terms of the transverse bundle $\tau_M^*\tau_M$, written in more detail as $(TM \times_M TM, \tau_1, TM)$, in other words, the pull-back bundle formalism is utilized.

The theory built on the first choice was founded by J. Grifone [36]. This direction is being followed and enriched by the work of N. L. Youssef, and the papers written during the last ten years by the present author and his collaborators also belong here. The exposition is based on the vertical bundle in a monograph by A. Bejancu [9] and in a monograph by M. Abate and G. Patrizio [1]. We find an exposition based on the pull-back bundle in the papers of H. Akbar-Zadeh and his followers as well as in the excellent theses by P. Dazord and by J.-G. Diaz. The foundations of such a theory are outlined in a brilliant review by P. Dombrowski [30], which inspired an interesting early paper of Z. I. Szabó [71]. Taking a great leap forward in time, impressive arguments for this approach are provided in a paper dating from 2000 by M. Crampin [20]. Let us point out here that the vector bundle associated to the Finsler bundle of Matsumoto is just the pull-back bundle $\tau_M^*\tau_M$.

Which vector bundle should then be chosen as the fundamental geometrical structure? To come to a decision, we should also consider the technical tools available as calculative apparati in the different cases. On τ_{TM} we can certainly use the whole arsenal of differential geometry; besides classical tensor calculus, also the calculus of differential forms and the apparatus of covariant derivatives. Further efficient and elegant tools are provided by the Frölicher–Nijenhuis calculus of vector-valued differential forms. It was J. Grifone who made a systematic use of this in Finsler geometry and elsewhere (see e.g. [37]).

It is a strong practical argument in favour of the pull-back bundle that it has rank n as opposed to rank $2n$ for τ_{TM}, so in calculations with coordinates they will not be duplicated. On the pull-back bundle, as on any vector bundle, covariant derivative operators may be defined, although the introduction of torsion is not self-evident. Problems arise in connection with the definition of exterior differentiation and hence of the Lie derivative. All these difficulties, however, had been solved by the beginning of the 1990's. Indeed, by that time E. Martínez, J. Cariñena and W. Sarlet had worked out the calculus of differential forms along the tangent bundle projection, which offers adequate technical tools for any geometric theory on the pull-back bundle $\tau_M^*\tau_M$. It is important that the calculus of Martínez, Cariñena and Sarlet can be identified with the Frölicher–Nijenhuis calculus if the latter is restricted to semibasic differential forms and tensors. (In applications to Finsler geometry, this restriction does not impair generality!)

All these considerations together give us convincing arguments in favour of the following decision:

> In this work the theory will be formulated in the pull-back vector bundle
> $$\tau_M^*\tau_M = (TM \times_M TM,\ p_1, TM).$$

As a consequence,

> the main technical tools are the Frölicher–Nijenhuis calculus as applied to semibasic scalar- and vector-valued forms over TM, and covariant derivatives in $\tau_M^*\tau_M$.

As to the covariant derivatives, let us remark here that in the majority of cases we apply a Berwald derivative coming from a nonlinear connection.

Structure

The structure of the work is more or less implied by the fundamental framework just set up. In shaping the exposition, that it be self-contained was one of our top priorities, especially since, to the best of our knowledge, this is the first systematic work to proceed along these lines.

Chapter 1 contains the fundamentals of manifolds, vector bundles, covariant derivative operators in vector bundles, basic differential operators on a manifold and all other notions to be used later. In addition, some indispensable results will also be stated here, so that the reader will not be forced to look for them in other books. Proofs are generally omitted; the reader is referred to our main sources [15], [35], [44], [61] concerning the basic theory. The Appendix is actually a supplement to this chapter (and partly to the next): it provides a glossary of notational conventions, set-theoretical and topological concepts and summarizes the necessary algebraic background.

Chapter 2 forms the backbone of the study. The general constructions presented in Chapter 1 are to be applied to the pull-back bundle $\tau_M^*\tau_M$. The main topics are:

(1) to work out, starting from scratch, a fairly comprehensive theory of nonlinear connections in the generality of vector bundles, specializing the treatment to the tangent bundle later when the set of tools available becomes larger;

(2) to work out the Frölicher–Nijenhuis theory;

(3) to present the elements of the Martínez–Cariñena–Sarlet theory and to relate it to the Frölicher–Nijenhuis theory;

(4) to discuss covariant derivative operators in $\tau_M^*\tau_M$, in particular, the Berwald derivative induced by a nonlinear connection. (This is what is traditionally called 'the theory of Finsler connections'.)

In this chapter, as well as in the next chapter, most of the results are proved in detail.

Chapter 3 demonstrates how the general theory works in practice. It is applied to a geometric treatment of second-order differential equations and (not independently!) to a deduction of the theorems providing the foundations of the theory of Finsler manifolds. What is called here, with a permissible sloppiness, a second-order differential equation, will appear in the main text as a second-order vector field. Some related notions are semisprays and sprays. (In the case of a second-order vector field we require smoothness everywhere, while semisprays and sprays are not necessarily differentiable and are of class C^1 in the zero vectors, respectively. In addition, sprays are homogeneous of degree 2.) Any second-order vector field (semispray, spray) generates a nonlinear connection, and hence a Berwald derivative in $\tau_M^*\tau_M$. This Berwald derivative is an efficient tool in the study of the linearizability properties of a second-order vector field as the theorems of Martínez and Cariñena show. We build up the theory of Finsler manifolds in an unconventional manner. Instead of a Lagrange function or an energy function, we start with a metric tensor which leads to a Finsler manifold if it satisfies the 'normality' condition of M. Hashiguchi. In the course of the exposition, the regularity conditions due to R. Miron are of great importance for a better understanding of the fine details of metric properties.

Intended audience

To comprehend this work no prior knowledge of connection theory or Finsler geometry is required, and the author believes that he indeed succeeded in presenting a self-contained treatment of the subject at the level of an interested graduate student. The work is, however, addressed also to interested physicists and biologists. Let me hope that it will in fact prove to be useful (at least as a work of reference) and stimulating for a broader readership not versed in dealing with the conceptual and technical apparatus exposed here. The foundation is firm, but the theory is still far from complete, and this may prove a challenge for further work.

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Chapter 1

The Background: Vector Bundles and Differential Operators

A. Manifolds

1.1. An n-dimensional *topological manifold* (or briefly a *topological n-manifold*) is a second countable Hausdorff space M which is locally homeomorphic to the Euclidean n-space $\mathbb{R}^n$. If $\mathcal{U} \subset M$ is an open set and u is a homeomorphism of $\mathcal{U}$ onto an open subset of $\mathbb{R}^n$ then the pair $(\mathcal{U}, u)$ is called a *chart* for M. If $(e^i)_{i=1}^n$ is the dual of the canonical basis of $\mathbb{R}^n$, the functions $u^i := e^i \circ u$ $(1 \leqq i \leqq n)$ are mentioned as the *coordinate functions* of the chart. Instead of $(\mathcal{U}, u)$ we frequently write $(\mathcal{U}, (u^i)_{i=1}^n)$.

A family $(\mathcal{U}_\alpha, u_\alpha)_{\alpha \in A}$ of charts on M is an *atlas* on M if $(\mathcal{U}_\alpha)_{\alpha \in A}$ forms a cover of M.

1.2. The requirement of second countability for a topological manifold M is equivalent (among others) to the following properties:

(1) M is *σ-compact*, i.e., M is the union of a sequence of compact parts of M.

(2) M is *paracompact* (i.e., every open covering of M has a locally finite refinement) and the number of connected components of M is at most countable.

Using topological dimension theory it can be shown that *any topological manifold admits a finite atlas.* For a proof of this important fact we refer to

[35], Vol. I, 1.1.

1.3. An atlas $\mathcal{A} = (\mathcal{U}_\alpha, u_\alpha)_{\alpha\in A}$ for a topological n-manifold M is said to be *smooth* if any two charts in $\mathcal{A}$ *overlap smoothly* in the following sense:

all of the chart changes
$u_\alpha \circ u_\beta^{-1} : u_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \longrightarrow u_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \qquad ((\alpha,\beta) \in A\times A,\ \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq 0)$
are smooth (between open subsets of $\mathbb{R}^n$).

A smooth atlas $\mathcal{A}$ on M is called *complete* if $\mathcal{A}$ contains each chart in M that overlaps smoothly with every chart in $\mathcal{A}$. A complete smooth atlas on M is said to be a *smooth structure* on M. It can easily be seen that each smooth atlas on M is contained in a unique complete atlas, and hence determines a unique smooth structure on M. A *smooth manifold* is a topological manifold furnished with a smooth structure.

By a typical abuse of notation, we usually write M for a smooth manifold, the presence of the smooth structure being understood.

1.4. Let M and N be smooth manifolds. A continuous map $f : M \longrightarrow N$ is said to be *smooth* if for each point $p \in M$ and each chart $(\mathcal{V}, x)$ on N with $f(p) \in \mathcal{V}$ there is a chart $(\mathcal{U}, u)$ on M with $p \in \mathcal{U}$ such that $f(\mathcal{U}) \subset \mathcal{V}$ and the map $x\circ f\circ u^{-1}$ is smooth (as a mapping of open subsets of Euclidean spaces, see A.4.). The set of smooth maps $M \longrightarrow N$ is denoted by $C^\infty(M,N)$. A smooth map $f \in C^\infty(M,N)$ is called a *diffeomorphism* if it is bijective and $f^{-1} \in C^\infty(N,M)$ is also smooth.

In particular, a *smooth function* on a manifold M is a smooth map $f : M \longrightarrow \mathbb{R}$. ($\mathbb{R}$, as a smooth manifold, will be described in a moment; see 1.8, Example.) For the set of all smooth real-valued functions on M we use the shorthand $C^\infty(M) := C^\infty(M,\mathbb{R})$. $C^\infty(M)$ is a real commutative algebra (as well as a commutative ring) under the usual pointwise operations.

Unless otherwise stated, in our forthcoming considerations

the term ‘manifold’ will mean ‘smooth manifold’ of dimension at least two and assumed to be connected; all ‘maps’ are also ‘smooth maps’

.

1.5. The *support* of a smooth function $f \in C^\infty(M)$ is

$$\operatorname{supp}(f) := \text{ closure of } \{p \in M \mid f(p) \neq 0\}.$$

The existence of a number of fundamental differential geometric objects (connections, covariant derivatives, Riemannian metrics, etc.) depends on the following fact:

Any smooth manifold M admits smooth partitions of unity: if $(\mathcal{U}_\alpha)_{\alpha\in A}$ is an open cover of M, then there exists a family $(f_\alpha)_{\alpha\in A}$ of smooth functions on M satisfying the following conditions:

PU 1. $\forall p \in M, \quad \forall \alpha \in A : f_\alpha(p) \geqq 0;$

PU 2. $\forall \alpha \in A : \mathrm{supp}(f_\alpha) \subset \mathcal{U}_\alpha$, i.e. (f_α) is subordinate to $(\mathcal{U}_\alpha)$;

PU 3. $(\mathrm{supp}(f_\alpha))_{\alpha\in A}$ is a locally finite family;

PU 4. $\forall p \in M : \sum_{\alpha\in A} f_\alpha(p) = 1.$

1.6. Cocycles. Let GL(k) be the group of all $k \times k$ invertible real matrices ($k \in \mathbb{N}^*$; cf. **A.5**(5)). A GL(k)-*cocycle* on a manifold M is a family $\gamma = (\mathcal{U}_i, \gamma_{ij})_{i,j\in I}$ such that

(i) $(\mathcal{U}_i)_{i\in I}$ is an open cover of M;

(ii) if $(i,j) \in I \times I$ and $\mathcal{U}_i \cap \mathcal{U}_j \neq 0$ then $\gamma_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow \mathrm{GL(k)}$ is a smooth map;

(iii) if $(i,j,k) \in I \times I \times I$ and $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$ then for each point $p \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$

$$\gamma_{ij}(p)\gamma_{jk}(p) = \gamma_{ik}(p).$$

As an immediate consequence of **(iii)** we obtain:

(iv) $\forall i \in I : \gamma_{ii}(p) = 1_k$; $\forall (i,j) \in I \times I : \gamma_{ij}(p) = (\gamma_{ji}(p))^{-1}$, for an appropriate choice of $p \in M$ (1_k is the unit $k \times k$ matrix).

Two GL(k)-cocycles on M are called *equivalent* if they are contained in a common GL(k)-cocyle. It can easily be checked that this is indeed an equivalence relation; the set of equivalence classes of GL(k)-cocycles on M will be denoted by $H^1(M; \mathrm{GL(k)})$.

1.7. Let M be an n-dimensional manifold. The *tangent space* to M at a point p is the real vector space of linear functions $v : C^\infty(M) \longrightarrow \mathbb{R}$ satisfying the Leibniz rule

$$v(fg) = v(f)g(p) + f(p)v(g) \qquad \text{for all } f,g \in C^\infty(M).$$

If $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ is a chart on M with $p \in \mathcal{U}$ then the functions

$$\left(\frac{\partial}{\partial u^i}\right)_p : f \in C^\infty(M) \mapsto \left(\frac{\partial}{\partial u^i}\right)_p (f) = \frac{\partial f}{\partial u^i}(p) := D_i(f \circ u^{-1})u(p) \quad (1 \leqq i \leqq n)$$

are tangent vectors to M and we have the following important

Basis theorem. $\left(\left(\frac{\partial}{\partial u^i}\right)_p\right)_{i=1}^n$ *is a basis for the tangent space* T_pM*, and*

$$v = \sum_{i=1}^n v(u^i)\left(\frac{\partial}{\partial u^i}\right)_p \qquad \textit{for all } v \in T_pM.$$

If $]a,b[\subset \mathbb{R}$ is an open interval and $c :]a,b[\longrightarrow M$ a smooth map, i.e. a *curve* in M, then the *velocity* of c at $t \in]a,b[$ is the tangent vector $\dot{c}(t) \in T_{c(t)}M$ given by

$$\dot{c}(t)(f) := (f\circ c)'(t) \qquad \text{for all } f \in C^\infty(M).$$

Suppose we are given a chart $(\mathcal{U},(u^i)_{i=1}^n)$ around $c(t)$. Then, applying the basis theorem, we get

$$\dot{c}(t) = \sum_{i=1}^n \dot{c}(t)(u^i)\left(\frac{\partial}{\partial u^i}\right)_{c(t)} = \sum_{i=1}^n (u^i\circ c)'(t)\left(\frac{\partial}{\partial u^i}\right)_{c(t)}.$$

1.8. Let M and N be manifolds and $f : M \longrightarrow N$ a smooth map. The *tangent map* to f at a point $p \in M$ is the map $(f_*)_p : T_pM \longrightarrow T_{f(p)}N$ given by

$$(f_*)_p(v)(h) := v(h\circ f) \quad \text{for all } v \in T_pM \text{ and } h \in C^\infty(N).$$

If P is another manifold and $h : N \longrightarrow P$ is a smooth map, then we have the following global version of the *chain rule*:

$$[(h\circ f)_*]_p = (h_*)_{f(p)}\circ (f_*)_p \quad \textit{for all } p \in M.$$

We say that f is an *immersion* at p if $(f_*)_p$ is injective, and a *submersion* if $(f_*)_p$ is surjective. The map f is called an *immersion* or a *submersion* if the relevant property holds at every $p \in M$.

A manifold M is a *submanifold* of a manifold $\overline{M}$ provided

(1) M is a topological subspace of $\overline{M}$;

(2) the canonical injection $i : M \longrightarrow \overline{M}$ is an immersion.

Example. Let $r := 1_{\mathbb{R}}$. Then $(\mathbb{R},(r))$ is a chart, $\{(\mathbb{R},(r))\}$ is an atlas for $\mathbb{R}$, which defines a smooth structure (called *canonical*) on $\mathbb{R}$. According to 1.7, let

$$\left(\frac{d}{dr}\right)_t(\varphi) := \varphi'(t) \quad \text{for all } \varphi \in C^\infty(\mathbb{R}),\ t \in \mathbb{R}.$$

Then $((\frac{d}{dr})_t)$ is a basis for $T_t\mathbb{R}$. Now suppose we are given a curve $c :]\,a, b\,[\longrightarrow M$ in M. For the velocities of c we have

$$\dot{c}(t) = (c_*)_t \left(\frac{d}{dr}\right)_t, \qquad t \in]\,a, b\,[\,.$$

Indeed, if f is any smooth function on M, then

$$\dot{c}(t)(f) := (f \circ c)'(t) = \left(\frac{d}{dr}\right)_t (f \circ c) =: (c_*)_t \left(\frac{d}{dr}\right)_t (f),$$

which proves our claim.

1.9. Let V be an n-dimensional real vector space, $\underline{b} = (b_i)_{i=1}^n$ a basis for V, and $\underline{b}^* := (b^i)_{i=1}^n$ the dual of $\underline{b}$. The map

$$\varphi_{\underline{b}} : V \longrightarrow \mathbb{R}^n,\ \ v \mapsto \varphi_{\underline{b}}(v) := (b^1(v), \dots, b^n(v))$$

is a linear isomorphism, hence $(V, \varphi_{\underline{b}})$ is a global chart for V, which makes it a manifold. (The smooth structure obtained in this way does not depend on the choice of $\underline{b}$.) Fix a point $p \in V$ and consider the map

$$\iota_p : V \longrightarrow T_pV, \quad v \mapsto \iota_p(v) := \dot{c}(0),$$

where $c : t \in \mathbb{R} \mapsto c(t) := p + tv$. Then

$$\iota_p(v) = \dot{c}(0) \overset{1.7}{=} \sum_{i=1}^n \dot{c}(0)(b^i) \left(\frac{\partial}{\partial b^i}\right)_p = \sum_{i=1}^n b^i(v) \left(\frac{\partial}{\partial b^i}\right)_p;$$

in particular

$$\iota_p(b_j) = \sum_{i=1}^n \delta_j^i \left(\frac{\partial}{\partial b^i}\right)_p = \left(\frac{\partial}{\partial b^j}\right)_p \quad (1 \leqq i \leqq n).$$

From this it follows that ι_p is a linear isomorphism; it will be mentioned as the *canonical identification* of V and the tangent space T_pV.

B. Vector bundles

1.10. Fibred manifolds. A triple (E, π, M) is said to be a *fibred manifold* if E and M are manifolds and $\pi : E \longrightarrow M$ is a surjective submersion. Then E is called the *total space*, π the *projection*, and M the *base space*. For each point $p \in M$, the subset $E_p := \pi^{-1}(p)$ of E is called the *fibre over* p.

An elementary but important example of fibred manifolds may be constructed as follows.

Let M and N be manifolds and let $M \times N$ have the product topology. If $(\mathcal{U}_\alpha, x_\alpha)_{\alpha \in A}$ and $(\mathcal{V}_\beta, y_\beta)_{\beta \in B}$ are atlases for M and N respectively, then $(\mathcal{U}_\alpha \times \mathcal{V}_\beta, x_\alpha \times y_\beta)_{(\alpha,\beta) \in A \times B}$ is an atlas for $M \times N$ and it gives rise to a smooth structure on the product. The smooth manifold obtained in this way is said to be the *product manifold* of $M \times N$. It follows immediately from the corresponding definitions that the natural projection maps

$$pr_1 : M \times N \longrightarrow M,\ (p, q) \mapsto p; \quad pr_2 : M \times N \longrightarrow N,\ (p, q) \mapsto q$$

are smooth. The triple $(M \times N, pr_1, M)$ is a fibred manifold called a *trivial fibred manifold* over M; its fibres $(M \times N)_p = \{p\} \times N$ $(p \in M)$ are canonically diffeomorphic to N.

Notes. (1) From now on, the product $M \times N$ of two manifolds will tacitly be assigned the product smooth structure described above.

(2) We shall frequently denote a fibred manifold by the same symbol as we use for its projection, thus the shorthand for (E, π, M) will 'officially' be π. However, by an abuse of language, we shall also speak of a 'fibred manifold E'. This usage needs some carefulness: there are several instances where the same manifold is the total space of different fibred manifolds.

1.11. Sections. Let (E, π, M) be a fibred manifold.

(**1**) A smooth map $s : M \longrightarrow E$ is said to be a *section* of π if it satisfies the condition $\pi \circ s = 1_M$. The set of all sections of π will be denoted by $\Gamma(\pi)$.

(**2**) Suppose that $\mathcal{U} \subset M$ is an open submanifold. A *local section* of π with domain $\mathcal{U}$ is a smooth map $s : \mathcal{U} \longrightarrow E$ satisfying the condition $\pi \circ s = 1_{\mathcal{U}}$. For sets of local sections we use the following notations:

$\Gamma_{\mathcal{U}}(\pi)$ – the set of all local sections of π with domain $\mathcal{U}$;

$\Gamma_{\mathrm{loc}}(\pi)$ – the set of all local sections of π regardless of domain;

$\Gamma_p(\pi)$ – the set of all local sections whose domains contain the point p.

An important property of fibred manifolds is that $\Gamma_{\mathrm{loc}}(\pi) \neq \emptyset$, i.e., *any fibred manifold admits local sections.* In fact, it can be shown that *a smooth surjection $\pi : E \longrightarrow M$ is a submersion if and only if for each point $p \in M$ $\Gamma_p(\pi) \neq \emptyset$.* For a proof the reader is referred to [13].

1.12. Vector bundles. (**1**) A fibred manifold (E, π, M) is called a *(real) vector bundle of rank k* ($k \in \mathbb{N}$) if the following conditions are satisfied:

VB 1. For each point $p \in M$, the fibre E_p is a k-dimensional real vector space.

VB 2. For each point $p \in M$, there exists an open neighbourhood $\mathcal{U}$ of p and a diffeomorphism $\varphi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{R}^k$ such that

(i) $pr_1 \circ \varphi = \pi \restriction \pi^{-1}(\mathcal{U})$, i.e., the diagram

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}) \arrow[rr, "\varphi"] \arrow[dr, "\pi \restriction \pi^{-1}(\mathcal{U})"'] & & \mathcal{U} \times \mathbb{R}^k \arrow[dl, "pr_1"] \\ & \mathcal{U} & \end{tikzcd}$$

is commutative;

(ii) for each $q \in \mathcal{U}$, the map

$$\varphi_2 := pr_2 \circ (\varphi \restriction E_q) : E_q \longrightarrow \mathbb{R}^k$$

is a linear isomorphism.

A pair $(\mathcal{U}, \varphi)$ having the properties formulated in **VB 2** is called a *vector bundle chart* for π. A family $(\mathcal{U}_i, \varphi_i)_{i \in I}$ is said to be a *vector bundle atlas* for π if $(\mathcal{U}_i)_{i \in I}$ is a cover of M.

(**2**) Let (E, π, M) be a vector bundle of rank k, and let $(\mathcal{V}, \varphi)$ be a vector bundle chart for π. Suppose that $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ is a chart on M such that $\mathcal{U} \subset \mathcal{V}$. Consider the canonical basis $(e_i)_{i=1}^k$ of $\mathbb{R}^k$ and its dual $(e^i)_{i=1}^k$. If

$$x^i := u^i \circ \pi, \ y^j := e^j \circ pr_2 \circ \varphi \quad (1 \leqq i \leqq n, \ 1 \leqq j \leqq k)$$

then $(\pi^{-1}(\mathcal{U}), \ ((x^i)_{i=1}^n, (y^j)_{j=1}^k))$ is a chart for the *manifold* E, called an *adapted chart* to the vector bundle chart $(\mathcal{V}, \varphi)$. The family

$$(x, y) := ((x^i)_{i=1}^n, (y^j)_{j=1}^k)$$

will be mentioned as an *adapted local coordinate system* on E.

(**3**) If (E, π, M) and (E', π', M') are vector bundles then a *vector bundle homomorphism*, or a *bundle map from* π *to* π', consists of a pair

$$f : E \longrightarrow E', \qquad \underline{f} : M \longrightarrow M'$$

of smooth maps, satisfying the following condition:

The diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow[\underline{f}]{} & M' \end{array}$$

is commutative, and the induced maps $f_p : E_p \longrightarrow E'_{\underline{f}(p)}$ *are linear for all* $p \in M$.

If, in addition, f is a diffeomorphism, then we speak of a *(vector bundle) isomorphism.* In the particular case $M' = M$ a bundle map of the form $(f, 1_M)$ is called a *strong bundle map* or M*-morphism* and denoted simply by f.

By abuse of language, instead of a bundle map $(f, \underline{f}) : \pi \longrightarrow \pi'$ we shall also speak of a 'bundle map $f : E \longrightarrow E'$ ', cf. 1.10, Note (2).

(**4**) A vector bundle (E', π', M) is called a *(vector) subbundle* of the vector bundle (E, π, M) if $E' \subset E$, $\pi' = \pi \restriction E'$ and the canonical inclusion $i : E' \longrightarrow E$ is a strong bundle map.

Notation. The category of vector bundles will be denoted by VB. More precisely, the category VB has as its objects all vector bundles and as its morphisms all bundle maps. For each manifold M, VB(M) denotes the subcategory constituted by vector bundles over M and M-morphisms. If k is a natural number, $\mathrm{V^kB}$ stands for the full subcategory of vector bundles of rank k. Finally, $\mathrm{V^kB(M)} := \mathrm{V^kB} \cap \mathrm{VB(M)}$ is the subcategory of all vector bundles of rank k over M with M-morphisms as morphisms. (We recall the notion of category and morphism in the Appendix; see A.10.)

1.13. Construction of vector bundles. (**1**) Let us first suppose that (E, π, M) is a vector bundle of rank k over the n-dimensional base space M, and let $(\mathcal{U}_i, \varphi_i)_{i \in I}$ be a vector bundle atlas for π. Then the mappings

$$\gamma_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow \mathrm{GL(k)},\ q \mapsto \gamma_{ij}(q) := (\varphi_i)_q \circ (\varphi_j)_q^{-1}$$
$$((i,j) \in I \times I;\ \mathcal{U}_i \cap \mathcal{U}_j \neq 0)$$

together with the cover $(\mathcal{U}_i)_{i\in I}$ constitute a GL(k)-cocycle

$$\gamma := (\mathcal{U}_i, \gamma_{ij})_{i,j\in I}$$

for M, called a *structure cocycle for* π.

(**2**) Our next goal is to sketch that, conversely, any GL(k)-cocycle γ on M determines, up to a bundle isomorphism, a vector bundle π of rank k over M such that a structure cocycle for π is the given γ.

Starting from the cocycle $\gamma := (\mathcal{U}_i, \gamma_{ij})_{i,j\in I}$, consider the disjoint union

$$\widetilde{E}_\gamma := \bigsqcup_{i\in I} \mathcal{U}_i \times \mathbb{R}^k := \bigcup_{i\in I} \{i\} \times \mathcal{U}_i \times \mathbb{R}^k.$$

Define a relation $\sim$ on $\widetilde{E}_\gamma$ by setting

$$(i,p,v) \sim (j,q,w) :\Leftrightarrow (p = q \in \mathcal{U}_i \cap \mathcal{U}_j) \text{ and } w = \gamma_{ji}(p)(v).$$

Due to the cocycle properties 1.6, **(i)**–**(iv)** this is an equivalence relation. Let $[(i,p,v)]$ be the equivalence class of (i,p,v), and $E_\gamma := \widetilde{E}_\gamma/\sim$ the set of all equivalence classes. Define the projection $\pi_\gamma : E_\gamma \longrightarrow M$ by $\pi_\gamma[(i,p,v)] := p$. Now it is not difficult to check that the triple $(E_\gamma, \pi_\gamma, M)$ obtained so is a vector bundle of rank k over M for which γ is a structure cocycle. If $\widetilde{\gamma}$ is also a GL(k)-cocycle on M and $\widetilde{\gamma}$ is equivalent to γ (in the sense of 1.6) then the vector bundle $(E_{\widetilde{\gamma}}, \pi_{\widetilde{\gamma}}, M)$ constructed from $\widetilde{\gamma}$ is isomorphic to $(E_\gamma, \pi_\gamma, M)$. This means that the isomorphism class $[\pi_\gamma]$ depends only on the equivalence class $[\gamma] \in H^1(M, \mathrm{GL(k)})$. Denoting by $\mathrm{Vect_k}(M)$ the set of strong bundle isomorphism classes of vector bundles of rank k over M, we can conclude:

Bundle classification theorem. *There is a canonical bijective correspondence*

$$\boxed{\mathrm{Vect_k}(M) \leftrightarrow H^1(M, \mathrm{GL(k)})}.$$

(**3**) The construction principle sketched a moment ago can frequently be realised in the following form.

Consider an n-dimensional manifold M. Let VS(k) be the category of k-dimensional real vector spaces, Ob(VS(k)) the class of its objects (cf. A.10). Assume that

(i) a map $M \longrightarrow \mathrm{Ob(VS(k))}$, $p \mapsto E_p$ is given;

(ii) there is an open cover $(\mathcal{U}_i)_{i\in I}$ of M and a family $((\varphi_i)_p)_{i\in I}$ of linear isomorphisms

$$(\varphi_i)_p : E_p \longrightarrow \mathbb{R}^k, \qquad p \in \mathcal{U}_i$$

such that

$$\begin{cases} \gamma := (\mathcal{U}_i, \gamma_{ij})_{i,j\in I}, \\ \gamma_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow \mathrm{GL(k)},\ q \mapsto \gamma_{ij}(q) := (\varphi_i)_q \circ (\varphi_j)_q^{-1} \quad (\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset) \end{cases}$$

is a GL(k)-cocycle for M.

If

$$E := \bigsqcup_{p\in M} E_p := \bigcup_{p\in M} \{p\} \times E_p;\ \pi : E \longrightarrow M,\ \pi(\{p\} \times E_p) := p,$$

then there is a unique smooth structure on E which makes (E, π, M) into a vector bundle of rank k with a structure cocycle γ.

1.14. Applications 1: pull-backs. Let (E, π, M) be a vector bundle and let $f : N \longrightarrow M$ be a smooth map. Assign to each point $q \in N$ the vector space $E_{f(q)}$ and consider the disjoint union

$$f^*E := \bigsqcup_{q\in N} E_{f(q)} = \bigcup_{q\in N} \{q\} \times E_{f(q)}.$$

Suppose that $(\mathcal{V}_i, \psi_i)_{i\in I}$ is a vector bundle atlas for π, and let $(\mathcal{V}_i, \gamma_{ij})_{i,j\in I}$ be the corresponding structure cocycle (1.13(1)). If $\varphi_i(p) := (\psi_i)_{f(p)}$ $(i \in I)$ then

$$(\varphi_i)_p \circ (\varphi_j)_p^{-1} = (\psi_i)_{f(p)} \circ (\psi_j)_{f(p)}^{-1} = (\gamma_{ij} \circ f)(p),$$

for every $(i, j) \in I \times I$ and $p \in \mathcal{U}_i \cap \mathcal{U}_j$, furthermore the family

$$(\mathcal{U}_i, \gamma_{ij} \circ f)_{i,j\in I}, \quad \mathcal{U}_i := f^{-1}(\mathcal{V}_i)$$

is a GL(k)-cocycle on N. Thus, by 1.13(3), there is a vector bundle

$$(f^*E,\ f^*\pi,\ N)$$

of rank k over N with a structure cocycle $(\mathcal{U}_i, \gamma_{ij} \circ f)_{i,j\in I}$. This vector bundle is called the *pull-back of* π *by* f. The total space of $f^*\pi$ is just the fibre product

$$N \times_M E := \{(q, z) \in N \times E \mid f(q) = \pi(z)\},$$

therefore f^*E is a (closed) submanifold of $N \times E$. The projection of the pull-back bundle is

$$pr_1 \restriction N \times_M E =: \pi_1,$$

so, by a slight abuse of notation, instead of $(f^*E, f^*\pi, N)$ we also write (f^*E, π_1, N) or $(N \times_M E, \pi_1, N)$. If

$$\pi_2 := pr_2 \upharpoonright N \times_M E,$$

then (π_2, f) is a bundle map from $f^*\pi$ to π:

$$\begin{tikzcd} f^*E = N \times_M E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}.$$

If we identify each fibre $(f^*E)_q$ with $E_{f(q)}$ itself, then we can characterize the pull-back construction as follows.

The pull-back of a vector bundle (E, π, M) by a smooth map $f : N \longrightarrow M$ is a vector bundle $(f^*E, f^*\pi, N)$ such that:

PB 1. $(f^*E)_q = E_{f(q)}$ for all $q \in N$.

PB 2. *The diagram*

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

where the top arrow means a fibrewise identical map is commutative.

PB 3. *If E is trivial, i.e. equal to $M \times \mathbb{R}^k$, then $f^*E = N \times \mathbb{R}^k$ and $f^*\pi = pr_1$.*

PB 4. *If $\mathcal{V} \subset M$ is an open subset, $\mathcal{U} := f^{-1}(\mathcal{V}) \subset N$, $E_{\mathcal{V}} := \pi^{-1}(\mathcal{V}) \subset E$, $(f^*E)_{\mathcal{U}} := (f^*\pi)^{-1}(\mathcal{U})$, then $f^*E_{\mathcal{V}} = (f^*E)_{\mathcal{U}}$, and we have the following commutative diagram:*

$$\begin{tikzcd} & f^*E_{\mathcal{V}} \arrow[rr] \arrow[dl] \arrow[dd] & & E_{\mathcal{V}} \arrow[dl] \arrow[dd] \\ f^*E \arrow[rr, crossing over] \arrow[dd] & & E & \\ & \mathcal{U} \arrow[rr] \arrow[dl] & & \mathcal{V} \arrow[dl] \\ N \arrow[rr] & & M \arrow[from=uu, crossing over] & \end{tikzcd}$$

1.15. Applications 2: linear algebra for vector bundles.
The process described in 1.13 allows us to construct new vector bundles from old ones using fibrewise the algebraic operations which one employs in linear algebra for vector spaces and homomorphisms. (These purely algebraic constructions are briefly summarized in A.5, A.6, A.8 and A.9.) In other words, any smooth functor (or multifunctor) can be canonically extended from the category VS of finite-dimensional real vector spaces to the category VB(M) to obtain new vector bundles. For later use, we list here some typical examples.

Let $(\pi_i)_{i=0}^m$ be a series of vector bundles in VB(M) ($m \in \mathbb{N}^*$, π_0 is the shorthand for (E, π, M)). Then we obtain the following vector bundles:

Bundle	**Total space**
π^* – the *dual* of π	$E^* := \bigsqcup_{p\in M} E_p^*$
$\pi_1 \oplus \pi_2$ – *Whitney sum* of π_1 and π_2	$E_1 \oplus E_2 := \bigsqcup_{p\in M} (E_1)_p \oplus (E_2)_p$
$\mathrm{Hom}(\pi_1, \pi_2)$ – *homomorphism bundle*	$\mathrm{Hom}(E_1, E_2) =$ $:= \bigsqcup_{p\in M} \mathrm{Hom}((E_1)_p, (E_2)_p)$
$\mathrm{End}(\pi) := \mathrm{Hom}(\pi, \pi)$ – *endomorphism bundle*	$\mathrm{End}(E) = \bigsqcup_{p\in M} \mathrm{End}(E_p)$
$\pi_1 \otimes \pi_2$ – *tensor product* of π_1 and π_2	$E_1 \otimes E_2 := \bigsqcup_{p\in M} (E_1)_p \otimes (E_2)_p$
$\wedge^s\pi$ – sth *exterior power* of π	$\wedge^s E := \bigcup_{p\in M} \wedge^s(E_p)$; $\wedge^\circ E := M \times \mathbb{R}$, $\wedge^s E := M \times \{0\}$, $s > \operatorname{rank}\pi$
$\wedge\pi := \overset{k}{\underset{m=0}{\otimes}} \wedge^m \pi$ – *exterior algebra bundle* of π ($k := \operatorname{rank}\pi$)	$\wedge E := \overset{k}{\underset{m=0}{\otimes}} \wedge^m E$
$\mathbf{T}_s\pi$ – *bundle of covariant tensors of order* s on E	$\mathbf{T}_s E := \bigsqcup_{p\in M} \mathbf{T}_s E_p = \bigsqcup_{p\in M} L^s(E_p) =$ $= \bigsqcup_{p\in M} \overset{s}{\otimes} E_p^* =: \overset{s}{\otimes} E^*$
$\mathbf{T}^r\pi$ – *bundle of contravariant tensors of order* r over π	$\mathbf{T}^r E := \bigsqcup_{p\in M} \mathbf{T}^r E_p =$ $= \bigsqcup_{p\in M} L^r(E_p^*) = \bigsqcup_{p\in M} \overset{r}{\otimes} E_p =: \overset{r}{\otimes} E$
$\mathbf{T}_s^r\pi := \mathbf{T}^r\pi \otimes \mathbf{T}_s\pi$ – *bundle of type* (r, s) *tensors* over π (of contravariant order r, covariant order s)	$\mathbf{T}_s^r E := \mathbf{T}^r E \otimes \mathbf{T}_s E =$ $= \underbrace{E \otimes \cdots \otimes E}_{r \text{ times}} \otimes \underbrace{E^* \otimes \cdots \otimes E^*}_{s \text{ times}}$
$\mathbf{A}_\ell\pi$ – *bundle of antisymmetric covariant tensors of order* ℓ over π; $\mathbf{A}_\ell\pi = \wedge^\ell\pi^*$	$\mathbf{A}_\ell E := \bigsqcup_{p\in m} \mathbf{A}_\ell(E_p) = \bigsqcup_{p\in M} \wedge^\ell E_p^* =$ $=: \wedge^\ell E^*$
$\mathbf{A}\pi := \overset{k}{\underset{\ell=0}{\oplus}} \mathbf{A}_\ell\pi$ – *bundle of forms over* π; $\mathbf{A}\pi = \wedge\pi^*$	
$\mathbf{S}_\ell\pi$ – *bundle of symmetric covariant tensors of order* ℓ over π	$\mathbf{S}_\ell E := \bigsqcup_{p\in M} L_{\mathrm{sym}}^\ell(E_p)$

Remark. The Whitney sum and the tensor product of several vector bundles in VB(M) may be defined by recursion.

The main steps in a unified justification of these constructions are the following:

(**i**) Let $\mathcal{F} : \mathrm{VS} \longrightarrow \mathrm{VS}$ be a covariant smooth functor. If $(\mathcal{U}_i, \gamma_{ij})_{i,j\in I}$ is a structure cocycle for the vector bundle (E, π, M), then the maps

$$\mathcal{F}(\gamma_{ij}) : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow GL(\mathcal{F}(\mathbb{R}^k)),\ p \mapsto \mathcal{F}(\gamma_{ij})(p) := \mathcal{F}(\gamma_{ij}(p))$$

$$(\ \mathcal{U}_i \cap \mathcal{U}_j \neq 0; \quad (i,j) \in I \times I\ )$$

are also smooth, and

$$(*) \qquad (\mathcal{U}_i,\ \mathcal{F}(\gamma_{ij}))_{i,j\in I}$$

is a $GL(\mathcal{F}(\mathbb{R}^k))$-cocycle for M. In view of 1.13(2) there exists a unique vector bundle $\mathcal{F}(\pi) \in \mathrm{VB(M)}$ with structure cocycle $(*)$ and fibres (isomorphic to) $\mathcal{F}(E_p)$, $p \in M$.

(**ii**) If $\mathcal{F} : \mathrm{VS} \longrightarrow \mathrm{VS}$ is a *contravariant* smooth functor, then in the preceding construction we have to consider the new cocycle $(\mathcal{U}_i, \mathcal{F}(\gamma_{ij}^{-1}))_{i,j\in I}$ instead of $(*)$; while if $\mathcal{F}$ is a contra-covariant smooth *bifunctor*, then we have to form the cocycle

$$(\mathcal{U}_i, \mathcal{F}((\gamma_1)_{ij}^{-1}, (\gamma_2)_{ij})_{i,j\in I}.$$

In light of these special cases, the extension of the construction to other functors and multi-functors is immediate.

1.16. Consider the vector bundles (E, π, M) and (F, ϱ, M). The canonical isomorphisms

$$E_p^* \otimes F_p \longrightarrow L(E_p, F_p) \quad (p \in M)$$

(cf. A.8, **1**(f)) induce a strong bundle isomorphism

$$\boxed{\pi^* \otimes \varrho \cong \mathrm{Hom}(\pi, \varrho)}\,.$$

More generally, let (E_i, π_i, M) $(1 \leqq i \leqq s)$ and (F, ϱ, M) be vector bundles. We may construct the bundle $L(\pi_1, \dots, \pi_s; \varrho)$ whose fibre at a point $p \in M$ is the vector space of s-linear maps $(E_1)_p \times \cdots \times (E_s)_p \longrightarrow F_p$. Then

$$\boxed{L(\pi_1, \dots, \pi_s; \varrho) \cong L(\pi_1 \otimes \cdots \otimes \pi_s; \varrho) \cong \pi_1^* \otimes \cdots \otimes \pi_s^* \otimes \varrho}\,.$$

1.17. To conclude this overview on the basic constructions we recall the following fundamental result:

> *For every vector bundle* $\pi \in \mathrm{VB(M)}$ *there exists a vector bundle* $\varrho \in \mathrm{VB(M)}$ *such that* $\pi \oplus \varrho$ *is trivial.*

For a well-readable proof (which strongly depends on the finite atlas theorem (1.12)) the reader is referred to [15] or [35], Vol. I, 2.23.

1.18. Let (E, π, M) and (E', π', M) be two vector bundles over M, and suppose that $f : \pi \longrightarrow \pi'$ is a strong bundle map. For any point $p \in M$, let us consider the linear map $f_p := f \restriction E_p$. The rank $rk(f_p)$ is called the *rank of f at p*. Put

$$\operatorname{Ker} f := \bigcup_{p \in M} \operatorname{Ker} f_p, \quad \operatorname{Im} f := \bigcup_{p \in M} \operatorname{Im} f_p.$$

With this notation we have:

(1) *The mapping $p \mapsto rk(f_p)$ of M into the discrete topological space $\mathbb{N}$ is lower semi-continuous.*

(2) *The following conditions are equivalent:*

  (i) *The function $p \in M \mapsto rk(f_p) \in \mathbb{N}$ is continuous, and therefore (since M is connected) constant.*
  (ii) $\operatorname{Ker} f$ *is a vector subbundle of E.*
  (iii) $\operatorname{Im} f$ *is a vector subbundle of E'.*

For a proof see [28], Vol. III, (16.17.5).

1.19. Exact sequences. Let a finite sequence

$$(E_i, \pi_i, M) \qquad (0 \leqq i \leqq m;\ m \geqq 2)$$

of vector bundles be given. A sequence (f_1, f_2) of two strong bundle maps written in the form

$$E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} E_2$$

is said to be *exact* or *exact at E_1* (or at π_1) if, for each point $p \in M$, the sequence $((f_1)_p, (f_2)_p)$ of two linear maps

$$(E_0)_p \xrightarrow{(f_1)_p} (E_1)_p \xrightarrow{(f_2)_p} (E_2)_p$$

of real vector spaces is exact, i.e. (see A.5(8)), $\operatorname{Im}(f_1)_p = \operatorname{Ker}(f_2)_p$.

A longer sequence

$$E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} E_2 \longrightarrow \dots \longrightarrow E_{m-1} \xrightarrow{f_m} E_m$$

of strong bundle maps is called an *exact sequence* if each sequence (f_i, f_{i+1}) is exact at E_i for $1 \leqq i \leqq m-1$. In particular, an exact sequence of the form

$$(*) \qquad 0 \longrightarrow E_1 \xrightarrow{f_2} E_2 \xrightarrow{f_3} E_3 \longrightarrow 0$$

with the trivial bundles $(M \times \{0\}, pr_1, M)$ and hence zero bundle maps at the ends, is called a *short exact sequence*. By a slight abuse of language, instead of an 'exact sequence of strong bundle maps' we also speak of an 'exact sequence of vector bundles'.

Notice that if $E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} E_2$ is an exact sequence then the strong bundle maps f_1 and f_2 satisfy the equivalent conditions of 1.18(2), and the bundles $\operatorname{Im} f_1$ and $\operatorname{Ker} f_2$ are therefore defined and equal. Exactness of the short sequence $(*)$ means that f_2 is injective, $\operatorname{Im} f_2 = \operatorname{Ker} f_3$, and that f_3 is surjective.

A strong bundle map $s_2 : E_3 \longrightarrow E_2$ is said to *split* the short exact sequence $(*)$ if

$$f_3 \circ s_2 = 1_{E_3};$$

then s_3 is also mentioned as a *(right) splitting* of $(*)$. (Second countability of the common base space guarantees that splittings of a short exact sequence do exist.) Since f_3 is surjective, s_2 is injective and the image of s_2 is a Whitney summand in E_2:

$$E_2 = \operatorname{Im} f_2 \oplus \operatorname{Im} s_2.$$

To any right splitting s_2 of $(*)$ there exists a unique (necessarily surjective) strong bundle map

$$s_1 : E_2 \longrightarrow E_1$$

such that

$$s_1 \circ f_2 = 1_{E_1}$$

and the sequence of strong bundle maps

$$0 \longleftarrow E_1 \underset{s_1}{\longleftarrow} E_2 \underset{s_2}{\longleftarrow} E_3 \longleftarrow 0$$

is also exact. s_1 is said to be the *(bundle) retraction* associated with f_2 and *complementary to* s_2 (cf. A.5, Proposition 2).

C. Sections of vector bundles

1.20. If (E, π, M) is a vector bundle then the set $\Gamma(\pi)$ of all sections of π is a $C^\infty(M)$-module under the pointwise operations

$$(s_1 + s_2)(p) := s_1(p) + s_2(p), \quad (fs)(p) := f(p)s(p)$$

$(s, s_1, s_2 \in \Gamma(\pi);\ f \in C^\infty(M);\ p \in M)$. The zero element of this module is the *zero section* o given by

$$p \in M \mapsto o(p) := \textit{ the zero vector } 0_p =: 0 \textit{ of } E_p.$$

If $\mathcal{U} \subset M$ is an open submanifold, then $\Gamma_{\mathcal{U}}(\pi)$ is a $C^\infty(\mathcal{U})$-module in the same way. The following result is of basic importance:

The $C^\infty(M)$-module $\Gamma(\pi)$ is finitely generated and projective.

Of course, this does not imply that $\Gamma(\pi)$ possesses a basis. However, for each point $p \in M$ there exists an open neighbourhood $\mathcal{U}$ of p such that $\Gamma_{\mathcal{U}}(\pi)$ has a basis, called a *frame* or a *local basis* for π over $\mathcal{U}$.

Indeed, choose a vector bundle chart $(\mathcal{U}, \varphi)$ around p, and let $(e_i)_{i=1}^k$ be the canonical basis of $\mathbb{R}^k$ $(k = \operatorname{rank}(\pi))$. Then the maps

$$\varepsilon_i : q \in \mathcal{U} \mapsto \varepsilon_i(q) := \varphi^{-1}(q, e_i) \quad (1 \leqq i \leqq k)$$

are clearly local sections of π, and $(\varepsilon_i(q))_{i=1}^k$ is a basis of the real vector space E_q for any point $q \in \mathcal{U}$. Hence $(\varepsilon_i)_{i=1}^k$ is a basis of the $C^\infty(\mathcal{U})$-module $\Gamma_{\mathcal{U}}(\pi)$. By shrinking $\mathcal{U}$ if necessary, we can suppose that $\pi^{-1}(\mathcal{U})$ is the domain of an adapted local coordinate system $((x^i)_{i=1}^n, (y^j)_{j=1}^k)$. Then for any section $\sigma \in \Gamma(\pi)$ we have the coordinate expression

$$\sigma \restriction \mathcal{U} = \sum_{j=1}^k \sigma^j \varepsilon_j, \text{ where } \sigma^j := y^j \circ \sigma \quad (1 \leqq j \leqq k).$$

1.21. Push-forwards of sections. If $(f, \underline{f})$ is a bundle map from the vector bundle (E, π, M) into the vector bundle (E', π', M') and $\underline{f}$ is a *diffeomorphism*, then for any (local) section s of π,

$$f_\#(s) := f \circ s \circ \underline{f}^{-1}$$

is a (local) section of π', called the *push-forward* of s by f:

$$
\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, shift right, "\pi"'] & E' \arrow[d, shift left, "\pi'"] \\
M \arrow[u, shift right, "s"'] \arrow[r, shift right, "\underline{f}"'] & M' \arrow[u, shift left] \arrow[l, shift right, "\underline{f}^{-1}"']
\end{tikzcd}
$$

The map $f_{\#} : \Gamma(\pi) \longrightarrow \Gamma(\pi')$ obtained so is a module homomorphism under the ring isomorphism

$$(\underline{f}^{-1})^* : C^\infty(M) \longrightarrow C^\infty(M'),\ h \mapsto h \circ \underline{f}^{-1}.$$

In particular, if f is a strong bundle map then $f_{\#}$ is a homomorphism of $C^\infty(M)$-modules.

Via push-forwards, each exact sequence of vector bundles gives rise to an exact sequence of modules of sections: if

$$E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2$$

is an exact sequence of M-morphisms, then

$$\Gamma(\pi_0) \xrightarrow{(f_0)_{\#}} \Gamma(\pi_1) \xrightarrow{(f_1)_{\#}} \Gamma(\pi_2)$$

is an exact sequence of ($C^\infty(M)$-linear maps of) $C^\infty(M)$-modules.

1.22. Let (E, π, M) be a vector bundle, and let $f : N \longrightarrow M$ be a smooth map. A *section of* π *along the map* f is a smooth map $\sigma : N \longrightarrow E$ such that $\pi \circ \sigma = f$. The set of sections of π along f will be denoted by $\Gamma_f(\pi)$. $\Gamma_f(\pi)$ is evidently a $C^\infty(N)$-module; now we show that *there is a canonical isomorphism*

$$\boxed{\Gamma(f^*\pi) \cong \Gamma_f(\pi)}$$

given by

$$(*) \qquad s \in \Gamma(f^*\pi) \mapsto \sigma := \pi_2 \circ s \in \Gamma_f(\pi),$$

where $\pi_2 := pr_2 \restriction N \times_M E$ (see 1.14):

$$\begin{tikzcd}
N \times_M E \arrow[r, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & E \arrow[d, "\pi"] \\
N \arrow[u, shift right, "s"'] \arrow[ur, "\sigma"] \arrow[r, "f"'] & M
\end{tikzcd}$$

In fact, $\sigma := \pi_2 \circ s$ is a smooth map, and it is indeed a section along f since

$$\pi \circ \sigma = \pi \circ \pi_2 \circ s = f \circ \pi_1 \circ s = f.$$

The correspondence $(*)$ is invertible, its inverse 'pulls $\sigma \in \Gamma_f(\pi)$ back' to the section $(1_N, \sigma) \in \Gamma(f^*\pi)$.

1.23. Canonical isomorphisms of modules of sections. For explicit or implicit later use, we list here some canonical isomorphisms of basic importance. Let (E, π, M) and (E', π', M) be vector bundles.

$$\boxed{\Gamma(\pi^*) \cong (\Gamma(\pi))^*} \tag{1}$$

The *idea of the proof* is quite clear. If $\sigma \in \Gamma(\pi^*)$, we define an element $\widehat{\sigma}$ in $(\Gamma(\pi))^*$ by

$$\widehat{\sigma}(s)(p) := \sigma(p)(s(p)),$$

where $s \in \Gamma(\pi)$ and $p \in M$. Then $\widehat{\sigma}(s)$ is indeed a smooth function on M, and it is easy to check that the map $\sigma \in \Gamma(\pi) \mapsto \widehat{\sigma} \in (\Gamma(\pi))^*$ is an injective homomorphism of $C^\infty(M)$-modules. After that, employing first a local argument and using a partition of unity in the next step, it may be shown that the correspondence $\sigma \mapsto \widehat{\sigma}$ is surjective as well.

$$\boxed{\Gamma(\pi \otimes \pi') \cong \Gamma(\pi) \otimes_{C^\infty(M)} \Gamma(\pi')} \tag{2}$$

This isomorphism can be established by the following reasoning. In view of the definition of the tensor product (see A.8), the module $\Gamma(\pi) \otimes_{C^\infty(M)} \Gamma(\pi')$ is generated by elements of the form $\sigma \otimes \sigma'$, where $\sigma \in \Gamma(\pi)$, $\sigma' \in \Gamma(\pi')$. We may therefore define a map

$$\alpha : \Gamma(\pi) \otimes_{C^\infty(M)} \Gamma(\pi') \longrightarrow \Gamma(\pi \otimes \pi')$$

by the rule that, for each point $p \in M$,

$$\alpha(\sigma \otimes \sigma')(p) = \sigma(p) \otimes \sigma'(p),$$

and using $C^\infty(M)$-linear extension to the whole module. Then α is an injective homomorphism of $C^\infty(M)$-modules. To check that α is also surjective, first we establish the result locally, and then, by a partition of unity we piece together the local sections to obtain the desired global section.

Using the fundamental relations (1) and (2), the following isomorphisms can easily be deduced:

(3a) $\Gamma(\mathbf{T}_s(\pi)) \cong \mathbf{T}_s(\Gamma(\pi)) = L^s_{C^\infty(M)}(\Gamma(\pi))$,

(3b) $\Gamma(\mathbf{T}^r(\pi)) \cong \mathbf{T}^r(\Gamma(\pi)) \cong L^r_{C^\infty(M)}((\Gamma(\pi))^*) \cong L^r_{C^\infty(M)}(\Gamma(\pi^*))$,

(3c) $\Gamma(\mathbf{T}^r_s(\pi)) \cong \mathbf{T}^r_s(\Gamma(\pi)) = \mathbf{T}^r(\Gamma(\pi)) \otimes \mathbf{T}_s(\Gamma(\pi))$,

(4a) $\Gamma(\wedge^r \pi) \cong \wedge^r_{C^\infty(M)}\Gamma(\pi)$,

(4b) $\Gamma(\wedge^s \pi^*) \cong \wedge^s_{C^\infty(M)}\Gamma(\pi^*) \cong \wedge^s_{C^\infty(M)}(\Gamma(\pi))^* \cong (L^s_{\text{skew}})_{C^\infty(M)}(\Gamma(\pi))$,

(5) $\Gamma(\mathbf{S}_\ell(\pi)) \cong \mathbf{S}_\ell(\Gamma(\pi)) = (L^\ell_{\text{sym}})_{C^\infty(M)}(\Gamma(\pi))$.

It will be convenient to denote the $C^\infty(M)$ -modules

$$\Gamma(\mathbf{T}_s(\pi)), \quad \Gamma(\mathbf{T}^r(\pi)), \quad \Gamma(\mathbf{T}^r_s(\pi)) \text{ and } (\wedge^s\pi^*)$$

by

$$\Gamma_s(\pi), \quad \Gamma^r(\pi), \quad \Gamma^r_s(\pi) \text{ and } \mathcal{A}^s(\pi),$$

respectively. Then $\Gamma^\bullet_\bullet(\pi)$ stands for the mixed tensor algebra of $\Gamma(\pi)$. Elements of $\Gamma^r_s(\pi)$ will also be called *π-tensor fields* of type (r,s) (r-fold contravariant, s-fold covariant) on the base manifold M.

We emphasize that in virtue of the listed results a π-tensor field of type (r,s) can always be regarded as a $C^\infty(M)$-multilinear map

$$\underbrace{[\Gamma(\pi)]^* \times \cdots \times [\Gamma(\pi)]^*}_{r \text{ times}} \times \underbrace{\Gamma(\pi) \times \cdots \times \Gamma(\pi)}_{s \text{ times}} \longrightarrow C^\infty(M).$$

1.24. Pseudo-Riemannian and Riemannian vector bundles. Let (E,π,M) be a vector bundle. A *pseudo-Riemannian metric* in π is a section g of $\mathbf{S}_2(\pi)$ such that for each point $p \in M$, the symmetric bilinear form $g_p : E_p \times E_p \longrightarrow \mathbb{R}$ is non-degenerate. Then the pair (π, g), or simply π, is called a *pseudo-Riemannian vector bundle*. If, in particular, the bilinear forms g_p are positive definite on the vector spaces E_p, then g is said to be a

Riemannian metric in π, and (π, g), or just π, is called a *Riemannian vector bundle*.

Every vector bundle admits Riemannian metrics. Indeed, local existence is clear, and we may glue the local sections with the help of a partition of unity on M, since the positive definite sections form a convex open set.

Any pseudo-Riemannian metric g in π determines a strong bundle isomorphism $\flat : z \in E \longrightarrow \flat(z) =: z^\flat \in E^*$ between π and π^* by the rule

$$z^\flat(w) := g_p(z, w) \quad \text{for all } p \in M;\ z, w \in E_p.$$

The inverse of $\flat$ is denoted by $\sharp$; $\flat$ and $\sharp$ are called the *musical isomorphisms with respect to g*.

D. Tangent bundle and tensor fields

In our forthcoming discussion M is an n-dimensional manifold, in accordance with 1.4.

1.25. If $TM := \bigsqcup_{p \in M} T_pM$ and

$$\tau_M : TM \longrightarrow M, \; v \mapsto \tau_M(v) := p, \quad \text{if } v \in T_pM$$

then (TM, τ_M, M) is a vector bundle of rank n. More precisely, there is a unique smooth structure on the set TM such that (TM, τ_M, M) is a fibred manifold (the tangent spaces T_pM being the fibres) and such that the following condition is satisfied:

TB. *For each chart $(\mathfrak{U}, (u^i)_{i=1}^n)$ on M, the mapping*

$$v \in \tau_M^{-1}(\mathfrak{U}) \mapsto (\tau_M(v), v(u^1), \ldots, v(u^n)) \in \mathfrak{U} \times \mathbb{R}^n$$

is a diffeomorphism.

From these it follows that (TM, τ_M, M) is indeed a vector bundle, called the *tangent vector bundle*, or more briefly the *tangent bundle* of the manifold M. The total space TM is also mentioned as the *tangent manifold* to M.

Next, suppose that $f : M \longrightarrow N$ is a smooth map. The *tangent map* of f is the map $f_* : TM \longrightarrow TN$ whose restriction to a tangent space T_pM is given in 1.8. Then (f_*, f) is a bundle map between the tangent bundles (TM, τ_M, M) and (TN, τ_N, N):

$$\begin{array}{ccc} TM & \xrightarrow{f_*} & TN \\ \tau_M \big\downarrow & & \big\downarrow \tau_N \\ M & \xrightarrow{f} & N \end{array}$$

1.26. Vector fields. The sections of τ_M are called *vector fields* on M. Thus a vector field X assigns to every point $p \in M$ a tangent vector $X(p) =: X_p$ of T_pM such that the map $M \longrightarrow TM$ obtained so is smooth. For the $C^\infty(M)$-module of vector fields on M we use the specific notation $\mathfrak{X}(M) := \Gamma(\tau_M)$.

A basic feature of the module $\mathfrak{X}(M)$ is that *it can be canonically identified with the $C^\infty(M)$-module,* $\operatorname{Der} C^\infty(M)$, *of derivations of the algebra $C^\infty(M)$.* Indeed, any vector field X on M acts as a derivation on $C^\infty(M)$ by the rule

$$f \in C^\infty(M) \mapsto Xf, \; (Xf)(p) := X(p)(f) \quad (p \in M).$$

Conversely, every derivation on $C^\infty(M)$ comes from a vector field.

Henceforth, whenever convenient, *we will consider vector fields to be derivations on* $C^\infty(M)$ without any comment or notational change. This interpretation enables us to define the *Lie bracket* $[X,Y]$ of two vector fields $X, Y \in \mathfrak{X}(M)$ by

$$[X,Y]f := X(Yf) - Y(Xf) \quad \textit{for all } f \in C^\infty(M).$$

This bracket operation is *skew-symmetric, bilinear over the real numbers* and satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad (X,Y,Z \in \mathfrak{X}(M)).$$

The Lie bracket, though $\mathbb{R}$-bilinear, is *not* $C^\infty(M)$-bilinear. In fact, for any function $f \in C^\infty(M)$ and vector fields $X, Y \in \mathfrak{X}(M)$ we have

$$[fX,Y] = f[X,Y] - (Yf)X \quad \text{and} \quad [X,fY] = f[X,Y] + (Xf)Y.$$

Anyway, we see that the pair $(\mathfrak{X}(M), [\,,\,])$ is a *Lie algebra over* $\mathbb{R}$; a prototype of Lie algebras.

Local description. With the conventions of 1.1, let $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ be a chart on M. The maps

$$\frac{\partial}{\partial u^i} : f \in C^\infty(\mathcal{U}) \mapsto \frac{\partial f}{\partial u^i} := D_i(f \circ u^{-1}) \circ u \quad (1 \leqq i \leqq n)$$

(cf. 1.7) are local vector fields defined on $\mathcal{U}$, called the *coordinate vector fields* of the chart $(\mathcal{U}, u)$. In view of the basis theorem in 1.7, $\left(\frac{\partial}{\partial u^i}\right)_{i=1}^n$ is a *frame* for τ_M over $\mathcal{U}$ in the sense of 1.20. Thus, for any vector field X on M, the restriction $X \restriction \mathcal{U}$ can uniquely be represented in the form

$$X \restriction \mathcal{U} = \sum_{i=1}^n X(u^i) \frac{\partial}{\partial u^i}.$$

More generally, consider a domain $\mathcal{V}$ of M. By a *frame field* on $\mathcal{V}$ we mean a family $(X_i)_{i=1}^n$ of vector fields on $\mathcal{V}$ such that $(X_i(p))_{i=1}^n$ is a basis of T_pM for each point $p \in \mathcal{V}$. The frame field $(X_i)_{i=1}^n$ is said to be *holonomic*, if there exists a family of charts $(\mathcal{U}_\alpha, (u_\alpha^i)_{i=1}^n)_{\alpha \in A}$ such that $(\mathcal{U}_\alpha)_{\alpha \in A}$ covers $\mathcal{V}$ and $X_i \restriction \mathcal{U}_\alpha = \frac{\partial}{\partial u_\alpha^i}$ $(1 \leqq i \leqq n)$. A frame field is *anholonomic* if it is not holonomic.

Remark. A frame field $(X_i)_{i=1}^n$ formed by *globally defined* vector fields $X_i \in \mathfrak{X}(M)$ is also called a *parallelization* of the manifold. A manifold which admits a global parallelization is said to be *parallelizable*. The real line $\mathbb{R}$ endowed with the canonical smooth structure (1.8, Example) is obviously parallelizable: the natural vector field

$$\frac{d}{dr} : \mathbb{R} \longrightarrow T\mathbb{R}, \quad t \mapsto \left(\frac{d}{dr}\right)_t$$

provides a parallelization for $\mathbb{R}$. More generally, the real vector spaces $\mathbb{R}^n$ ($n \in \mathbb{N}$) as manifolds (cf. 1.9) are parallelizable: if the maps

$$D_i : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}, \quad f \mapsto D_i f \quad (1 \leqq i \leqq n)$$

are the standard partial derivatives in $\mathbb{R}^n$ (see A.4), then $(D_i)_{i=1}^n$ is a frame field (called *natural*) on $\mathbb{R}^n$. Further well-known examples of parallelizable manifolds are the spheres S^1, S^3 and S^7. For a good introduction to the subject the reader is referred to [15].

A frame field defined on a (proper) domain of a manifold M is also mentioned as a *local parallelization* of M.

1.27. f-relatedness. Let $f : M \longrightarrow N$ be a smooth map. We say that two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are *f-related*, denoted as $X \underset{f}{\sim} Y$, if $f_* \circ X = Y \circ f$, i.e., if the diagram

$$\begin{tikzcd} TM \arrow[r, "f_*"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"'] & N \arrow[u, "Y"'] \end{tikzcd}$$

is commutative. It can readily be seen that

$$X \underset{f}{\sim} Y \Leftrightarrow (\forall h \in C^\infty(N) : Yh \circ f = X(h \circ f)).$$

Using this observation, we obtain: if $X \underset{f}{\sim} X_1$ and $Y \underset{f}{\sim} Y_1$, then

(1) $\quad \lambda X + \mu Y \underset{f}{\sim} \lambda X_1 + \mu Y_1 \quad (\lambda, \mu \in \mathbb{R})$;

(2) $\quad (h \circ f)X \underset{f}{\sim} hX_1 \quad (h \in C^\infty(N))$;

(3) $\quad [X, Y] \underset{f}{\sim} [X_1, Y_1]$.

Now suppose that f is a *diffeomorphism*. Then (f_*, f) is a bundle isomorphism between τ_M and τ_N, and the general construction described in 1.21 yields for any vector field X on M the push-forward vector field

$$f_\# X := f_* \circ X \circ f^{-1} \in \mathfrak{X}(N).$$

Then, obviously, $X \underset{f}{\sim} f_\# X$.

Let us finally consider a fibred manifold (E, π, M). A vector field $\xi \in \mathfrak{X}(E)$ is said to be *projectable* on M if there is a smooth map $X : M \longrightarrow TM$ such that $X \circ \pi = \pi_* \circ \xi$:

$$\begin{tikzcd} E \arrow[r, "\xi"] \arrow[d, "\pi"'] & TE \arrow[r, "\tau_E"] \arrow[d, "\pi_*"] & E \arrow[d, "\pi"] \\ M \arrow[r, "X"'] & TM \arrow[r, "\tau_M"'] & M \end{tikzcd} .$$

Then

$$\tau_M \circ X \circ \pi = \tau_M \circ \pi_* \circ \xi = \pi \circ \tau_E \circ \xi = \pi.$$

Since π is surjective and therefore it has a right inverse, we conclude that $\tau_M \circ X = 1_M$. This means that X is actually a vector field on M. In view of the construction, $\xi \underset{\pi}{\sim} X$.

1.28. Flows.

In this subsection I denotes a nonempty open interval of the real line.

(**1**) Let X be a vector field on the manifold M. A curve $c : I \longrightarrow M$ is said to be an *integral curve* of X if $\dot{c} = X \circ c$.

Local description. Choose a chart $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ for M. If $X \restriction \mathcal{U} = \sum_{i=1}^n X^i \frac{\partial}{\partial u^i}$; $f^i := X^i \circ u^{-1}$, $c^i := u^i \circ c$, then c is an integral curve of X if, and only if,

$$c^{i\prime} = f^i \circ (c^1, \dots, c^n) \qquad (1 \leqq i \leqq n),$$

or, equivalently, if the curve

$$u \circ c : c^{-1}(\mathcal{U}) \subset I \longrightarrow u(\mathcal{U}) \subset \mathbb{R}^n$$

in $\mathbb{R}^n$ is a solution of the ordinary differential equation (abbreviated as ODE)

$$x' = f(x), \qquad f := (f^1, \ldots, f^n).$$

(**2**) A smooth map $\varphi : \mathbb{R} \times M \longrightarrow M$ is said to be a *dynamical system* or *flow* on M if for all $p \in M$ and $s, t \in \mathbb{R}$ we have

FLOW 1. $\varphi(0, p) = p$.

FLOW 2. $\varphi(s, \varphi(t, p)) = \varphi(s + t, p)$.

For any fixed point $p \in M$, the curve

$$c_p : \mathbb{R} \longrightarrow M, \; t \mapsto c_p(t) := \varphi(t, p)$$

is called the *flow line* of p, the image $c_p(\mathbb{R})$ of the flow line c_p is called the *orbit* of p. The map

$$\dot{\varphi} : M \longrightarrow TM, \; p \mapsto \dot{\varphi}(p) := \dot{c}_p(0)$$

is a vector field, the *velocity field* of the flow φ.

Observe that *the flow lines are integral curves of the velocity field of a flow.* Indeed, let us denote by X the velocity field of φ. Consider the flow line c_p of a point $p \in M$. Then

$$X(c_p(0)) = X(p) := \dot{\varphi}(p) := \dot{c}_p(0),$$

so the assertion holds automatically for the parameter 0. If $t \in \mathbb{R}$ is arbitrary and $q := c_p(t)$, then for all $s \in \mathbb{R}$ we have

$$c_q(s) := \varphi(s, q) = \varphi(s, \varphi(t, p)) = \varphi(s + t, p) = c_p(s + t),$$

therefore $\dot{c}_q(0) = \dot{c}_p(t)$, and hence

$$X[c_p(t)] = X(q) = \dot{\varphi}(q) = \dot{c}_q(0) = \dot{c}_p(t),$$

as we claimed.

(**3**) A subset $W \subset \mathbb{R} \times M$ is called *radial,* if it contains $\{0\} \times M$ and the intersection $W \cap (\mathbb{R} \times \{p\})$ is connected for all $p \in M$. A *local flow* on the manifold M is a smooth map from a radial open set $W \subset \mathbb{R} \times M$ into M such that the conditions **FLOW 1, 2** are satisfied for all s, t, p for which both sides of the relations are defined. The flow lines and the velocity field of a local flow may be introduced in the same way as in the global case. (Of course, then the domain of a flow line is a proper open interval of $\mathbb{R}$, in general.)

Now the fundamental existence and uniqueness theorem for ODE may be translated into the language of manifolds as follows.

Integrability theorem for vector fields. *Every vector field is the velocity field of a unique, maximal local flow; on a compact manifold even a global one.*

('Maximality' means the maximality of the domains of flow lines.)

1.29. One-forms. The dual (T^*M, τ_M^*, M) of the tangent bundle τ_M is called the *cotangent bundle* of M ($T^*M := (TM)^*$; cf. 1.15). Similarly to the foregoing, the shorthand for the cotangent bundle of M will be τ_M^*. The $C^\infty(M)$-module of sections of τ_M^*, denoted by $\mathcal{A}^1(M) := \Gamma(\tau_M^*)$, is the module of the *one-forms* on M. In view of 1.23(1) we obtain:

$$\mathcal{A}^1(M) \cong (\Gamma(\tau_M))^* = (\mathfrak{X}(M))^* = \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M))$$
$$=: L_{C^\infty(M)}(\mathfrak{X}(M)).$$

Taking into account that the tangent spaces T_pM may be identified with the second dual spaces $(T_pM)^{**}$, we may also write

$$(\mathcal{A}^1(M))^* = [\Gamma(\tau_M^*)]^* \overset{1.23(1)}{\cong} \Gamma((\tau_M^*)^*) \cong \Gamma(\tau_M) = \mathfrak{X}(M),$$

hence $[\mathfrak{X}(M)]^{**} \cong \mathfrak{X}(M)$.

For any smooth function f on M, the map

$$X \in \mathfrak{X}(M) \mapsto Xf \in C^\infty(M)$$

is an element of $(\mathfrak{X}(M))^*$, therefore the isomorphism $\mathcal{A}^1(M) \cong (\mathfrak{X}(M))^*$ assures the existence of a unique one-form $df \in \mathcal{A}^1(M)$ such that

$$df(X) = Xf \quad \textit{for all } X \in \mathfrak{X}(M).$$

df is called the *exterior derivative*, or simply the *differential*, of f.

It may easily be seen that the differential of a constant function is zero. Conversely, due to the connectedness of M, if $f \in C^\infty(M)$ satisfies $df = 0$, then f is constant.

We also recall a much deeper result:

The $C^\infty(M)$-module $\mathcal{A}^1(M)$ is generated by differentials.

For a proof the reader is referred to [35] Vol. I, pp. 117–118.

1.30. Tensor fields. Let $(r,s) \in \mathbb{N} \times \mathbb{N}$, $r+s \neq 0$. A section of the tensor bundle $\mathbf{T}^r_s(\tau_M)$ is called a *tensor field* (or, by abuse of language, a *tensor*) of type (r,s) *on* M (contravariant of order r, covariant of order s). The $C^\infty(M)$-module of type (r,s) tensor fields on M is denoted by $\mathcal{T}^r_s(M)$; $\mathcal{T}^0_0(M) := C^\infty(M)$. The direct sum $\mathcal{T}^\bullet_\bullet(M) := \bigoplus_{(r,s)\in\mathbb{N}\times\mathbb{N}} \mathcal{T}^r_s(M)$ is said to be the *(mixed) tensor algebra of* M.

Examples.

(1)
$$\mathcal{T}^1_0(M) := \Gamma(\mathbf{T}^1_0\tau_M) \overset{1.23(3b)}{\cong} \mathbf{T}^1_0(\mathfrak{X}(M))$$
$$\cong \operatorname{Hom}_{C^\infty(M)}((\mathfrak{X}(M))^*, C^\infty(M)) = (\mathfrak{X}(M))^{**} \cong \mathfrak{X}(M).$$

(2)
$$\mathcal{T}^0_1(M) := \Gamma(\mathbf{T}^0_1\tau_M) \cong \mathbf{T}^0_1(\mathfrak{X}(M)) = \operatorname{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M))$$
$$= (\mathfrak{X}(M))^* \cong \mathcal{A}^1(M).$$

(3) Let $s \geqq 1$. Then

$$\mathcal{T}_s(M) := \mathcal{T}^0_s(M) := \Gamma(\mathbf{T}_s\tau_M) \cong \mathbf{T}_s\mathfrak{X}(M) = L^s_{C^\infty(M)}(\mathfrak{X}(M)),$$

i.e., $\mathcal{T}_s(M)$ may be identified with the module of $C^\infty(M)$-multilinear maps $(\mathfrak{X}(M))^s \longrightarrow C^\infty(M)$.

Consider a smooth map $f : M \longrightarrow N$. If $A \in \mathcal{T}^0_s(N)$, and

$$(f^*A)_p(v_1, \dots, v_s) := A_{f(p)}((f_*)_p(v_1), \dots, (f_*)_p(v_s))$$

for each point $p \in M$ and vectors $v_i \in T_pM$ $(1 \leqq i \leqq s)$, then $f^*A \in \mathcal{T}^0_s(M)$, is called the *pull-back* of A by f.

(4) Keep the assumption $s \geqq 1$. We also have the canonical isomorphism

$$\mathcal{T}^1_s(M) \cong L^s_{C^\infty(M)}(\mathfrak{X}(M), \mathfrak{X}(M)).$$

Indeed,

$$\mathcal{T}^1_s(M) := \Gamma(\mathbf{T}^1_s\tau_M) \overset{1.15}{=} \Gamma(\tau_M \otimes \tau^*_M \otimes \cdots \otimes \tau^*_M) \overset{1.23(2)}{\cong}$$
$$\cong \mathfrak{X}(M) \otimes \mathcal{A}^1(M) \otimes \cdots \otimes \mathcal{A}^1(M)$$
$$\cong (\mathcal{A}^1(M) \otimes \cdots \otimes \mathcal{A}^1(M)) \otimes \mathfrak{X}(M)$$
$$\cong \operatorname{Hom}_{C^\infty(M)}((\mathcal{A}^1(M) \otimes \cdots \otimes \mathcal{A}^1(M))^*, \mathfrak{X}(M)) \overset{1.29}{\cong}$$
$$\cong \operatorname{Hom}_{C^\infty(M)}(\mathfrak{X}(M) \otimes \ldots \otimes \mathfrak{X}(M), \mathfrak{X}(M)) \cong L^s_{C^\infty(M)}(\mathfrak{X}(M), \mathfrak{X}(M)),$$

taking into account the definition of the tensor product of modules (see A.8(**1**)) in the last step. Thus, the elements of $\mathcal{T}^1_s(M)$ may be interpreted as $C^\infty(M)$-multilinear maps

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s \text{ times}} \longrightarrow \mathfrak{X}(M).$$

For this reason, tensor fields of type $(1, s)$ will also be mentioned as *vector valued (covariant) tensor fields* of order s. In particular, if $s = 1$, we see that there is a canonical isomorphism

$$\mathcal{T}^1_1(M) \cong \operatorname{End}_{C^\infty(M)} \mathfrak{X}(M).$$

The unique tensor field ι_M of type $(1, 1)$ which corresponds to the identity homomorphism $1_{\mathfrak{X}(M)}$ is called the *unit tensor field* on M. ι_M, as a section of $\tau_M \otimes \tau^*_M$, acts by the rule

$$p \in M \mapsto \iota_M(p) \in \mathbf{T}^1_1(T_pM);\ \iota_M(p)(\alpha, v) := \alpha(v); \quad v \in T_pM,\ \alpha \in (T_pM)^*.$$

(**5**) A *(pseudo-) Riemannian manifold* (M, g) is a manifold M with a (pseudo-) Riemannian metric g on τ_M (cf. 1.24). Then g is also called a *(pseudo-) Riemannian metric on* M. In view of the cited definition, a (pseudo-) Riemannian metric $g \in \mathcal{T}^0_2(M)$ is a symmetric tensor field such that

$$g_p : T_pM \times T_PM \longrightarrow \mathbb{R} \text{ is } \begin{cases} \textit{non-degenerate}, \text{ resp.} \\ \textit{positive definite} \end{cases}$$

at any point $p \in M$. As we have learnt above (see 1.24) in the general case, a pseudo-Riemannian metric g gives rise to the musical (strong bundle) isomorphisms

$$\flat : TM \longrightarrow T^*M \quad \text{and} \quad \sharp : T^*M \longrightarrow TM,$$

inverse to each other. They induce isomorphisms between the modules of sections, denoted by the same letter, in a natural manner. To be explicit,

$$\flat : X \in \mathfrak{X}(M) \mapsto X^\flat \in \mathcal{A}^1(M),\ X^\flat(Y) = g(X, Y) \qquad \text{for all } Y \in \mathfrak{X}(M);$$

$$\sharp : \theta \in \mathcal{A}^1(M) \mapsto \theta^\sharp \in \mathfrak{X}(M), \quad g(\theta^\sharp, Y) = \theta(Y) \qquad \text{for all } Y \in \mathfrak{X}(M).$$

The *gradient* of a function $f \in C^\infty(M)$ (with respect to g) is the vector field $\operatorname{grad} f := (df)^\sharp$. Then

$$g(\operatorname{grad} f, Y) = (df)(Y) = Yf \quad \text{for all } Y \in \mathfrak{X}(M).$$

1.31. Contractions. There is a unique $C^\infty(M)$-linear map

$$tr : \operatorname{End}\mathfrak{X}(M) \cong \mathcal{T}_1^1(M) \longrightarrow C^\infty(M),$$

called *trace* or $(1,1)$-*contraction* such that

$$tr(X \otimes \theta) := \iota_M(\theta, X) \quad \textit{for all } X \in \mathfrak{X}(M) \textit{ and } \theta \in \mathcal{A}^1(M).$$

This map can be extended to tensors of higher order as follows. Assume that $A \in \mathcal{T}_s^r(M)$ $(r, s \in \mathbb{N}^*)$, and let two indices i, j be given such that $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$. Consider the map

$$\widehat{A}_j^i : (\mathcal{A}^1(M))^{r-1} \times (\mathfrak{X}(M))^{s-1} \longrightarrow \mathcal{T}_1^1(M),$$

$$(\theta^1, \dots, \theta^{r-1}, X_1, \dots, X_{s-1}) \mapsto \widehat{A}_j^i(\theta^1, \dots, \theta^{r-1}, X_1, \dots, X_{s-1})$$

defined by

$$\begin{aligned}\widehat{A}_j^i(\theta^1, \dots, \theta^{r-1}, X_1, \dots, X_{s-1})(\theta, X) &:= \\ &= A(\theta^1, \dots, \overset{i}{\smile}{\!\!\!\!\!\theta}, \dots, \theta^{r-1}, X_1, \dots, \overset{j}{\smile}{\!\!\!\!\!X}, \dots, X_{s-1})\end{aligned}$$

for any one-form θ and vector field X on M. Then the map

$$c_j^i : \mathcal{T}_s^r(M) \longrightarrow \mathcal{T}_{s-1}^{r-1}(M), \quad A \mapsto c_j^i A := tr \circ \widehat{A}_j^i$$

is $C^\infty(M)$-linear. This map is called the *contraction* of the contravariant index i and the covariant index j.

1.32. Tensor derivations. A map $\mathcal{D} : \mathcal{T}_\bullet^\bullet(M) \longrightarrow \mathcal{T}_\bullet^\bullet(M)$ is said to be a *tensor derivation* on M if it satisfies the following conditions:

DO 1. $\mathcal{D}$ *is* $\mathbb{R}$-*linear.*

DO 2. $\mathcal{D}$ *is type-preserving*, i.e., $\mathcal{D}(\mathcal{T}_s^r(M)) \subset \mathcal{T}_s^r(M)$ for each $(r, s) \in \mathbb{N} \times \mathbb{N}$.

DO 3. $\mathcal{D}$ obeys the *Leibniz rule* $\mathcal{D}(A \otimes B) = \mathcal{D}A \otimes B + A \otimes \mathcal{D}B$ for any tensor fields A and B on M.

DO 4. $\mathcal{D}$ *commutes with all contractions.*

Now we summarize some of the most important consequences of **DO 1**–**DO 4**.

(1) *Tensor derivations are local operators or differential operators* in the following sense: if $\mathcal{D}$ is a tensor derivation on M, $A \in \mathcal{T}^{\bullet}_{\bullet}(M)$ and $\mathcal{U} \subset M$ is an open set, then

$$A \restriction \mathcal{U} = 0 \;\Rightarrow\; \mathcal{D}A \restriction \mathcal{U} = 0.$$

Indeed, let p be any point in $\mathcal{U}$. There is a smooth function f on M such that $f(p) = 0$ and $f(q) = 1$, if $q \in M \setminus \mathcal{U}$. Then $A = fA = f \otimes A$, and in view of **DO 3** and **DO 2** we have

$$\mathcal{D}A = \mathcal{D}(f \otimes A) = \mathcal{D}f \otimes A + f \otimes \mathcal{D}A = (\mathcal{D}f)A + f\mathcal{D}A.$$

Thus $(\mathcal{D}A)(p) = (\mathcal{D}f)(p)A(p) + f(p)(\mathcal{D}A)(p) = 0$; hence $\mathcal{D}A \restriction \mathcal{U} = 0$.

(2) *Tensor derivations are natural with respect to restrictions*. That is, if $\mathcal{D}$ is a tensor derivation on M and $\mathcal{U}$ is an open set of M, then there is a unique tensor derivation $\mathcal{D}_{\mathcal{U}}$ on $\mathcal{U}$ such that

$$\mathcal{D}_{\mathcal{U}}(A \restriction \mathcal{U}) = (\mathcal{D}A) \restriction \mathcal{U} \quad \textit{for all tensors } A \textit{ on } M$$

or the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}^r_s(M) & \xrightarrow{\restriction_{\mathcal{U}}} & \mathcal{T}^r_s(\mathcal{U}) \\ \mathcal{D}\Big\downarrow & & \Big\downarrow \mathcal{D}_{\mathcal{U}} \\ \mathcal{T}^r_s(M) & \xrightarrow[\restriction_{\mathcal{U}}]{} & \mathcal{T}^r_s(\mathcal{U}) \end{array}$$

$\mathcal{D}_{\mathcal{U}}$ is called the *restriction* of $\mathcal{D}$ to $\mathcal{U}$, and henceforth we omit the subscript $\mathcal{U}$.

(3) **Product rule.** Let $\mathcal{D}$ be a tensor derivation on M. If $A \in \mathcal{T}^r_s(M)$ then

$$\boxed{\begin{aligned} (\mathcal{D}A)(\theta^1, \dots, \theta^r, X_1, \dots, X_s) &= \mathcal{D}[A(\theta^1, \dots, \theta^r, X_1, \dots, X_s)] \\ &\quad - \sum_{i=1}^{r} A(\theta^1, \dots, \mathcal{D}\theta^i, \dots, \theta^r, X_1, \dots, X_s) \\ &\quad - \sum_{j=1}^{s} A(\theta^1, \dots, \theta^r, X_1, \dots, \mathcal{D}X_j, \dots, X_s) \end{aligned}}$$

($\theta^i \in \mathcal{A}^1(M)$, $1 \leqq i \leqq r$, $X_j \in \mathfrak{X}(M)$, $1 \leqq j \leqq s$).

(4) **Theorem of T. J. Willmore.** *Any tensor derivation of the mixed tensor algebra* $\mathcal{T}^\bullet_\bullet(M)$ *is completely determined by its action over the smooth functions and vector fields on* M. *Conversely, given a vector field* $Z \in \mathfrak{X}(M)$ *and an* $\mathbb{R}$*-linear map* $\mathcal{D}_0 : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ *satisfying the condition*

$$\mathcal{D}_0(fX) = (Zf)X + f\mathcal{D}_0(X) \quad \text{for all } f \in C^\infty(M),\ X \in \mathfrak{X}(M),$$

there exists a (necessarily unique) tensor derivation $\mathcal{D}$ *on* M *such that*

$$\mathcal{D} \restriction C^\infty(M) = Z \quad \textit{and} \quad \mathcal{D} \restriction \mathfrak{X}(M) = \mathcal{D}_0.$$

For an enjoyable proof the reader is referred to [61].

(5) **Extension to π-tensor fields.** Let (E, π, M) be a vector bundle and let us consider the tensor algebra $\Gamma^\bullet_\bullet(\pi)$ of π-tensor fields on M. We also define a *tensor derivation* $\mathcal{D} : \Gamma^\bullet_\bullet(\pi) \longrightarrow \Gamma^\bullet_\bullet(\pi)$ by the requirements **DO 1**–**DO 4** of 1.32. Then the properties 1.32, (1)–(3) are valid without any change. A strict analogue of Willmore's theorem is also true in this generality. For its fundamental importance we restate the result:

Theorem (generalized Willmore's theorem).

(i) *Any tensor derivation of the mixed tensor algebra* $\Gamma^\bullet_\bullet(\pi)$ *is completely determined by its action over the smooth functions of the base space and the sections in* $\Gamma(\pi)$.

(ii) *Given a vector field* X *on* M *and an* $\mathbb{R}$*-linear map* $\mathcal{D}_0 : \Gamma(\pi) \longrightarrow \Gamma(\pi)$ *such that*

$$\mathcal{D}_0(f\sigma) = (Xf)\sigma + f\mathcal{D}_0(\sigma) \quad \textit{for all } f \in C^\infty(M),\ \sigma \in \Gamma(\pi)$$

there exists a unique tensor derivation $\mathcal{D}$ *along* π *satisfying*

$$\mathcal{D} \restriction C^\infty(M) = X \quad \text{and} \quad \mathcal{D} \restriction \Gamma(\pi) = \mathcal{D}_0.$$

1.33. The Lie derivative. Let a vector field $X \in \mathfrak{X}(M)$ be given. *There exists a unique tensor derivation* d_X *on* M *such that*

$$d_X f := Xf \qquad \text{for all } f \in C^\infty(M);$$

$$d_X Y := [X, Y] \quad \text{for all } Y \in \mathfrak{X}(M).$$

The operator d_X is called the *Lie derivative* with respect to X.

In fact, for all functions $f \in C^\infty(M)$ and vector fields $X \in \mathfrak{X}(M)$ we have

$$d_X fY := [X, fY] = (Xf)Y + f[X, Y] = (Xf)Y + f d_X Y;$$

therefore the assumptions of Willmore's theorem are satisfied with the choice

$$\mathcal{D}_0 : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),\ Y \mapsto \mathcal{D}_0(Y) := [X, Y].$$

Corollaries. (1) For any vector fields X, Y on M,

$$d_{[X,Y]} = [d_X, d_Y] := d_X \circ d_Y - d_Y \circ d_X.$$

Indeed, it may be immediately checked that $d_{[X,Y]}$ and $[d_X, d_Y]$ coincide on $C^\infty(M)$ and $\mathfrak{X}(M)$, and therefore they are equal by Willmore's theorem.

(2) Let $A \in \mathcal{T}_s^0(M)$, $B \in \mathcal{T}_s^1(M) \cong L^s\ (\mathfrak{X}(M), \mathfrak{X}(M))$ $(s \geqq 1)$. In view of the product rule 1.32(3), for the Lie derivative $d_X A$ of A and $d_X B$ of B we obtain:

$$\boxed{\begin{gathered} d_X A(X_1, \dots, X_s) = X[A(X_1, \dots, X_s)] - \sum_{i=1}^{s} A(X_1, \dots, [X, X_i], \dots, X_s), \\[2ex] d_X B(X_1, \dots, X_s) = [X, B(X_1, \dots, X_s)] - \sum_{i=1}^{s} B(X_1, \dots, [X, X_i], \dots, X_s) \\[2ex] (X_i \in \mathfrak{X}(M),\ 1 \leqq i \leqq s) \end{gathered}}\ .$$

Dynamic interpretation. Let two vector fields, X and Y be given on M. Suppose that X is the velocity field of the local flow $\varphi : W \subset \mathbb{R} \times M \longrightarrow M$ (see 1.28), and denote by φ_t the map $p \mapsto \varphi(t, p)$ if $(t, p) \in W$ for a fixed $t \in \mathbb{R}$. Then we have

$$\boxed{d_X Y = [X, Y] = \lim_{t \longrightarrow 0} \frac{1}{t}((\varphi_t^{-1})_\# Y - Y) = \lim_{t \longrightarrow 0} \frac{1}{t}(Y - (\varphi_t)_\# Y)}\ .$$

For a proof the reader is referred to [61] or [80].

E. Differential forms

We continue to let M be an n-dimensional, connected manifold.

1.34. A *differential form of degree k* ($\in \mathbb{N}^*$) on the manifold M or an *exterior form* of degree k, or a *k-form* for short, is a section of the vector bundle $\wedge^k \tau_M^* = \mathbf{A}_k \tau_M$. According to 1.23(4b),

$$\Gamma(\wedge^k \tau_M^*) \cong (L^k_{\text{skew}})_{C^\infty(M)}(\mathfrak{X}(M)),$$

therefore any k-form α on M may be regarded as a $C^\infty(M)$-multilinear map $(\mathfrak{X}(M))^k \longrightarrow C^\infty(M)$ which is skew-symmetric:

$$\alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)}) = \varepsilon(\sigma)\alpha(X_1, \dots, X_k)$$

for every permutation $\sigma \in \mathfrak{S}_k$ and vector fields $X_i \in \mathfrak{X}(M)$ $(1 \leqq i \leqq k)$. For the $C^\infty(M)$-module of k-forms on M we use the notation

$$\mathcal{A}^k(M) := \Gamma(\wedge^k \tau_M^*).$$

We put $\mathcal{A}^\circ(M) := C^\infty(M)$, then the direct sum

$$\mathcal{A}(M) := \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$$

becomes a graded algebra over the ring $C^\infty(M)$ with *the wedge product* defined by

$$(\alpha \wedge \beta)(X_1, \dots, X_{k+\ell}) :=$$
$$= \frac{1}{k!\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} \varepsilon(\sigma)\alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)})\beta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})$$

($\alpha \in \mathcal{A}^k(M)$, $\beta \in \mathcal{A}^\ell(M)$; $X_i \in \mathfrak{X}(M)$, $1 \leqq i \leqq k+\ell$) cf. A.9(2)). This algebra is said to be the *exterior* or *Grassmann algebra of the manifold M*. The Grassmann algebra is *associative* and *graded commutative*:

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha \quad \text{for all } \alpha \in \mathcal{A}^k(M) \text{ and } \beta \in \mathcal{A}^\ell(M).$$

$\mathcal{A}(M)$, considered as an algebra over $\mathbb{R}$, is generated by smooth functions and their differentials since, as we indicated in 1.29, the $C^\infty(M)$-module $\mathcal{A}^1(M)$ is generated by differentials.

1.35. Volume manifolds. A *volume form* on an n-dimensional manifold M is an n-form $\mu \in \mathcal{A}^n(M)$ such that $\mu(p) \neq 0$ for all $p \in M$; M is called *orientable* if there exists a volume form on M. A volume form μ on M assigns an orientation $\mu_p \in A_n(T_pM)$ to each tangent space of M; then a basis $(b_i)_{i=1}^n$ of T_pM is called *positive* if $\mu_p(b_1, \ldots, b_p) > 0$.

Due to the connectedness of M, we have the following results:

(1) *M is orientable if, and only if, there is an n-form $\mu \in \mathcal{A}^n(M)$ such that any n-form $\nu \in \mathcal{A}^n(M)$ may be written in the form $\nu = f\mu$, for some $f \in C^\infty(M)$.*

(2) *If M is orientable then M has exactly two orientations.*

A pair (M, μ) is said to be a *volume manifold* if μ is a volume form on M.

Now assume that (M, g) is an oriented (pseudo-) Riemannian manifold. Then there exists a unique volume form μ_g on M such that for any point $p \in M$ and any positive orthonormal basis $(b_i)_{i=1}^n$ of T_pM we have

$$(\mu_g)_p(b_1, \ldots, b_n) = 1.$$

The volume form μ_g is said to be the *Riemannian volume* form on M.

1.36. Divergence and Laplace–Beltrami operator. Let (M, μ) be a volume manifold. If X is a vector field on M, then the Lie derivative $d_X\mu$ is an n-form again. Hence, taking into account 1.35(1), there exists a unique function $\operatorname{div}_\mu X \in C^\infty(M)$ such that

$$d_X\mu = (\operatorname{div}_\mu X)\mu;$$

this function is called the *divergence* of the vector field X with respect to μ.

If f is a nowhere zero smooth function on M then $f\mu$ is also a volume form, and for the divergence of X with respect to $f\mu$ we have

$$\operatorname{div}_{f\mu} X = \operatorname{div}_\mu X + \frac{1}{f} Xf.$$

Indeed, using the Leibniz rule **DO 3** in 1.32,

$$(\operatorname{div}_{f\mu} X)f\mu := d_X(f\mu) = (Xf)\mu + f d_X\mu = \frac{1}{f}(Xf)f\mu + (\operatorname{div}_\mu X)f\mu,$$

whence the result.

Now let, in particular, (M, g) be a (pseudo-) Riemannian manifold. Then we define the *divergence* $\operatorname{div}_g X$ of a vector field $X \in \mathfrak{X}(M)$ *with respect to*

g to be the divergence of X with respect to the Riemannian volume form μ_g, i.e.,

$$(\operatorname{div}_g X)\mu_g := d_X \mu_g.$$

Finally, the *Laplace–Beltrami operator on functions* on an orientable (pseudo-) Riemannian manifold (M, g) is defined by the formula

$$\Delta := \operatorname{div}_g \circ \operatorname{grad}.$$

If a metric tensor g is fixed on M, we omit the subscript g.

1.37. Bundle-valued forms, wedge product, wedge-bar product. Let a vector bundle (E, π, M) be given, and consider the tangent bundle τ_M of M.

(**1**) Using the construction principle sketched in 1.13(3), we can build a vector bundle

$$\mathbf{A}_{\mathrm{k}}(\tau_M, \pi) \in \mathrm{VB(M)} \quad (k \in \mathbb{N}^*)$$

whose fibre at a point $p \in M$ consists of the skew-symmetric k-linear maps

$$T_pM \times \cdots \times T_pM \longrightarrow E_p.$$

A π*-valued* (or, less consistently, E-valued) k*-form on* M is a section of the bundle $\mathbf{A}_{\mathrm{k}}(\tau_M, \pi)$;

$$\mathcal{A}^k(M, \pi) := \Gamma(\mathbf{A}_{\mathrm{k}}(\tau_M, \pi))$$

is the $C^\infty(M)$-module of π-valued k-forms on M. (An alternative notation: $\mathcal{A}^k(M, E)$.) We extend the definition to the case $k = 0$ by

$$\mathcal{A}^0(M, \pi) := \Gamma(\pi).$$

Any π-valued k-form K on M may be canonically identified with the skew-symmetric k-linear map

$$\widetilde{K} : (\mathfrak{X}(M))^k \longrightarrow \Gamma(\pi)$$

given by

$$\big[\widetilde{K}(X_1, \ldots, X_k)\big](p) := K(p)(X_1(p), \ldots, X_k(p))$$

for every point $p \in M$ and vector fields $X_1, \ldots, X_k$ on M. (For a proof we refer to [35], Vol. I, 2.24.) Henceforth we shall use the canonical isomorphism

$$\mathcal{A}^k(M, \pi) \cong L^k_{\mathrm{skew}}(\mathfrak{X}(M), \Gamma(\pi))$$

obtained in this way without any comment.

(**2**) Next, consider the direct sum

$$\mathcal{A}(M,\pi) := \bigoplus_{k=0}^{n} \mathcal{A}^k(M,\pi)$$

and define the *wedge product* $\alpha \wedge L$ of a k-form $\alpha \in \mathcal{A}^k(M)$ and a π-valued ℓ-form $L \in \mathcal{A}^\ell(M,\pi)$ by the formula

$$(\alpha \wedge L)(X_1, \dots, X_{k+\ell}) := \\ = \frac{1}{k!\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} \varepsilon(\sigma)\alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)})L(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})$$

($X_i \in \mathfrak{X}(M)$, $1 \leqq i \leqq k+\ell$). Then $\alpha \wedge L \in \mathcal{A}^{k+\ell}(M,\pi)$, and $\mathcal{A}(M,\pi)$ becomes a graded left module over the graded ring $\mathcal{A}(M)$ with the wedge product as the scalar multiplication. This action of $\mathcal{A}(M)$ on $\mathcal{A}(M,\pi)$ is *effective* in the following sense: if $\alpha, \beta \in \mathcal{A}(M)$ and

$$\alpha \wedge K = \beta \wedge K \text{ for all } K \in \mathcal{A}(M,\pi)$$

then $\alpha = \beta$. Notice that the $C^\infty(M)$-bilinear correspondence

$$\alpha \otimes \sigma \in \mathcal{A}(M) \otimes_{C^\infty(M)} \Gamma(\pi) \mapsto \alpha \wedge \sigma \in \mathcal{A}(M,\pi)$$

induces a canonical isomorphism of graded $\mathcal{A}(M)$-modules

$$\mathcal{A}(M) \otimes_{C^\infty(M)} \Gamma(\pi) \xrightarrow{\cong} \mathcal{A}(M,\pi).$$

We shall consider these modules identical under this isomorphism and write $\alpha \otimes \sigma = \alpha \wedge \sigma$ whenever it is convenient.

(**3**) Now consider the endomorphism bundle $\mathrm{End}(\pi)$ (see 1.15) and the module $\mathcal{A}(M, \mathrm{End}(\pi))$ of $\mathrm{End}(\pi)$-valued forms on M. For later use, we define a seemingly artifical 'multiplication'

$$(\Omega, L) \in \mathcal{A}(M, \mathrm{End}(\pi)) \times \mathcal{A}(M,\pi) \mapsto \Omega[L] \in \mathcal{A}(M,\pi)$$

as follows: if $\Omega \in \mathcal{A}^k(M, \mathrm{End}(\pi))$, $L \in \mathcal{A}^\ell(M,\pi)$, then

$$\Omega[L](X_1, \dots, X_{k+\ell}) =: \\ = \frac{1}{k!\ell!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} \varepsilon(\sigma)\Omega(X_{\sigma(1)}, \dots, X_{\sigma(k)})(L(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}))$$

for all vector fields $X_1, \dots, X_{k+\ell}$ on M. With the help of this product, each $\Omega \in \mathcal{A}(M, \operatorname{End}(\pi))$ determines a $C^\infty(M)$-linear map

$$\widehat{\Omega} : \sigma \in \Gamma(\pi) \mapsto \widehat{\Omega}(\sigma) := \Omega[\sigma] \in \mathcal{A}(M, \pi),$$

and the correspondence

$$\Omega \in \mathcal{A}(M, \operatorname{End}(\pi)) \mapsto \widehat{\Omega} \in \operatorname{Hom}_{C^\infty(M)}(\Gamma(\pi), \mathcal{A}(M, \pi))$$

defines a natural isomorphism of $C^\infty(M)$-modules.

More generally, consider the map

$$a : \Omega \in \mathcal{A}^k(M, \operatorname{End}(\pi)) \mapsto a_\Omega,$$

given by

$$a_\Omega(L) := \Omega[L] \quad \text{for all } L \in \mathcal{A}(M, \pi).$$

It can easily be checked that a_Ω is a graded $\mathcal{A}(M)$-module homomorphism of $\mathcal{A}(M, \pi)$, i.e., if $\alpha \in \mathcal{A}^j(M)$ and $L \in \mathcal{A}^\ell(M, \pi)$, then

$$a_\Omega(\alpha \wedge L) = (-1)^{j\ell} \alpha \wedge a_\Omega(L).$$

Conversely, it may be shown that any graded $\mathcal{A}(M)$-homomorphism of $\mathcal{A}(M, \pi)$ is of the form a_Ω.

(**4**) Next we turn to the special case when the role of the vector bundle π is played by the tangent bundle τ_M. Then a τ_M-valued form on M is called a *vector-valued form on* M and we write $\mathcal{B}(M)$ as a shorthand for $\mathcal{A}(M, \tau_M)$. In detail,

$$\mathcal{B}^k(M) := \mathcal{A}^k(M, \tau_M) := \Gamma(\mathbf{A}_{\mathrm{k}}(\tau_M, \tau_M)) \quad (k > 0);$$

$$\mathcal{B}^0(M) := \mathfrak{X}(M), \quad \mathcal{B}(M) := \bigoplus_{k=0}^{n} \mathcal{B}^k(M) = \mathcal{A}(M, \tau_M).$$

Now we are in a position to define an important *action of* $\mathcal{B}(M)$ *on* $\mathcal{A}(M)$. Namely, let the *wedge-bar product* of a k-form $\alpha \in \mathcal{A}^k(M)$ and a vector-valued ℓ-form $L \in \mathcal{B}(M)$ be the $(k + \ell - 1)$-form $\alpha \barwedge L$ given by

$$(\alpha \barwedge L)(X_1, \dots, X_{k+\ell-1}) :=$$

$$= \frac{1}{\ell!(k-1)!} \sum_{\sigma \in \mathfrak{S}_{k+\ell-1}} \varepsilon(\sigma) \alpha(L(X_{\sigma(1)}, \dots, X_{\sigma(\ell)}), X_{\sigma(\ell+1)}, \dots, X_{\sigma(\ell+k-1)}),$$

where $X_i \in \mathfrak{X}(M)$; $1 \leqq i \leqq k + \ell - 1$, and $k > 0$;

$$\alpha \barwedge L := 0, \quad \text{if } \alpha \in \mathcal{A}^0(M) = C^\infty(M).$$

Examples. 1. Let $L \in \mathcal{B}^1(M) = \mathcal{A}^1(M, \tau_M) \cong \mathcal{T}^1_1(M)$. Then

$$(\alpha \barwedge L)(X_1, \dots, X_k) = \sum_{i=1}^{k} \alpha(X_1, \dots, L(X_i), \dots, X_k)$$

for any k ($\geqq 1$)-form α and vector fields $X_1, \dots, X_k$ on M. In particular, for the unit tensor field $\iota_M \in \mathcal{T}^1_1(M)$ we have the relation

$$\alpha \barwedge \iota_M = k\alpha.$$

2. Suppose that $L \in \mathcal{B}^\ell(M) = \mathcal{A}^\ell(M, \tau_M)$ ($\ell \in \mathbb{N}$), and let α be a one-form on M. Then $\alpha \barwedge L \in \mathcal{A}^\ell(M)$ and for any vector fields $X_1, \dots, X_\ell$ on M we have

$$\alpha \barwedge L(X_1, \dots, X_\ell) = \frac{1}{\ell!} \sum_{\sigma \in \mathfrak{S}_\ell} \varepsilon(\sigma) \alpha\big[L(X_{\sigma(1)}, \dots, X_{\sigma(\ell)})\big] = \alpha\big[L(X_1, \dots, X_\ell)\big],$$

therefore

$$\alpha \barwedge L = \alpha \circ L \quad \text{for all } \alpha \in \mathcal{A}^1(M).$$

In particular, $\alpha \barwedge L = \alpha(L)$ if $\ell = 0$, i.e., if $L \in \mathfrak{X}(M)$. (Compare this with 1.38(1) !)

(**5**) We also define the *wedge-bar product of two vector-valued forms:*

(i) $K \barwedge L := 0$, if $K \in \mathcal{B}^0(M) = \mathfrak{X}(M)$;

(ii) if $K \in \mathcal{B}^k(M)$, $k \in \mathbb{N}^*$ and $L \in \mathcal{B}^\ell(M)$, then $K \barwedge L \in \mathcal{B}^{\ell+k-1}(M)$, given by

$$K \barwedge L(X_1, \dots, X_{k+\ell-1}) :=$$
$$= \frac{1}{\ell!(k-1)!} \sum_{\sigma \in \mathfrak{S}_{k+\ell-1}} \varepsilon(\sigma) K\big[L(X_{\sigma(1)}, \dots, X_{\sigma(\ell)}), X_{\sigma(\ell+1)}, \dots, X_{\sigma(\ell+k-1)}\big]$$

($X_i \in \mathfrak{X}(M)$, $1 \leqq i \leqq \ell + k - 1$). Then, in particular,

$$K \barwedge X = K(X), \quad \text{if } K \in \mathcal{B}^1(M) \text{ and } X \in \mathcal{B}^0(M) = \mathfrak{X}(M);$$

$$K \barwedge L = K \circ L, \quad \text{if } K \in \mathcal{B}^1(M),\ L \in \mathcal{B}(M).$$

Lemma. *If α is a one-form and A is a vector-valued one-form on M, then for any vector-valued forms K, L on M we have the 'associativity' properties*

$$\alpha \barwedge (K \barwedge L) = (\alpha \barwedge K) \barwedge L, \quad A \barwedge (K \barwedge L) = (A \barwedge K) \barwedge L.$$

Proof. Since $\alpha \barwedge K = \alpha \circ K$, $\alpha \barwedge (K \barwedge L) = \alpha \circ (K \barwedge L)$, $A \barwedge K = A \circ K$, $A \barwedge (K \barwedge L) = A \circ (K \barwedge L)$, the assertion is an immediate consequence of the definitions of the wedge-bar products. □

1.38. The substitution operator. (**1**) Let X be a vector field on the manifold M. If A is a k-form or vector-valued k-form on M $(k \geqq 1)$ and

$$i_X A(X_2, \dots, X_k) := A(X, X_2, \dots, X_k) \quad (X_i \in \mathfrak{X}(M),\ 2 \leqq i \leqq k),$$

then $i_X A$ is a differential form (resp. a vector-valued form) of degree $k-1$. We extend this definition to the case $k = 0$ by putting

$$i_X f = 0, \quad \text{if } f \in C^\infty(M) \text{ and } i_X Y := 0, \quad \text{if } Y \in \mathcal{B}^0(M) = \mathfrak{X}(M).$$

Notice that for a one-form $\alpha \in \mathcal{A}^1(M)$ we have

$$i_X \alpha = \alpha(X);$$

in particular, for the differential of a function $f \in C^\infty(M)$,

$$i_X df = X(f).$$

The map $i_X : \mathcal{A}(M) \longrightarrow \mathcal{A}(M)$ (resp. $\mathcal{B}(M) \longrightarrow \mathcal{B}(M)$) defined in this way is called the *substitution operator* induced by X. *i_X is a graded derivation of degree -1 of the graded algebra $\mathcal{A}(M)$* in the sense of A.7(**2**), therefore

$$i_X(\alpha \wedge \beta) = i_X \alpha \wedge \beta + (-1)^k \alpha \wedge i_X \beta \quad \text{for all } \alpha \in \mathcal{A}^k(M),\ \beta \in \mathcal{A}(M).$$

As for the wedge product of a 'scalar' k-form $\alpha \in \mathcal{A}(M)$ and a vector-valued form $L \in \mathcal{B}(M)$, we have a completely similar relation:

$$i_X(\alpha \wedge L) = i_X \alpha \wedge L + (-1)^k \alpha \wedge i_X L.$$

It follows immediately from the definition that

$$i_X^2 \alpha = 0,\ i_X^2 L = 0 \quad \text{for all } \alpha \in \mathcal{A}(M),\ L \in \mathcal{B}(M).$$

It may also readily be seen that *if $\alpha \in \mathcal{A}^k(M)$ (resp. $L \in \mathcal{B}^\ell(M)$) $(k, \ell \geqq 1)$ satisfies $i_X \alpha = 0$ (resp. $i_X L = 0$) for every vector field X on M, then $\alpha = 0$ (resp. $L = 0$).*

(**2**) Next we offer a straightforward generalization of the notion of a substitution operator to bundle-valued forms. Let (E, π, M) be a vector bundle over M and X a vector field on M. The substitution operator $i_X : \mathcal{A}(M,\pi) \longrightarrow \mathcal{A}(M,\pi)$ induced by X is defined word for word as above:

$$i_X\sigma := 0, \quad \text{if } \sigma \in \mathcal{A}^0(M,\pi) = \Gamma(\pi);$$

$$i_X L(X_2,\dots,X_k) := L(X,X_2,\dots,X_k), \quad \text{if } L \in \mathcal{A}^k(M,\pi),\ k \geqq 1$$

($X_i \in \mathfrak{X}(M)$; $2 \leqq i \leqq k$). Then i_X is clearly $C^\infty(M)$-linear, and for any bundle-valued $k(\geqq 1)$-forms L, $i_X L \in \mathcal{A}^{k-1}(M,\pi)$. As for the action $\mathcal{A}(M)$ on $\mathcal{A}(M,\pi)$ (see 1.37(2)), we have:

$$i_X(\alpha \wedge L) = i_X\alpha \wedge L + (-1)^k \alpha \wedge i_X L \quad \text{for all } \alpha \in \mathcal{A}^k(M), L \in \mathcal{A}(M,\pi).$$

1.39. The exterior derivative. Let α be a k-form on the manifold M, and define a map

$$d\alpha : [\mathfrak{X}(M)]^{k+1} \longrightarrow C^\infty(M)$$

as follows:

(i) if α is a 0-form, i.e. a smooth function on M, then $d\alpha$ is the differential of α described in 1.29;

(ii) if $\alpha \in \mathcal{A}^k(M)$, $k \geqq 1$, then

$$d\alpha(X_1,\dots,X_{k+1}) := \sum_{i=1}^{k+1}(-1)^{i+1}X_i[\alpha(X_1,\dots,\widehat{X}_i,\dots,X_{k+1})] + \sum_{1 \leqq i<j \leqq k+1}(-1)^{i+j}\alpha([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})$$

($X_i \in \mathfrak{X}(M)$, $1 \leqq i \leqq k+1$, $\widehat{X}_i$ means that X_i has to be deleted).

In both cases $d\alpha$ is a $(k+1)$-form on M, called the *exterior derivative* of α. The map

$$d : \mathcal{A}(M) \longrightarrow \mathcal{A}(M), \quad \alpha \mapsto d\alpha$$

is said to be the *exterior differential* on the Grassmann algebra $\mathcal{A}(M)$. The exterior differential is a *graded derivation of degree 1,* i.e., we have

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta \quad \text{for all } \alpha \in \mathcal{A}^k(M),\ \beta \in \mathcal{A}(M).$$

1.40. Fundamental identities. To conclude this section, here is a short list of standard results, all of them expressed also in terms of the graded commutator (see A.7, Lemma 2). Let X and Y be vector fields on the manifold M. Then

$$
\begin{array}{lll}
(1) & i_X \circ i_Y + i_Y \circ i_X = 0 & \iff [i_X, i_Y] = 0, \\
(2) & i_{[X,Y]} = d_X \circ i_Y - i_Y \circ d_X & \iff i_{[X,Y]} = [d_X, i_Y], \\
(3) & d_X = i_X \circ d + d \circ i_X & \iff d_X = [i_X, d], \\
(4) & d_{[X,Y]} = d_X \circ d_Y - d_Y \circ d_X & \iff d_{[X,Y]} = [d_X, d_Y], \\
(5) & d_X \circ d = d \circ d_X & \iff [d_X, d] = 0, \\
(6) & d^2 = 0 & \iff \dfrac{1}{2}[d, d] = 0.
\end{array}
$$

Formula (3), a 'magic' formula of H. Cartan, is particularly useful.

F. Covariant derivatives

1.41. Definition. Let (E, π, M) be a vector bundle. A *covariant derivative operator* or simply a *covariant derivative* in π is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(\pi) \longrightarrow \Gamma(\pi), \quad (X, \sigma) \mapsto \nabla_X \sigma$$

which, for any vector fields X and Y on M, any sections σ, σ_1, σ_2 in $\Gamma(\pi)$, and any function f in $C^\infty(M)$, satisfies

COVD 1. $\qquad \nabla_{X+Y}\sigma = \nabla_X \sigma + \nabla_Y \sigma.$

COVD 2. $\qquad \nabla_{fX}\sigma = f\nabla_X \sigma.$

COVD 3. $\qquad \nabla_X(\sigma_1 + \sigma_2) = \nabla_X \sigma_1 + \nabla_X \sigma_2.$

COVD 4. $\qquad \nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X \sigma.$

Then the section $\nabla_X \sigma$ is called the *covariant derivative of* σ *with respect to* X.

Remarks. (1) Let ∇ be a covariant derivative in π. Then, for any section σ of π, the map

$$\nabla\sigma : \mathfrak{X}(M) \longrightarrow \Gamma(\pi), \quad X \mapsto (\nabla\sigma)(X) := \nabla_X \sigma$$

is $C^\infty(M)$-linear, in view of **COVD 2**. $\nabla\sigma$ is called the *covariant differential* of σ. A section of π is said to be *parallel* if its covariant differential vanishes.

Observe that, accoording to 1.37(1), $\nabla\sigma \in \mathcal{A}^1(M, \pi)$. From **COVD 4** we obtain that the map

$$\sigma \in \Gamma(\pi) \mapsto \nabla\sigma \in \mathcal{A}^1(M, \pi)$$

has the following property:

$$(*) \qquad \nabla(f\sigma) = df \wedge \sigma + f\nabla\sigma \quad \text{for all } f \in C^\infty(M),\ \sigma \in \Gamma(\pi).$$

Conversely, a covariant derivative operator in π may also be defined as an $\mathbb{R}$-*linear* map $\nabla : \Gamma(\pi) \longrightarrow \mathcal{A}^1(M, \pi)$ satisfying the condition $(*)$; then the operator $\nabla_X := i_X \circ \nabla : \Gamma(\pi) \longrightarrow \Gamma(\pi)$ is the covariant derivative with respect to the vector field X on M.

(2) Before proceeding, we point out two important features of a covariant derivative operator. First, it is clear from our preceding discussion that $\nabla_X \sigma$

is tensorial in X, therefore the value of $\nabla_X\sigma$ at a point $p \in M$ depends only on $X(p)$. Hence we may define $\nabla_v\sigma$ for any individual tangent vector $v \in TM$ as follows:

$$\nabla_v\sigma := \big(\nabla_X\sigma\big)(p), \quad \text{if } X \in \mathfrak{X}(M) \text{ satisfies } X(p) = v.$$

Secondly, the dependence of $\nabla_X\sigma$ on σ is more delicate. However, by a standard 'bump function reasoning' (see e.g. [62], Lemma 1.3), it can easily be shown that the map $\sigma \in \Gamma(\pi) \mapsto \nabla\sigma \in \mathcal{A}^1(M,\pi)$ is a *(first order) differential operator* analogously to 1.32(1), i.e., if a section $\sigma \in \Gamma(\pi)$ vanishes on a neighbourhood of a point $p \in M$ then $\nabla\sigma$ is also zero 'near p'.

(3) Supppose that ∇^1 and ∇^2 are two covariant derivative operators in (E,π,M); then their difference

$$\nabla^1 - \nabla^2 : \Gamma(\pi) \longrightarrow \mathcal{A}^1(M,\pi)$$

is $C^\infty(M)$-linear. Hence, in view of 1.37(3), there is a unique bundle-valued form $\mathcal{D} \in \mathcal{A}^1(M, \mathrm{End}(\pi))$ such that

$$\nabla^1\sigma - \nabla^2\sigma = \mathcal{D}[\sigma] \quad \text{for all } \sigma \in \Gamma(\pi).$$

$\mathcal{D}$ is called the *difference form* of the covariant derivatives ∇^1 and ∇^2. Conversely, given any covariant derivative ∇ in π and any form $\mathcal{D} \in \mathcal{A}^1(M, \mathrm{End}(\pi))$, the map

$$\overline{\nabla} : \sigma \in \Gamma(\pi) \mapsto \overline{\nabla}\sigma := \nabla\sigma + \mathcal{D}[\sigma]$$

is again a covariant derivative in π.

(4) Suppose that π is a trivial vector bundle with a total space $M \times \mathbb{R}^k$. Then there exists a unique covariant derivative ∇ in π such that the constant sections (i.e., sections of the form $p \in M \mapsto (p,v) \in M \times \mathbb{R}^k$ with a constant v) are parallel. ∇ is called the *trivial covariant derivative* in π. Using this fact and a partition of unity argument, it follows that *every vector bundle admits a covariant derivative operator.*

1.42. Induced covariant derivatives. (**1**) A covariant derivative operator ∇ in π induces naturally a covariant derivative ∇^* in the dual bundle π^* such that for any sections $s^* \in \Gamma(\pi^*)$, $\sigma \in \Gamma(\pi)$ and any vector field X on M

$$\big(\nabla^*_X s^*\big)(\sigma) := X[s^*(\sigma)] - s^*\big(\nabla_X\sigma\big).$$

Clearly, this relation determines ∇^* uniquely.

Taking into account that by the canonical isomorphisms listed in 1.23 we have

$$\Gamma(\mathrm{End}(\pi)) \cong \Gamma(\pi^* \otimes \pi) \cong \Gamma(\pi^*) \otimes \Gamma(\pi) \cong [\Gamma(\pi)]^* \otimes \Gamma(\pi)$$
$$\cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(\pi), \Gamma(\pi)),$$

a covariant derivative $\widehat{\nabla}$ may be obtained in $\mathrm{End}(\pi)$ as follows:

$$\big(\widehat{\nabla}_X L\big)(\sigma) := \nabla_X L(\sigma) - L\big(\nabla_X \sigma\big),$$

for all $L \in \Gamma(\mathrm{End}(\pi))$, $\sigma \in \Gamma(\pi)$, $X \in \mathfrak{X}(M)$. We shall write ∇ instead of ∇^* and $\widehat{\nabla}$, this abuse of notation is common and should not lead to confusion.

More generally, using the second version of Willmore's theorem (1.32(5)), we conclude that *given a vector field X on M, there exists a unique tensor derivation ∇_X of $\Gamma^\bullet_\bullet(\pi)$ such that*

$$\nabla_X f = Xf \quad \text{for all } f \in C^\infty(M),$$

and $\nabla_X \sigma$ is the covariant derivative of σ with respect to X for all $\sigma \in \Gamma(\pi)$.

It may easily be checked that if $A \in \Gamma^r_s(\pi)$ is a π-tensor field on M, then the map

$$X \in \mathfrak{X}(M) \mapsto \nabla_X A \in \Gamma^r_s(\pi)$$

is $C^\infty(M)$-linear. This fact enables us to adopt the following

Definition. The *covariant differential* of a π-tensor field $A \in \Gamma^r_s(\pi)$ is the $C^\infty(M)$-multilinear map ∇A given by

$$\nabla A(X, s^1, \dots, s^r, \sigma_1, \dots, \sigma_s) := \big(\nabla_X A\big)(s^1, \dots, s^r, \sigma_1, \dots, \sigma_s)$$

for all $X \in \mathfrak{X}(M)$, $s^i \in \big[\Gamma(\pi)\big]^*$ $(1 \leqq i \leqq r)$, $\sigma_j \in \Gamma(\pi)$ $(1 \leqq j \leqq s)$. If $\nabla A = 0$, then A is said to be *parallel* (with respect to ∇).

(**2**) Given vector bundles $\pi_i \in \mathrm{VB(M)}$ with covariant derivative operators ∇^i $(1 \leqq i \leqq 2)$, there is a unique covariant derivative ∇ on the tensor product bundle $\pi_1 \otimes \pi_2$ such that for any vector field X on M we have

$$\nabla_X(\sigma_1 \otimes \sigma_2) = \big(\nabla^1_X \sigma_1\big) \otimes \sigma_2 + \sigma_1 \otimes \nabla^2_X \sigma_2; \quad \sigma_1 \in \Gamma(\pi_1),\ \sigma_2 \in \Gamma(\pi_2).$$

Similarly, if Φ is a section of $\mathrm{Hom}(\pi_1, \pi_2)$, the formula

$$\big(\nabla_X \Phi\big)(\sigma) := \nabla^2_X(\Phi(\sigma)) - \Phi\big(\nabla^1_X(\sigma)\big); \quad \sigma \in \Gamma(\pi),\ X \in \mathfrak{X}(M)$$

defines a covariant derivative operator in $\mathrm{Hom}(\pi_1, \pi_2)$.

1.43. Covariant exterior derivative. Let ∇ be a covariant derivative operator in the vector bundle (E, π, M). The map

$$d^\nabla : \mathcal{A}(M,\pi) \longrightarrow \mathcal{A}(M,\pi),\ K \in \mathcal{A}^k(M,\pi) \mapsto d^\nabla K \in \mathcal{A}^{k+1}(M,\pi)$$

given by

(i) $d^\nabla K := \nabla K$, if $K \in \mathcal{A}^0(M,\pi) = \Gamma(\pi)$;

(ii) $$d^\nabla K(X_1,\dots,X_{k+1}) := \sum_{i=1}^{k+1}(-1)^{i+1}\nabla_{X_i}\left[K(X_1,\dots,\widehat{X}_i,\dots,X_{k+1})\right]$$
$$+ \sum_{1\leqq i<j\leqq k+1}(-1)^{i+j}K([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})$$
$$(X_i \in \mathfrak{X}(M),\ 1 \leqq i \leqq k+1)$$

is called the *covariant exterior derivative* (or *gauge exterior derivative*) with respect to ∇.

An illustrative particular case: if $K \in \mathcal{A}^1(M,\pi)$, then

$$d^\nabla K(X,Y) = \nabla_X[K(Y)] - \nabla_Y[K(X)] - K[X,Y] \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

As for the wedge product of forms and bundle-valued forms defined in 1.37(2) and the operator

$$a : \mathcal{A}(M, \mathrm{End}(\pi)) \longrightarrow \mathrm{Hom}_{\mathcal{A}(M)}(\mathcal{A}(M,\pi),\mathcal{A}(M,\pi)),$$

we obtain:

(iii) $d^\nabla(\alpha\wedge L) = d\alpha\wedge L + (-1)^k\alpha\wedge d^\nabla L;\ \alpha\in\mathcal{A}^k(M),\ L\in\mathcal{A}(M,\pi).$

(iv) $d^\nabla(a_\Omega(L)) = a_{d^\nabla\Omega}(L) + (-1)^k a_\Omega d^\nabla L,$ i.e.,

$$d^\nabla(\Omega[L]) = (d^\nabla\Omega)[L] + (-1)^k\Omega[d^\nabla L];\ \Omega\in\mathcal{A}^k(M,\mathrm{End}(\pi)),\ L\in\mathcal{A}(M,\pi).$$

(Notice that in the term $d^\nabla\Omega$ ∇ means the operator $\widehat{\nabla}$ described in 1.42(1).)

1.44. Curvature. The *curvature* R^∇ of a covariant derivative operator ∇ in a vector bundle (E,π,M) is the $\mathrm{End}(\pi)$-valued 2-form on M defined by

$$R^\nabla(X,Y)(\sigma) = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma$$

for any vector fields X, Y on M and any section σ in $\Gamma(\pi)$. It is easy to verify that the map

$$(X,Y,\sigma)\in\mathfrak{X}(M)\times\mathfrak{X}(M)\times\Gamma(\pi)\mapsto R^\nabla(X,Y)\sigma\in\Gamma(\pi)$$

is indeed trilinear over $C^\infty(M)$ and skew-symmetric in the first two arguments.

Lemma 1. *For any section $\sigma \in \Gamma(\pi)$ we have*

$$d^\nabla(d^\nabla\sigma) = R^\nabla[\sigma].$$

Proof. Let $X, Y \in \mathfrak{X}(M)$. By the definition of d^∇ and using the skew-symmetry of R^∇ in X and Y in the last step, we obtain:

$$\big[d^\nabla(d^\nabla\sigma)\big](X,Y) = \nabla_X\big[(d^\nabla\sigma)(Y)\big] - \nabla_Y\big[(d^\nabla\sigma)(X)\big] - (d^\nabla\sigma)[X,Y]$$
$$= \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma = R(X,Y)(\sigma) = R[\sigma](X,Y). \qquad \square$$

Lemma 2. *For any bundle-valued form $K \in \mathcal{A}(M,\pi)$,*

$$d^\nabla(d^\nabla K) = R^\nabla[K].$$

Proof. In view of the isomorphism $\mathcal{A}(M) \otimes_{C^\infty(M)} \Gamma(\pi) \cong \mathcal{A}(M,\pi)$ (see 1.37(2)), it is enough to check that the assertion is true for $K := \alpha \wedge \sigma$; $\alpha \in \mathcal{A}^k(M)$, $\sigma \in \Gamma(\pi)$. Using formula (iii) of 1.43 and Lemma 1, we obtain:

$$d^\nabla(d^\nabla K) = d^\nabla(d^\nabla(\alpha\wedge\sigma)) = d^\nabla(d\alpha\wedge\sigma + (-1)^k\alpha\wedge d^\nabla\sigma) = d^2\alpha\wedge\sigma$$
$$+ (-1)^{k+1}d\alpha\wedge d^\nabla\sigma + (-1)^k d\alpha\wedge d^\nabla\sigma + (-1)^{2k}\alpha\wedge d^\nabla(d^\nabla\sigma)$$
$$= \alpha\wedge d^\nabla(d^\nabla\sigma) = \alpha\wedge R^\nabla[\sigma] = R^\nabla[\alpha\wedge\sigma] = R^\nabla[K]. \qquad \square$$

Proposition (Differential Bianchi identity). *The curvature satisfies the relation*

$$d^\nabla R^\nabla = 0.$$

First proof. Let $\sigma \in \Gamma(\pi)$. By 1.43(iv), $d^\nabla(R[\sigma]) = (d^\nabla R)[\sigma] + R[d^\nabla\sigma]$. Hence, by Lemmas 1 and 2,

$$(d^\nabla R)[\sigma] = d^\nabla(d^\nabla(d^\nabla\sigma)) - d^\nabla(d^\nabla(d^\nabla\sigma)) = 0.$$

Second proof. We apply a slightly longer but more direct argument. Consider R^∇ as a map

$$(X,Y) \in \mathfrak{X}(M)\times\mathfrak{X}(M) \mapsto R^\nabla(X,Y) \in \mathrm{End}(\Gamma(\pi)) \cong \Gamma(\mathrm{End}(\pi))$$

and extend ∇ to $\mathrm{End}(\pi)$ according to 1.42. Our task is to verify that

$$(d^\nabla R^\nabla)(X,Y,Z) = \nabla_X(R^\nabla(Y,Z)) - \nabla_Y(R^\nabla(X,Z)) + \nabla_Z(R^\nabla(X,Y))$$
$$- R^\nabla([X,Y],Z) + R^\nabla([X,Z],Y) - R^\nabla([Y,Z],X) = 0.$$

Since the expression on the left is tensorial in X, Y and Z, we may assume that $[X,Y]=[X,Z]=[Y,Z]=0$. Thus, for any section $\sigma \in \Gamma(\pi)$,

$$
\begin{aligned}
&\big[(d^\nabla R^\nabla)(X,Y,Z)\big](\sigma) = \big[\nabla_X(R^\nabla(Y,Z))\big](\sigma) - \big[\nabla_Y(R^\nabla(X,Z))\big](\sigma)\\
&\quad + \big[\nabla_Z(R^\nabla(X,Y))\big](\sigma) = \nabla_X(R^\nabla(Y,Z)\sigma) - R^\nabla(Y,Z)(\nabla_X\sigma)\\
&\quad - \nabla_Y(R^\nabla(X,Z)\sigma) + R^\nabla(X,Z)(\nabla_Y\sigma) + \nabla_Z(R^\nabla(X,Y)\sigma)\\
&\quad - R^\nabla(X,Y)(\nabla_Z\sigma) = \nabla_X\nabla_Y\nabla_Z\sigma - \nabla_X\nabla_Z\nabla_Y\sigma - \nabla_Y\nabla_Z\nabla_X\sigma\\
&\quad + \nabla_Z\nabla_Y\nabla_X\sigma - \nabla_Y\nabla_X\nabla_Z\sigma + \nabla_Y\nabla_Z\nabla_X\sigma + \nabla_X\nabla_Z\nabla_Y\sigma - \nabla_Z\nabla_X\nabla_Y\sigma\\
&\quad + \nabla_Z\nabla_X\nabla_Y\sigma - \nabla_Z\nabla_Y\nabla_X\sigma - \nabla_X\nabla_Y\nabla_Z\sigma + \nabla_Y\nabla_X\nabla_Z\sigma = 0. \qquad \Box
\end{aligned}
$$

1.45. Proposition. *If a vector bundle has a covariant derivative operator whose curvature is zero, then any point of the base manifold has a neighbourhood on which there exists a frame consisting of parallel sections.*

Strategy of proof. Consider a vector bundle (E,π,M) of rank k over an n-dimensional base manifold M. Let ∇ be a covariant derivative operator in π with vanishing curvature R^∇. Let us pick a point $p \in M$ and choose a chart $(\mathcal{U},(u^i)_{i=1}^n)$ around p. By Remark 2 in 1.41, the covariant derivatives $\nabla_i := \nabla_{\frac{\partial}{\partial u^i}}$ with respect to the local vector fields $\frac{\partial}{\partial u^i}$ have a well-defined meaning. The condition that $R^\nabla = 0$ is equivalent to the property that the operators ∇_i all commute with each other, i.e.,

$$[\nabla_i, \nabla_j] = 0, \qquad 1 \leqq i,j \leqq n.$$

Next, we apply the classical theorem of Frobenius's on the local existence of an integral manifold with prescribed tangent spaces. (For an appropriate formulation of the theorem the reader is referred to [80] Chapter 1, §9; see also Exercise 5 there). This allows us to solve the system of partial differential equations

$$\nabla_i\sigma = 0, \qquad 1 \leqq i \leqq n$$

with prescribed $\sigma(p) \in E_p$. Hence, assigning a basis $(\sigma_j(p))_{j=1}^k$ of E_p we obtain a family of sections $(\sigma_j)_{j=1}^k$ which provides a local frame on a neighbourhood of p and has the property $\nabla_i\sigma_j = 0$; $1 \leqq i \leqq n$, $1 \leqq j \leqq k$. Since the last property implies that the sections σ_j are parallel, the truth of the Proposition follows.

1.46. Metric derivatives. Let (π, g) be a pseudo-Riemannian vector bundle over the base space M. A covariant derivative ∇ in π is said to be *compatible with the metric* g, or *metric derivative* if the metric g is parallel with respect to ∇, i.e., $\nabla g = 0$. This means that the following *Ricci identity* holds:

$$Xg(\sigma_1, \sigma_2) = g(\nabla_X \sigma_1, \sigma_2) + g(\sigma_1, \nabla_X \sigma_2) \quad \text{for all } X \in \mathfrak{X}(M);\ \sigma_1, \sigma_2 \in \Gamma(\pi).$$

Proposition and definition. *Every pseudo-Riemannian vector bundle admits a metric derivative.* Notably, let $\overset{\circ}{\nabla}$ be any covariant derivative in (π, g); consider the covariant differential $\mathcal{C}_\flat := \overset{\circ}{\nabla} g$, and define $\mathcal{C} \in \mathcal{A}^1(M, \operatorname{End}(\pi))$ by

$$g(\mathcal{C}(X, \sigma_1), \sigma_2) := \mathcal{C}_\flat(X, \sigma_1, \sigma_2) \quad \text{for all } X \in \mathfrak{X}(M); \sigma_1, \sigma_2 \in \Gamma(\pi).$$

Then $\nabla := \overset{\circ}{\nabla} + \frac{1}{2}\mathcal{C}$, i.e., the map

$$\sigma \in \Gamma(\pi) \mapsto \nabla\sigma := \overset{\circ}{\nabla}\sigma + \frac{1}{2}\mathcal{C}[\sigma]$$

is a metric covariant derivative in π. $\mathcal{C}_\flat$ and $\mathcal{C}$ are called the *lowered Cartan tensor* and the *Cartan tensor* of (π, g) with respect to $\overset{\circ}{\nabla}$, respectively. (The common term *Cartan tensor* will also be used for $\mathcal{C}_\flat$ and $\mathcal{C}$.) $\mathcal{C}_\flat$ is symmetric in its second two variables, i.e.,

$$\mathcal{C}_\flat(X, \sigma_1, \sigma_2) = \mathcal{C}_\flat(X, \sigma_2, \sigma_1) \quad \text{for all} \quad X \in \mathfrak{X}(M);\ \sigma_1, \sigma_2 \in \Gamma(\pi).$$

Proof of the statement. In virtue of Remarks (4) and (3) in 1.41, a covariant derivative $\overset{\circ}{\nabla}$ certainly exists in (π, g) and $\nabla := \overset{\circ}{\nabla} + \frac{1}{2}\mathcal{C}$ is a covariant derivative again. From the definitions (see also 1.42 (2)),

$$\begin{aligned} \mathcal{C}_\flat(X, \sigma_1, \sigma_2) &:= \big(\overset{\circ}{\nabla} g\big)(X, \sigma_1, \sigma_2) = \big(\overset{\circ}{\nabla}_X g\big)(\sigma_1, \sigma_2) = X[g(\sigma_1, \sigma_2)] \\ &\quad - g\big(\overset{\circ}{\nabla}_X \sigma_1, \sigma_2\big) - g\big(\sigma_1, \overset{\circ}{\nabla}_X \sigma_2\big) \quad (X \in \mathfrak{X}(M);\ \sigma_1, \sigma_2 \in \Gamma(\pi)), \end{aligned}$$

hence $\mathcal{C}_\flat$ is indeed symmetric in its second two variables. By using 1.42(2) repeatedly we obtain readily

$$\begin{aligned} &(\nabla_X g)(\sigma_1, \sigma_2) = X[g(\sigma_1, \sigma_2)] - g(\nabla_X \sigma_1, \sigma_2) - g(\sigma_1, \nabla_X \sigma_2) = X[g(\sigma_1, \sigma_2)] \\ &\quad - g\Big(\overset{\circ}{\nabla}_X \sigma_1 + \frac{1}{2}\mathcal{C}(X, \sigma_1), \sigma_2\Big) - g\Big(\sigma_1, \overset{\circ}{\nabla}_X \sigma_2 + \frac{1}{2}\mathcal{C}(X, \sigma_2)\Big) = X[g(\sigma_1, \sigma_2)] \\ &\quad - g\big(\overset{\circ}{\nabla}_X \sigma_1, \sigma_2\big) - g\big(\sigma_1, \overset{\circ}{\nabla}_X \sigma_2\big) - \frac{1}{2}\mathcal{C}_\flat(X, \sigma_1, \sigma_2) - \frac{1}{2}\mathcal{C}_\flat(X, \sigma_2, \sigma_1) \\ &= \big(\overset{\circ}{\nabla}_X g\big)(\sigma_1, \sigma_2) - \mathcal{C}_\flat(X, \sigma_1, \sigma_2) = \big(\overset{\circ}{\nabla} g - \mathcal{C}_\flat\big)(X, \sigma_1, \sigma_2) = 0 \end{aligned}$$

$(X \in \mathfrak{X}(M);\ \sigma_1, \sigma_2 \in \Gamma(\pi))$, therefore ∇ is metric. $\square$

Lemma. *Let ∇ be a metric derivative in a pseudo-Riemannian vector bundle (π, g). Then the curvature R^∇ has the following skew-symmetry property:*

$$g(R^\nabla(X,Y)\sigma_1, \sigma_2) = -g(R^\nabla(X,Y)\sigma_2, \sigma_1)$$

for all $X, Y \in \mathfrak{X}(M)$; $\sigma_1, \sigma_2 \in \Gamma(\pi)$.

Proof. Let X and Y be fixed vector fields on the base space of π, and let σ be any section in $\Gamma(\pi)$. By the tensoriality of R^∇ we may assume that $[X,Y] = 0$. Then, using the metric property of ∇, we have

$$\begin{aligned} &g(R^\nabla(X,Y)\sigma,\sigma) = g(\nabla_X\nabla_Y\sigma,\sigma) - g(\nabla_Y\nabla_X\sigma,\sigma) = X[g(\nabla_Y\sigma,\sigma)] \\ &- g(\nabla_Y\sigma,\nabla_X\sigma) - Y\big[g(\nabla_X\sigma,\sigma)\big] + g(\nabla_X\sigma,\nabla_Y\sigma) \\ &= \frac{1}{2}\big[X(Yg(\sigma,\sigma)) - Y(Xg(\sigma,\sigma))\big] = \frac{1}{2}[X,Y]g(\sigma,\sigma) = 0, \end{aligned}$$

whence the statement.

1.47. D-manifolds. If D is covariant derivative operator in the tangent bundle τ_M, then, by a quite common abuse of language, D is said to be a *covariant derivative operator on the manifold M*. Thus D is a map from $\mathfrak{X}(M)\times\mathfrak{X}(M)$ into $\mathfrak{X}(M)$ satisfying the axioms **COVD 1**–**COVD 4**; or, equivalently, D is an $\mathbb{R}$-linear map

$$X \in \mathfrak{X}(M) \mapsto DX \in \mathcal{A}^1(M, \tau_M) \cong \mathfrak{T}^1_1(M)$$

such that

$$D(fX) = df \otimes X + fDX \quad \text{for all } f \in C^\infty(M),\ X \in \mathfrak{X}(M).$$

Following the terminology of Lang's book [45], a pair (M, D) consisting of a manifold and a covariant derivative operator on the manifold will be called a *D-manifold.*

Definition. Let (M, D) be a D-manifold. The exterior covariant derivative of the unit tensor field $\iota_M \in \mathfrak{T}^1_1(M)$ is said to be the *torsion tensor field* (briefly the *torsion*) of D, and it is denoted by T^D:

$$\boxed{T^D := d^D\iota_M}.$$

D is called *torsion-free*, if $T^D = 0$.

It follows immediately that for any vector fields X, Y on M we have

$$T^D(X,Y) = D_XY - D_YX - [X,Y].$$

Corollary. (The algebraic Bianchi identity for D-manifolds).

If (M, D) is a D-manifold, then its curvature tensor field R^D and torsion tensor field T^D satisfy

$$\boxed{d^D T^D = R^D[\iota_M]}.$$

Indeed, according to 1.44, Lemma 2, $d^D T^D = d^D d^D \iota_M = R^D[\iota_M]$.

Remark 1. Evaluating both sides of the relation $d^D T^D = R^D[\iota_M]$ on a triple (X, Y, Z) of vector fields on M we obtain

$$\underset{(X,Y,Z)}{\mathfrak{S}}((D_X T^D)(Y, Z) + T^D(T^D(X, Y), Z)) = \underset{(X,Y,Z)}{\mathfrak{S}} R^D(X, Y)Z.$$

In particular, if D is torsion-free, we have

$$R^D(X, Y)Z + R^D(Y, Z)X + R^D(Z, X)Y = 0 \quad \text{for all } X, Y, Z \in \mathfrak{X}(M).$$

($\underset{(X,Y,Z)}{\mathfrak{S}}$ means cyclic sum for the ‘variables’ X, Y, Z; i.e., if we are given an expression of the form $A(X, Y, Z)$, then

$$\underset{(X,Y,Z)}{\mathfrak{S}} A(X, Y, Z) := A(X, Y, Z) + A(Y, Z, X) + A(Z, X, Y).)$$

Remark 2. If D is a torsion-free covariant derivative on M, then the Lie bracket of two vector fields may be expressed in terms of covariant derivatives. This enables one to express the exterior derivative of a differential form in terms of D. Namely, 1.39(ii) leads to the formula

$$d\alpha(X_1, \dots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} (D_{X_i}\alpha)(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}),$$

where $\alpha \in \mathcal{A}^k(M)$ $(k \in \mathbb{N}^*)$, $X_i \in \mathfrak{X}(M)$, $1 \leqq i \leqq k + 1$.

1.48. Locally affine structures.

Definition. An atlas $\mathcal{A} = (\mathfrak{U}_\alpha, (u^i_\alpha)^n_{i=1})_{\alpha \in A}$ of a manifold M is said to be a *locally affine structure* on M if all its transition maps

$$\gamma_{\alpha\beta} : p \in \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \mapsto \gamma_{\alpha\beta}(p) := \left(\frac{\partial u^i_\beta}{\partial u^j_\alpha}(p) \right) \in \mathrm{GL}(n) \quad ((\alpha, \beta) \in A \times A)$$

are constant.

Lemma and definition. *Assume that* $\mathcal{A} = (\mathcal{U}_\alpha, (u^i_\alpha)^n_{i=1})_{\alpha \in A}$ *is a locally affine structure on the manifold* M. *For all* $\alpha \in A$, *let*

$$D^\alpha_X Y := \sum_{i=1}^n X(Y^i) \frac{\partial}{\partial u^i_\alpha} \quad \textit{if } X, Y \in \mathfrak{X}(\mathcal{U}_\alpha),\ Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial u^i_\alpha}.$$

Then the family $(D^\alpha)_{\alpha \in A}$ *determines a well-defined covariant derivative operator* D *on* M, *called the* covariant derivative arising from the locally affine structure $\mathcal{A}$. D *is torsion-free and has zero curvature.*

The proof is a routine verification.

Proposition. *Any covariant derivative operator* D *on a manifold* M *which has zero curvature and torsion arises from a locally affine structure on* M.

Proof. Since D has zero curvature, 1.45 Proposition shows that any given $p \in M$ has a neighbourhood which admits a local frame $(X_i)^n_{i=1}$ with the property $D_{X_i} X_j = 0$ $(1 \leqq i, j \leqq n)$. Since D is torsion-free we have $[X_i, X_j] = 0$ $(1 \leqq i, j \leqq n)$. This guarantees (see e.g. [13], Proposition 11.5.2) that there is a chart around p such that $X_i = \frac{\partial}{\partial u^i}$ $(1 \leqq i \leqq n)$. These charts give rise to a locally affine structure on M with the desired property. □

1.49. The Levi-Civita derivative.

> *We conclude this preparatory chapter with a very well-known but crucial result. The method of its proof, the 'Christoffel trick' is almost equally important and will appear several times later on.*

The fundamental lemma of (pseudo-) Riemannian geometry. *Let* (M, g) *be a pseudo-Riemannian manifold. There exists a unique covariant derivative operator* D *on* M, *called the* Levi-Civita derivative *(of* g*) such that*

LC 1. D *is metric, i.e.* $Dg = 0$.

LC 2. D *is torsion-free, i.e.* $T^D = 0$.

Proof. (1) Let X, Y, Z be arbitrary vector fields on M. For the *uniqueness*, we express $g(D_X Y, Z)$ in terms which do not involve the covariant derivative operator D. To do this, we write down the metric property **LC 1** three times, permuting cyclically the vector fields X, Y, Z:

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z),$$

$$Yg(Z, X) = g(D_Y Z, X) + g(Z, D_Y X),$$

$$Zg(X, Y) = g(D_Z X, Y) + g(X, D_Z Y).$$

Adding the first two relations, subtracting the third, and using the property **LC 2** we obtain:

$$\boxed{\begin{array}{rcl} 2g(D_XY,Z) & = & Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z]) \\ & + & g(Y,[Z,X])+g(Z,[X,Y]) \end{array}}\,.$$

This is the so-called *Koszul formula*, which implies the uniqueness.

(2) As to the *existence*, for fixed vector fields X, Y on M let α be the map

$$Z \in \mathfrak{X}(M) \mapsto \alpha(Z) := \textit{the right-hand side of the Koszul formula.}$$

It may be easily checked that α is a $C^\infty(M)$-linear map from $\mathfrak{X}(M)$ into $C^\infty(M)$, i.e. $\alpha \in \mathcal{A}^1(M)$. The musical isomorphism between $\mathcal{A}^1(M)$ and $\mathfrak{X}(M)$ described in 1.30(**5**) guarantees that there is a unique vector field, namely $D_XY := \alpha^\#$, such that

$$2g(D_XY,Z) = \alpha(Z) \quad \textit{for all} \quad Z \in \mathfrak{X}(M).$$

Now a straightforward, but tedious calculation shows that this way of defining D actually leads to a covariant derivative operator on M which satisfies **LC 1** and **LC 2**. $\square$

Chapter 2

Calculus of Vector-valued Forms and Forms along the Tangent Bundle Projection

A. Vertical bundle to a vector bundle

In this part (E, π, M) is a vector bundle of rank k over the n-dimensional base manifold M; $k, n \in \mathbb{N}^*$.

2.1. The vertical subbundle of τ_E.

Consider the tangent map $\pi_* : TE \longrightarrow TM$. The pair (π_*, π) is a bundle map between the tangent bundles τ_E and τ_M (see 1.25). Let $z \in E$. The vector space

$$V_z E := \operatorname{Ker}(\pi_*)_z \subset T_z E$$

is called the *vertical subspace* of $T_z E$, and the vectors of $V_z E$ are mentioned as *vertical vectors* at z. As the linear maps $(\pi_*)_z$ are all surjective, we see that

$$\dim V_z E = \dim T_z E - \dim T_{\pi(z)} M = n + k - n = k = \operatorname{rank} \pi.$$

For each point $p \in M$ the fibre E_p is a submanifold of E. Denote the canonical inclusion $E_p \longrightarrow E$ by j_p. Then $\pi \circ j_p$ is the constant map $E_p \longrightarrow \{p\}$, therefore

$$(\pi_*)_z \circ \left[(j_p)_*\right]_z = 0 \quad \text{for all } z \in E_p.$$

From this it follows that

$$\operatorname{Im} \left[(j_p)_*\right]_z \subset V_z E \quad \text{for all } z \in E_p.$$

On the other hand, the map

$$\left[(j_p)_*\right]_z : T_z E_p \longrightarrow T_z E$$

is obviously injective, hence

$$\dim \operatorname{Im} \left[(j_p)_*\right]_z = \dim T_z E_p = k = \dim V_z E.$$

Thus we conclude the following result:

Lemma. $V_z E = \operatorname{Im} \left[(j_p)_*\right]_z$ *for all* $p \in M$ *and* $z \in E_p$*; therefore we have a canonical isomorphism* $V_z E \cong T_z E_p$. □

Now consider the subset

$$VE := \bigsqcup_{z \in E} V_z E$$

of TE, and let $V\pi$ be the natural projection

$$VE \longrightarrow E, \quad w \in V_z E \mapsto V\pi(w) := z.$$

Then VE carries a unique smooth structure such that $(VE, V\pi, E)$ becomes a vector subbundle of τ_E. This vector bundle (abbreviated, according to our practice, by $V\pi$) is said to be the *vertical bundle* to π, or, as suggested by the Lemma, the *bundle along the fibres.*

2.2. Canonical maps. The map

$$\mathbf{i} : (z, z') \in E \times_M E \mapsto \mathbf{i}(z, z') := \left[(j_p)_*\right]_z (\iota_z(z')) \quad (p := \pi(z) = \pi(z')),$$

where ι_z is the canonical identification of $E_p = E_{\pi(z)}$ with $T_z E_p$ (see 1.9), defines a strong bundle isomorphism

$$\pi^* \pi \xrightarrow{\mathbf{i}} V\pi,$$

displayed more vividly by the diagram

$$\begin{array}{ccc} E \times_M E & \xrightarrow{\mathbf{i}} & VE \\ \pi_1 \big\downarrow & & \big\downarrow V\pi \\ E & \underset{1_E}{=\!=\!=} & E \end{array}$$

(see also 1.14). The canonical isomorphism $\mathbf{i}$ is said to be the *big vertical morphism.* It may naturally be prolonged to a strong bundle map $E \times_M E \longrightarrow TE$, this prolongation will also be denoted by $\mathbf{i}$. In particular, we have the useful map

$$v\ell_E : E \longrightarrow TE, \quad z \mapsto v\ell_E(z) := \mathbf{i}(0_{\pi(z)}, z)$$

($0_{\pi(z)}$ is the zero vector of $E_{\pi(z)}$). $v\ell_E$ is called the *small vertical* lift (a term of P. Michor, see [56]).

A further important canonical map is the *vertical projection*

$$vpr_E := \pi_2 \circ \mathbf{i}^{-1} : VE \longrightarrow E.$$

The pair (vpr_E, π) yields a bundle map $V\pi \longrightarrow \pi$, called the *canonical surjection* of $V\pi$ onto π. These maps may be arranged in the diagram

$$\begin{tikzcd}
 & E \times_M E \arrow[dr, "\pi_2"] & \\
VE \arrow[ur, "\mathbf{i}^{-1}"] \arrow[rr, dashed, "vpr_E"'] \arrow[d, "V\pi"'] & & E \arrow[d, "\pi"] \\
E \arrow[rr, "\pi"'] & & M
\end{tikzcd}$$

Roughly speaking, vpr_E picks off the 'second component' z' of $\mathbf{i}(z, z')$, whereas $V\pi : VE \longrightarrow E$ yields the 'first component' z. (As for the latter remark, the precise relation is $V\pi = \tau_E \upharpoonright VE = \pi_1 \circ \mathbf{i}^{-1}$).

2.3. The canonical short exact sequence arising from π. Let us first consider the pull-back bundle $\pi^*\tau_M$ of the tangent bundle τ_M by π. This bundle is also said to be the *transverse bundle* to π. In view of 1.14, the total space of the transverse bundle is $E \times_M TM$, the fibre over a point $z \in E$ is the vector space $\{z\} \times T_{\pi(z)}M \cong T_{\pi(z)}M$, and we have the commutative diagram

$$\begin{tikzcd}
E \times_M TM \arrow[r, "p_2"] \arrow[d, "p_1"'] & TM \arrow[d, "\tau_M"] \\
E \arrow[r, "\pi"'] & M
\end{tikzcd}$$

where $p_1 := \pi^*\tau_M := pr_1 \upharpoonright E \times_M TM$, $p_2 := pr_2 \upharpoonright E \times_M TM$.

Lemma and definition. The map

$$\mathbf{j} : TE \longrightarrow E \times_M TM,\ w \in T_z E \mapsto \mathbf{j}(w) := (z, (\pi_*)_z(w))$$

defines a surjective strong bundle map between τ_E and the transverse bundle $\pi^*\tau_M$. If, by a slight abuse of notation, $\mathbf{i} : E \times_M E \longrightarrow TE$ means the big vertical morphism composed with the canonical inclusion $VE \hookrightarrow TE$, then the sequence

$$0 \longrightarrow E \times_M E \xrightarrow{\mathbf{i}} TE \xrightarrow{\mathbf{j}} E \times_M TM \longrightarrow 0$$

or, more accurately

$$0 \longrightarrow \pi^*\pi \xrightarrow{\mathbf{i}} \tau_E \xrightarrow{\mathbf{j}} \pi^*\tau_M \longrightarrow 0$$

is a short exact sequence of vector bundles, called the *canonical short exact sequence* constructed from π.

Proof. $\mathbf{j}$ is clearly a strong bundle map. The surjectivity of $\mathbf{j}$ may easily be seen using local coordinates. Thus the sequence is exact at $\pi^*\tau_M$. As $\mathbf{i}$ is injective, the sequence is exact at $\pi^*\pi$. $\operatorname{Im}\mathbf{i} = V\pi$ by 2.2. For any point $z \in E$ and vector $w \in T_z E$,

$$\mathbf{j}(w) = (z, (\pi_*)_z(w)) = 0 \in \{z\} \times T_{\pi(z)}M \Leftrightarrow w \in V_z E,$$

hence $\operatorname{Im}\mathbf{i} = V\pi = \operatorname{Ker}\mathbf{j}$, and the sequence is exact at τ_E. □

Remark. The strong bundle map $\mathbf{j} : \tau_E \longrightarrow \pi^*\tau_M$ may be interpreted as a $\pi^*\tau_M$-valued one-form on E, i.e. we can write

$$\mathbf{j} \in \mathcal{A}^1(E, \pi^*\tau_M) := \Gamma(\mathbf{A}_1(\tau_E, \pi^*\tau_M)).$$

Indeed, $\mathbf{j}$ may be considered as the map

$$z \in E \mapsto \mathbf{j}_z := (\pi_*)_z \in \operatorname{Hom}(T_z E, T_{\pi(z)}M) \cong \operatorname{Hom}(T_z E, (\pi^*\tau_M)_z).$$

2.4. Vertical vector fields. A section of the vertical bundle $V\pi$ is called a *vertical vector field* on E. For the $C^\infty(E)$-module of vertical vector fields we shall use the convenient notation $\mathfrak{X}^{\mathrm{v}}(E)$ instead of $\Gamma(V\pi)$. Now we summarize some basic facts concerning vertical vector fields which we shall need in subsequent sections.

(**1**) *A vector field* $\xi \in \mathfrak{X}(E)$ *is vertical if, and only if,* $\xi \underset{\pi}{\sim} 0$. *The Lie bracket of two vertical vector fields is vertical, hence* $\mathfrak{X}^{\mathrm{v}}(E)$ *is a subalgebra of the Lie algebra* $\mathfrak{X}(E)$.

These assertions are immediate consequences of 1.27(1), (3).

(**2**) $\xi \in \mathfrak{X}(E)$ *is vertical if, and only if, for each* $f \in C^\infty(M)$, $\xi(f \circ \pi) = 0$.

This is obtained directly from the definitions (see also 1.8).

(**3**) Consider the big vertical morphism $\mathbf{i} : E \times_M E \longrightarrow VE$. The map

$$C_E : E \longrightarrow VE, \quad z \mapsto C_E(z) := \mathbf{i}(z, z)$$

is a vertical vector field, called the *canonical vector field*, or the *radial vector field*, or the *Liouville vector field* on E. Obviously

$$vpr_E(C_E(z)) = z \quad \text{for all } z \in E,$$

and C_E is the unique vertical vector field on E which satisfies this relation. Using an adapted chart $\left(\pi^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^j)_{j=1}^k)\right)$ on E (see 1.12(2)) it is easily verified that

$$\boxed{C_E \restriction \pi^{-1}(\mathcal{U}) = \sum_{j=1}^k y^j \frac{\partial}{\partial y^j}}.$$

Now we show that C_E *is the velocity field of the flow*

$$\varphi : \mathbb{R} \times E \longrightarrow E, \quad (t, z) \mapsto e^t z.$$

Let z_0 be a point of E, and let us consider the flow line

$$c_{z_0} : t \in \mathbb{R} \mapsto e^t z_0 \in E$$

of z_0. Then

$$\begin{aligned}\dot{c}_{z_0}(0) &= \sum_{i=1}^n (x^i \circ c_{z_0})'(0) \left(\frac{\partial}{\partial x^i}\right)_{z_0} + \sum_{i=1}^k (y^i \circ c_{z_0})'(0) \left(\frac{\partial}{\partial y^i}\right)_{z_0} \\ &= \sum_{i=1}^k y^i(z_0) \left(\frac{\partial}{\partial y^i}\right)_{z_0},\end{aligned}$$

since the functions $x^i \circ c_{z_0} = u^i \circ \pi \circ c_{z_0}$ are obviously constant, while $y^i \circ c_{z_0}(t) = e^t y^i(z_0)$ $(t \in \mathbb{R})$. Thus

$$\dot{\varphi} \restriction \pi^{-1}(\mathcal{U}) = C_E \restriction \pi^{-1}(\mathcal{U}),$$

as we claimed.

(**4**) More generally, let $z \in E$ be a fixed point. Then the map

$$\mathbf{i}_z : u \in E_{\pi(z)} \mapsto \mathbf{i}_z(u) := \mathbf{i}(z,u) \in V_z E$$

is a linear isomorphism, called the *vertical lift* from $E_{\pi(z)}$ into $V_z E$. The image $\mathbf{i}_z(u)$ of an individual vector $u \in E_{\pi(z)}$ is also mentioned as the vertical lift of u into $V_z E$, and we use for $\mathbf{i}_z(u)$ the more vivid notation $u^{\uparrow}(z)$.

(**5**) The *vertical lift of a section* $\sigma \in \Gamma(\pi)$ is the vertical vector field $\sigma^{\mathrm{v}} \in \mathfrak{X}^{\mathrm{v}}(E)$ defined by

$$\sigma^{\mathrm{v}}(z) := \big[\sigma(\pi(z))\big]^{\uparrow}(z) \quad \text{for all } z \in E.$$

Then

$$vpr_E \circ \sigma^{\mathrm{v}} = \sigma \circ \pi,$$

therefore σ^{v} is *fibrewise constant.* Its expression in an adapted local coordinate system $((x^i)_{i=1}^n, (y^j)_{j=1}^k)$ on $\pi^{-1}(\mathfrak{U})$ is

$$\sigma^{\mathrm{v}} \restriction \pi^{-1}(\mathfrak{U}) = \sum_{j=1}^{k} (\sigma^j \circ \pi) \frac{\partial}{\partial y^j}, \quad \sigma^j := y^j \circ \sigma \quad (1 \leqq j \leqq k).$$

Granting this, the following formulae are easy to see:

(i) $(\sigma_1 + \sigma_2)^{\mathrm{v}} = (\sigma_1)^{\mathrm{v}} + (\sigma_2)^{\mathrm{v}} \quad (\sigma_1, \sigma_2 \in \Gamma(\pi))$,

(ii) $(f\sigma)^{\mathrm{v}} = (f \circ \pi)\sigma^{\mathrm{v}} \quad (f \in C^\infty(M),\ \sigma \in \Gamma(\pi))$,

(iii) $\big[\sigma_1^{\mathrm{v}}, \sigma_2^{\mathrm{v}}\big] = 0 \quad (\sigma_1, \sigma_2 \in \Gamma(\pi))$.

(**6**) *A vertical vector field* $\xi \in \mathfrak{X}^{\mathrm{v}}(E)$ *is the vertical lift of a section if, and only if, for any section* σ *in* $\Gamma(\pi)$, $[\sigma^{\mathrm{v}}, \xi] = 0$.

Necessity is evident from the last relation (iii). Using an adapted local coordinate system on E, sufficiency may also be easily verified.

Remark. As to the vertical lift $X^{\mathrm{v}} \in \mathfrak{X}^{\mathrm{v}}(TM)$ of a vector field X on M, a more conceptual reasoning will be presented in 2.31(2).

2.5. The deleted bundle for π**.** Consider the zero section

$$o : p \in M \mapsto o(p) := 0_p \in E_p$$

of π. Then $o(M)$ is a closed submanifold of E (see e.g. [35], Vol I., 3.10 Ex. 5); by abuse of language, this submanifold will also be mentioned as the zero section of π. Let

$$\overset{\circ}{E} := \bigsqcup_{p\in M} E_p \setminus \{0_p\}, \quad \overset{\circ}{\pi} := \pi \restriction \overset{\circ}{E}.$$

Then $\overset{\circ}{E}$ is the complement of $o(M)$ in E, hence it is an open submanifold. The triple $(\overset{\circ}{E}, \overset{\circ}{\pi}, M)$ is obviously a fibred manifold, moreover a *fibre bundle;* i.e., it has the following property:

Local triviality. *For each point $p \in M$ there exists an open neighbourhood $\mathfrak{U}$ of p in M and a diffeomorphism $\varphi : \overset{\circ}{\pi}^{-1}(\mathfrak{U}) \longrightarrow \mathfrak{U} \times (\mathbb{R}^k \setminus \{0\})$ such that $pr_1 \circ \varphi = \overset{\circ}{\pi} \restriction \overset{\circ}{\pi}^{-1}(\mathfrak{U})$.* (C.f. 1.12, **VB 2**.)

The fibre bundle $(\overset{\circ}{E}, \overset{\circ}{\pi}, M)$ is called the *deleted bundle for* π and is usually denoted by $\overset{\circ}{\pi}$.

2.6. Homogeneity. Consider the deleted bundle $(\overset{\circ}{E}, \overset{\circ}{\pi}, M)$ and let r be a real number.

Definition 1. A real-valued function $f : \overset{\circ}{E} \longrightarrow \mathbb{R}$ is said to be

(i) *positive-homogeneous of degree* r, if for each positive $t \in \mathbb{R}$ and each $z \in \overset{\circ}{E}$
$$f(tz) = t^r f(z);$$

(ii) *homogeneous of degree* r, where r is an integer, if the above condition holds for all real $t \neq 0$.

Lemma 1. *Let $f : \overset{\circ}{E} \longrightarrow \mathbb{R}$ be a smooth function. In order that f is positive-homogeneous of degree r, it is necessary and sufficient that $C_E f = rf$.*

Proof. The statement is merely a transcription of Euler's classical theorem on homogeneous functions into a vector bundle context. □

We note in particular that if f is homogeneous of degree r, then the 'Euler relation' $C_E f = rf$ holds. On the other hand, this implies only positive-homogeneity, and not homogeneity.

The following simple observations will play an important role in subsequent sections.

Lemma 2. *Let f be a real-valued function on E which is smooth on $\overset{\circ}{E}$.*

(1) *If f is positive-homogeneous of degree 0 on $\overset{\circ}{E}$ and continuous on E, then f is constant along the fibres; hence there is a smooth function f_0 on M such that $f = f_0 \circ \pi$.*

(2) *If f is positive-homogeneous of degree 1 on $\overset{\circ}{E}$ and continuously differentiable (i.e., of class C^1) on E, then f is linear on the fibres* (P. Dombrowski's 'clever observation' [29]).

(3) *If f is positive-homogeneous of degree r on $\overset{\circ}{E}$ and r-times continuously differentiable on E, then f is a polynomial of degree r on the fibres.*

Example 1. If $((x^i)_{i=1}^n, (y^j)_{j=1}^k)$ is an adapted local coordinate system on E, then the x^i's are homogeneous functions of degree 0 and the y^j's of degree 1 on their domain.

Definition 2. (a) A map $\xi : E \longrightarrow TE$, $z \mapsto \xi(z) \in T_zE$ is said to be *homogeneous of degree r*, where r is an integer, if

(i) ξ is smooth on $\overset{\circ}{E}$ (and hence $\xi \restriction \overset{\circ}{E} \in \mathfrak{X}(\overset{\circ}{E})$),

(ii) $[C_E, \xi] = (r-1)\xi$.

(b) The maps $\alpha : z \in E \mapsto \alpha(z) \in \mathbf{A}(T_zE)$ and

$$A : z \in E \mapsto A(z) \in L_{\text{skew}}((T_zE)^k, T_zE) \quad (k \in \mathbb{N}^*)$$

are said to be *homogeneous of degree r* ($\in \mathbb{Z}$) if

(i) α and A are smooth on $\overset{\circ}{E}$ (i.e. $\alpha \restriction \overset{\circ}{E} \in \mathcal{A}(\overset{\circ}{E}), A \restriction \overset{\circ}{E} \in \mathcal{B}(\overset{\circ}{E})$),

(ii) $d_{C_E}\alpha = r\alpha$, $d_{C_E}A = (r-1)A$.

Example 2. Continuing Example 1, the coordinate vector fields $\frac{\partial}{\partial x^i}$ $(1 \leqq i \leqq n)$ are homogeneous of degree 1, the vector fields $\frac{\partial}{\partial y^j}$ $(1 \leqq j \leqq k)$ are homogeneous of degree 0. Dually, the one-forms dx^i are homogeneous of degree 0; the one-forms dy^j are homogeneous of degree 1. The Liouville vector field C_E is homogeneous of degree 1.

In general, a vector field $\xi \in \mathfrak{X}(\overset{\circ}{E})$ and a one-form $\alpha \in \mathcal{A}^1(\overset{\circ}{E})$ are homogeneous of degree r ($r \in \mathbb{Z}$) if, and only if, their components, relative to any adapted local coordinate system $((x^i)_{i=1}^n, (y^j)_{j=1}^k)$ for E, have the following homogeneity properties:

$$\begin{cases} \xi x^i \text{ is positive-homogeneous of degree } r-1, \\ \xi y^j \text{ is positive-homogeneous of degree } r; \end{cases}$$

$$\begin{cases} \alpha\left(\frac{\partial}{\partial x^i}\right) \text{ is positive-homogeneous of degree } r, \\ \alpha\left(\frac{\partial}{\partial y^j}\right) \text{ is positive-homogeneous of degree } r-1 \ (1 \leqq i \leqq n,\ 1 \leqq j \leqq k). \end{cases}$$

Proposition. *A continuous vector field $\xi : E \longrightarrow TE$, smooth on $\overset{\circ}{E}$, is*

(1) *a vertical lift if, and only if, ξ is vertical and homogeneous of degree* 0*;*

(2) *projectable on M if, and only if, $[C_E, \xi]$ is vertical.*

Expressing ξ in an adapted local coordinate system, both assertions can be verified by a straightforward calculation.

2.7. Basic and semibasic forms. (**1**) A differential form $\alpha \in \mathcal{A}(E)$ is called *basic*, if it is the pull-back of a form on M by π; *semibasic* if $i_\xi \alpha = 0$ for every vertical vector field $\xi \in \mathfrak{X}^{\mathrm{v}}(E)$.

(**2**) A vector-valued form $K \in \mathcal{B}(E) = \mathcal{A}(E, \tau_E)$ is called *semibasic* if it is vertical-valued and $i_\xi K = 0$ for all $\xi \in \mathfrak{X}^{\mathrm{v}}(E)$.

Semibasic (resp. vector-valued semibasic) forms constitute a subalgebra of $\mathcal{A}(E)$ (resp. a submodule of $\mathcal{B}(E)$), denoted by $\mathcal{A}_0(E)$ (resp. $\mathcal{B}_0(E)$).

Lemma. *A differential form on E is basic if, and only if, it is semibasic and homogeneous of degree* 0*.* ◪

Remark. We shall also need the analogous concept of a semibasic and a vector-valued semibasic covariant tensor field on E. The definition is self-evident: $A \in \mathcal{T}_s^0(E)$ and $B \in \mathcal{T}_s^1(E)$ are called semibasic if A and B kill every sequence $(\xi_1, \dots, \xi_s) \in [\mathfrak{X}(E)]^s$ which contains (at least one) vertical vector field and B is vertical valued. We agree that elements of $C^\infty(E)$ are semibasic tensors as well.

2.8. Vector fields along π. A section of the transverse bundle $\pi^*\tau_M$ is called a *vector field along π*. The terminology is justified by 1.22, which assures that

$$\Gamma(\pi^*\tau_M) \cong \Gamma_\pi(\tau_M).$$

So any section $\widetilde{\sigma}$ in $\Gamma(\pi^*\tau_M)$ can be considered as a smooth map $\widetilde{\sigma} : E \longrightarrow TM$ satisfing $\tau_M \circ \widetilde{\sigma} = \pi$. For the $C^\infty(E)$-module of vector fields along π we shall use the convenient notation $\mathfrak{X}(\pi)$. Now, in view of 1.21, the canonical short exact sequence arising from π leads to the exact sequence of $C^\infty(E)$-modules

$$0 \longrightarrow \Gamma(\pi^*\pi) \xrightarrow{\mathbf{i}_\#} \mathfrak{X}(E) \xrightarrow{\mathbf{j}_\#} \mathfrak{X}(\pi) \longrightarrow 0.$$

To simplify the symbolism, we shall usually omit the push-forward sign # and write, by a slight abuse of notation,

$$0 \longrightarrow \Gamma(\pi^*\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}(E) \xrightarrow{\mathbf{j}} \mathfrak{X}(\pi) \longrightarrow 0.$$

Remark. Let $\mathrm{Der}(\pi)$ denote the real vector space of $\mathbb{R}$-linear maps $\theta : C^\infty(M) \longrightarrow C^\infty(E)$ satisfying

$$\theta(fg) = \theta f(g \circ \pi) + (f \circ \pi)\theta g \quad \text{for all } f, g \in C^\infty(M).$$

Then *Der*(π) *is canonically isomorphic to* $\mathfrak{X}(\pi)$ (cf. 1.26); the canonical isomorphism is the map

$$\theta \in \mathrm{Der}(\pi) \mapsto \widetilde{X} \in \mathfrak{X}(\pi),$$
$$\widetilde{X} : z \in E \mapsto \widetilde{X}_z \in (E \times_M TM)_z = \{z\} \times T_{\pi(z)}M \cong T_{\pi(z)}M$$

given by

$$\widetilde{X}_z f := (\theta f)(z) \quad \text{for all } f \in C^\infty(M).$$

Indication of proof. A routine calculation shows that $\widetilde{X}_z$, defined by the above rule, is indeed a tangent vector to M at $\pi(z)$. Using coordinate expressions, it is easily verified that the map

$$\widetilde{X} : z \in E \mapsto \widetilde{X}_z \in (E \times_M TM)_z \cong T_{\pi(z)}M$$

is smooth, therefore $\widetilde{X} \in \Gamma(\pi^*\tau_M) =: \mathfrak{X}(\pi)$.

Conversely, let $\widetilde{X} \in \mathfrak{X}(\pi)$. If

$$(\widetilde{X}f)(z) := \widetilde{X}_z f \quad \text{for all } z \in E$$

(regarding $\widetilde{X}_z$ as an element of $T_{\pi(z)}M$), then for any functions f, g in $C^\infty(M)$ and point z in E we have

$$\big[\widetilde{X}(fg)\big](z) := \widetilde{X}_z(fg) \overset{1.7}{=} \big(\widetilde{X}_z f\big)g(\pi(z)) + f(\pi(z))\widetilde{X}_z g =$$
$$= \big[\widetilde{X}f(g \circ \pi) + (f \circ \pi)\widetilde{X}g\big](z).$$

This means that $\widetilde{X}$ may be interpeted as an element of $\mathrm{Der}(\pi)$.

B. Nonlinear connections in a vector bundle

We keep the hypotheses and notations of the preceding section.

2.9. Horizontal subbundles, horizontal vector fields. A subbundle $H\pi$ of τ_E is said to be *horizontal* if $\tau_E = H\pi \oplus V\pi$. We denote the total space of $H\pi$ by HE, the fibre over a point z by H_zE. The subspace H_zE of T_zE is called the *horizontal subspace* at z (with respect to the choice of $H\pi$). Since, for each point $z \in E$, the map

$$\pi_* \restriction H_zE : H_zE \longrightarrow T_{\pi(z)}M$$

is obviously a linear isomorphism, it follows that

$$\operatorname{rank}(H\pi) = \dim M = n,$$

while the manifold HE is $(2n+k)$-dimensional.

The $C^\infty(E)$-module of sections of $H\pi$ is denoted by $\mathfrak{X}^h(E)$, the elements of $\mathfrak{X}^h(E)$ are called *horizontal vector fields* on E. The horizontal vector fields constitute a finitely generated projective module over $C^\infty(E)$. However, in general, $\mathfrak{X}^h(E)$ does not form a Lie subalgebra of the Lie algebra $\mathfrak{X}(E)$.

The Whitney decomposition $\tau_E = H\pi \oplus V\pi$ leads to the direct sum decomposition

$$\mathfrak{X}(E) = \mathfrak{X}^h(E) \oplus \mathfrak{X}^{\mathrm{v}}(E)$$

of $C^\infty(E)$ -modules.

2.10. Horizontal maps. A (right) splitting of the short exact sequence

$$0 \longrightarrow \pi^*\pi \xrightarrow{\mathbf{i}} \tau_E \xrightarrow{\mathbf{j}} \pi^*\tau_M \longrightarrow 0$$

is said to be a *horizontal map for* π. In other words, a strong bundle map

$$\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E \quad (\text{or } \mathcal{H} : E \times_M TM \to TE)$$

is a horizontal map for π if

$$\mathbf{j} \circ \mathcal{H} = 1_{\pi^*\tau_M}.$$

The *retraction* associated with $\mathbf{i}$, complementary to $\mathcal{H}$ (cf. A.5, Proposition 2) is called the *vertical map* belonging to $\mathcal{H}$ and denoted by $\mathcal{V}$. Then the sequence

$$0 \longleftarrow \pi^*\pi \underset{\mathcal{V}}{\longleftarrow} \tau_E \underset{\mathcal{H}}{\longleftarrow} \pi^*\tau_M \longleftarrow 0$$

is also a short exact sequence of strong bundle maps. So we have

$$\mathcal{V} \circ \mathbf{i} = 1_{\pi^*\pi}, \quad \mathcal{V} \circ \mathcal{H} = 0.$$

Due to the second countability of the base space, *horizontal maps for a vector bundle π do exist.* However, in general there is no canonical way of specifying a horizontal map.

The relation between the horizontal subbundles and the horizontal maps is quite obvious. *If $\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E$ is a horizontal map for π, then* $\operatorname{Im}\mathcal{H}$ *is a horizontal subbundle of τ_E. Conversely, any horizontal subbundle of τ_E may be obtained as the image of a horizontal map.*

2.11. Basic geometric data. We associate the following objects to any horizontal map $\mathcal{H}$.

(**1**) $\mathbf{h} := \mathcal{H} \circ j$ – the *horizontal projector*. It has the properties

$$\mathbf{h}^2 = \mathbf{h}, \quad \operatorname{Im}\mathbf{h} = \operatorname{Im}\mathcal{H}, \quad \operatorname{Ker}\mathbf{h} = V\pi,$$

therefore $\mathbf{h}$ is indeed a projector which projects τ_E onto $\mathcal{H}\pi$ along $V\pi$.

(**2**) $\mathbf{v} := 1_{TE} - \mathbf{h}$ – the *vertical projector* belonging to $\mathcal{H}$. Obviously,

$$\mathbf{v} = i \circ \mathcal{V}, \quad \mathbf{v}^2 = \mathbf{v}, \quad \operatorname{Im}\mathbf{v} = V\pi, \quad \operatorname{Ker}\mathbf{v} = \operatorname{Im}\mathcal{H}.$$

(**3**) $K = vpr_E \circ \mathbf{v}$ – the *connector* or *Dombrowski map* belonging to $\mathcal{H}$. Then, evidently, (K, π) is a bundle map from τ_E into π, so we have the commutative diagram

$$\begin{array}{ccc} TE & \xrightarrow{K} & E \\ {\scriptstyle \tau_E}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\pi]{} & M. \end{array}$$

Furthermore,

$$K \restriction VE = vpr_E \quad \text{or, equivalently,} \ K \circ \sigma^{\mathrm{v}} = \sigma \circ \pi \quad (\sigma \in \Gamma(\pi)).$$

Indeed, for each point $z \in E$,

$$K[\sigma^{\mathrm{v}}(z)] := vpr_E \circ \mathbf{v}(\sigma^{\mathrm{v}}(z)) \overset{2.2}{=} \pi_2 \circ \mathbf{i}^{-1}(\sigma^{\mathrm{v}}(z)) \overset{2.4(4),(5)}{=}$$

$$= \pi_2 \circ \mathbf{i}^{-1} \circ \mathbf{i}(z, \sigma(\pi(z))) = \sigma(\pi(z)).$$

Note. Specifying a horizontal map $\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E$ for π, the notation $\mathbf{h}_{\mathcal{H}}$, $\mathbf{v}_{\mathcal{H}}$ and $K_{\mathcal{H}}$ would be more accurate for the horizontal projector, the vertical projector and the connector belonging to $\mathcal{H}$ than $\mathbf{h}$, $\mathbf{v}$ and K, respectively. However, the suffix $\mathcal{H}$ may be omitted without any risk of confusion in most cases.

(4) $\ell^h : X \in \mathfrak{X}(M) \mapsto \ell^h(X) =: X^h \in \mathfrak{X}^h(E)$ – the *horizontal lift* with respect to $\mathcal{H}$, defined by

$$X^h(z) := \mathcal{H}(z, X[\pi(z)]) \quad \text{for all } z \in E.$$

Then X^h is called the ($\mathcal{H}$ -) *horizontal lift of* X. Notice that

$$\pi_* \circ X^h = X \circ \pi, \quad \text{i.e. } X^h \underset{\pi}{\sim} X.$$

Now, using 1.27(2), it follows that

$$(fX)^h = (f \circ \pi)X^h \quad \text{for all } f \in C^\infty(M).$$

On the other hand, employing a local argument involving coordinates, it is straightforward to check that

$$X^h(f \circ \pi) = (Xf) \circ \pi \quad \text{for all } f \in C^\infty(M).$$

Lemma 1. *Suppose given a horizontal map $\mathcal{H}$ for π. Then the map*

$$(X, \sigma) \in \mathfrak{X}(M) \times \Gamma(\pi) \mapsto [X^h, \sigma^{\mathrm{v}}] \in \mathfrak{X}(E)$$

has the following properties:

(i) $[X^h, \sigma^{\mathrm{v}}] \in \mathfrak{X}^{\mathrm{v}}(E)$,

(ii) $\mathbb{R}$-*bilinear*,

(iii) $\left[(fX)^h, \sigma^{\mathrm{v}}\right] = (f \circ \pi)[X^h, \sigma^{\mathrm{v}}] \quad (f \in C^\infty(M))$,

(iv) $\left[X^h, (f\sigma)^{\mathrm{v}}\right] = (f \circ \pi)[X^h, \sigma^{\mathrm{v}}] + ((Xf) \circ \pi)\sigma^{\mathrm{v}}$.

Proof. Routine verification.

(5) $\Omega : (\xi, \eta) \in \mathfrak{X}(E) \times \mathfrak{X}(E) \mapsto \Omega(\xi, \eta) := -\mathbf{v}[\mathbf{h}\,\xi, \mathbf{h}\,\eta] \in \mathfrak{X}(E)$

– the *curvature* of the horizontal map $\mathcal{H}$. Ω is evidently $\mathfrak{X}^{\mathrm{v}}(E)$-valued and skew-symmetric. It may be seen in a moment that Ω is $C^\infty(E)$-bilinear as well. Hence $\Omega \in \mathcal{A}^2(E, V\pi)$, i.e., Ω is a $V\pi$-valued 2-form on E. The curvature of $\mathcal{H}$ measures 'how far the Lie bracket of two $\mathcal{H}$-horizontal vector fields deviates from the horizontal'. The horizontal subbundle $\operatorname{Im}\mathcal{H}$, as an n-plane field on E, is integrable if and only if the curvature of $\mathcal{H}$ vanishes; in this case the horizontal subspaces are tangent to a submanifold of E (cf. 1.45).

Lemma 2. *Let Ω be the curvature of the horizontal map $\mathcal{H}$ and consider the map*

$$\varrho : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}^{\mathrm{v}}(E), \quad (X, Y) \mapsto \varrho(X, Y) := \Omega(X^h, Y^h).$$

Then

(i) $\varrho(X, Y) = [X, Y]^h - [X^h, Y^h]$,

(ii) $vpr_E \circ \varrho(X, Y) = -K \circ [X^h, Y^h]$, *where K is the connector belonging to $\mathcal{H}$.*

Proof. As $X^h \underset{\pi}{\sim} X$, $Y^h \underset{\pi}{\sim} Y$, it follows that

$$[X^h, Y^h] = \mathbf{h}[X^h, Y^h] + \mathbf{v}[X^h, Y^h] \sim [X, Y].$$

This implies that $\mathbf{h}[X^h, Y^h] \underset{\pi}{\sim} [X, Y]$, since $\mathbf{v}[X^h, Y^h] \sim 0$. On the other hand, $[X, Y]^h \underset{\pi}{\sim} [X, Y]$; therefore

$$\mathbf{h}[X^h, Y^h] = [X, Y]^h,$$

and $\varrho(X, Y) := -\mathbf{v}[X^h, Y^h] = \mathbf{h}[X^h, Y^h] - [X^h, Y^h] = [X, Y]^h - [X^h, Y^h]$, whence (i). The relation (ii) is an immediate consequence of the definitions. □

2.12. Definition. (1) A vector-valued 1-form $\mathbf{h} \in \mathcal{A}^1(E, \tau_E) \cong \mathcal{T}_1^1(E)$ is said to be a *horizontal projector for* π, if it is a projector with kernel VE, i.e.,

$$\mathbf{h}^2 = \mathbf{h} \quad \text{and} \quad \operatorname{Ker} \mathbf{h} = VE.$$

(2) A *connector* K for π is a map from TE to E such that

CON 1. (K, π) is a bundle map from τ_E onto π.

CON 2. $K \restriction VE = vpr_E$.

Lemma 1. *If $\mathbf{h} \in \mathcal{A}^1(E, \tau_E)$ is a horizontal projector for π, then there is a unique horizontal map $\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E$ for π such that $\mathcal{H} \circ \mathbf{j} = \mathbf{h}$.*

Proof. Uniqueness. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two horizontal maps satisfying $\mathcal{H}_1 \circ \mathbf{j} = \mathbf{h}$ and $\mathcal{H}_2 \circ \mathbf{j} = \mathbf{h}$. Then $(\mathcal{H}_1 - \mathcal{H}_2) \circ \mathbf{j} = 0$. Since $\mathbf{j}$ has right inverses, we conclude that $\mathcal{H}_1 = \mathcal{H}_2$.

Existence. Let $\mathbf{v} := 1_{TE} - \mathbf{h}$. Choose any horizontal map $\mathcal{H}_0 : \pi^*\tau_M \longrightarrow \tau_E$, and define $\mathcal{H} := \mathbf{h} \circ \mathcal{H}_0$. Since $\operatorname{Im}\mathbf{v} = VE = \operatorname{Im}\mathbf{i}$ and $\mathbf{j} \circ \mathbf{i} = 0$, it follows that

$$\mathbf{j} \circ \mathbf{h} = \mathbf{j} \circ (1_{TE} - \mathbf{v}) = \mathbf{j} - \mathbf{j} \circ \mathbf{v} = \mathbf{j},$$

and hence

$$\mathbf{j} \circ \mathcal{H} = \mathbf{j} \circ \mathbf{h} \circ \mathcal{H}_0 = \mathbf{j} \circ \mathcal{H}_0 = 1_{\pi^*\tau_M}.$$

Thus $\mathcal{H}$ is a horizontal map for π. Let $\mathbf{h}_0$ be the horizontal projector belonging to $\mathcal{H}_0$, and $\mathbf{v}_0 = 1_{TE} - \mathbf{h}_0$. Then

$$\mathcal{H} \circ \mathbf{j} = \mathbf{h} \circ \mathcal{H}_0 \circ \mathbf{j} = \mathbf{h} \circ \mathbf{h}_0 = \mathbf{h} \circ (1_{TE} - \mathbf{v}_0) = \mathbf{h},$$

as was to be shown.

The horizontal map constructed in Lemma 1 is called the *horizontal map belonging to* (or *induced by*, etc.) the *horizontal projector* $\mathbf{h}$.

Lemma 2. *Let $K : TE \longrightarrow E$ be a connector for π. There is a unique horizontal map $\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E$ for π such that*

$$vpr_E \circ (1_{TE} - \mathcal{H} \circ \mathbf{j}) = K,$$

and therefore $\operatorname{Im}\mathcal{H} = \operatorname{Ker} K$.

Proof. Uniqueness. Suppose that the horizontal maps $\mathcal{H}_1$ and $\mathcal{H}_2$ have the desired property. Let us denote by $\mathbf{h}_1$, $\mathbf{v}_1$ and $\mathbf{h}_2$, $\mathbf{v}_2$ the horizontal and the vertical projectors belonging to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $vpr_E \circ (\mathbf{v}_1 - \mathbf{v}_2) = 0$. Since vpr_E is an isomorphism along the fibres of $V\pi$, it follows that $\mathbf{v}_1 = \mathbf{v}_2$, and hence $\mathbf{h}_1 = \mathbf{h}_2$, $\mathcal{H}_1 = \mathcal{H}_2$.

Existence. Using **CON 1**, consider the map

$$\mathbf{v} : z \in E \mapsto \mathbf{v}_z := \mathbf{i}_z \circ K_z \in \operatorname{Hom}(T_zE, V_zE).$$

In virtue of **CON 2**, for every point $z \in E$ and section $\sigma \in \Gamma(\pi)$ we have

$$K_z \circ \mathbf{i}_z[\sigma(\pi(z))] = K_z[\sigma^{\mathrm{v}}(z)] = \sigma(\pi(z));$$

therefore

$$\mathbf{v}_z^2 = \mathbf{i}_z \circ (K_z \circ \mathbf{i}_z) \circ K_z = \mathbf{i}_z \circ K_z = \mathbf{v}_z.$$

Hence $\mathbf{v}$ projects TE onto VE, and consequently, $\mathbf{h} = 1_{TE} - \mathbf{v}$ is a horizontal projector for π. In view of Lemma 1 there is a unique horizontal map $\mathcal{H}$ such that $\mathcal{H} \circ \mathbf{j} = \mathbf{h}$. Then for any vector $w \in T_zE$ we have

$$vpr_E \circ (1_{TE} - \mathcal{H} \circ \mathbf{j})(w) = vpr_E(\mathbf{v}(w)) = \pi_2 \circ \mathbf{i}^{-1} \circ \mathbf{i}(z, K_z(w)) = K_z(w),$$

which ends the proof.

The horizontal map constructed in Lemma 2 is called the *horizontal map belonging to* (*induced by*, etc.) the *connector* K. Next we point out that a horizontal subbundle of τ_E; a horizontal map, a horizontal projector and a connector for π are all 'equivalent' geometric concepts.

Corollary. *Among the sets of horizontal subbundles $H\pi$ of τ_E, horizontal maps, horizontal projectors and connectors for π, a bijective correspondence is established by the following scheme:*

$$
\begin{array}{lll}
 & & \mathbf{h}_{\mathcal{H}} := \mathcal{H} \circ \mathbf{j} \\
 & \nearrow & \text{horizontal projector for } \pi \\
H\pi \overset{\operatorname{Im}\mathcal{H}=H\pi}{\longleftrightarrow} \text{horizontal map } \mathcal{H} & & \\
 & \searrow & K_{\mathcal{H}} := vpr_E \circ (1_{TE} - \mathcal{H} \circ \mathbf{j}) \\
 & & \text{connector for } \pi
\end{array}
$$

$$
\begin{array}{lll}
 & & \mathcal{H}_{\mathbf{h}} := \mathbf{h} \circ \mathcal{H}_0 \\
 & \nearrow & \text{horizontal map for } \pi \\
H\pi \overset{\operatorname{Im}\mathbf{h}=H\pi}{\longleftrightarrow} \text{horizontal projector } \mathbf{h} & & \\
 & \searrow & K_{\mathbf{h}} := vpr_E \circ (1_{TE} - \mathbf{h}) \\
 & & \text{connector for } \pi
\end{array}
$$

$$
\begin{array}{lll}
 & & \mathbf{h}_K : w \in T_zE \mapsto \mathbf{h}_K(w) := \\
 & & = w - \mathbf{i}_z(K_z(w)) \\
 & \nearrow & \text{horizontal projector for } \pi \\
H\pi \overset{\operatorname{Ker}K=H\pi}{\longleftrightarrow} \text{connector } K & & \\
 & \searrow & \mathcal{H}_K := \mathbf{h}_K \circ \mathcal{H}_0 \\
 & & \text{horizontal map for } \pi
\end{array}
$$

($\mathcal{H}_0$ is an arbitrary horizontal map for π).

Terminology. (1) For the equivalent concepts described by the Corollary the common term *nonlinear connection* will be used as well. By a slight abuse of language we shall also speak of a *nonlinear connection given by* a horizontal map, or a horizontal projector, and so on.

(2) Given a manifold M, a *nonlinear connection* (a *horizontal map*, etc.) *on* M is (by abuse of language again) a nonlinear connection in the tangent vector bundle τ_M (a horizontal map for τ_M, etc.).

2.13. Behaviour on the zero section. Let the zero section $o : M \longrightarrow E$ of π be given, and suppose that $((x^i)_{i=1}^n, (y^j)_{j=1}^k)$ is an adapted local coordinate system on an open subset $\pi^{-1}(\mathcal{U})$ of E. A trivial calculation shows that

$$(o_*)_p \left(\frac{\partial}{\partial u^i}\right)_p = \left(\frac{\partial}{\partial x^i}\right)_{o(p)} \quad (1 \leqq i \leqq n) \quad \text{for all } p \in \mathcal{U}.$$

Since $\left(\left(\frac{\partial}{\partial y^j}\right)_{o(p)}\right)_{j=1}^k$, $1 \leqq j \leqq k$, is a basis for $V_{o(p)}E$, it follows that

$$T_{o(p)}E = (o_*)_p(T_pM) \oplus V_{o(p)}E.$$

Thus we have a natural horizontal map

$$\mathcal{H}_o : \{o(p)\} \times T_pM \mapsto (o_*)_p(T_pM) \subset T_{o(p)}E$$

over the zero section. It is hardly to be expected that any horizontal map acts in this way on $o(M)$, however, this property would be convenient for our purposes. So we make the following compromise.

> We drop the requirement of differentiability of a horizontal map (horizontal projector, connector, horizontal subbundle) over the zero section, and assume that all horizontal maps coincide with the map
>
> $$\mathcal{H}_o : \{o(p)\} \times T_pM \mapsto (o_*)_p(T_pM)$$
>
> on the zero section.

We shall see in the course of our investigations that this convention is in harmony with the demands and some characteristic features of Finsler geometry.

2.14. Homogeneity of nonlinear connections. (**1**) Let $t \in \mathbb{R}^*$. The map

$$c_E^t : E \longrightarrow E, \quad z \mapsto c_E^t(z) := tz$$

is called a *dilation* of E. (As we have seen in 2.4(3), the vector field generating the flow of positive dilations $c_E^{\exp t}$ is just the canonical vector field C_E. This motivates the choice of notation c_E^t.) Analogously, the dilations of $\pi^*TM := E \times_M TM$ are the maps

$$c_{\pi^*TM}^t : (z, v) \in E \times_M TM \mapsto (tz, v) \in E \times_M TM \quad (t \in \mathbb{R}^*).$$

Lemma 1. *For every $t \in \mathbb{R}^*$ we have*

$$c_E^t \circ vpr_E = vpr_E \circ \big((c_E^t)_* \restriction VE\big). \qquad \boxslash$$

(**2**) A horizontal map $\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E$ is said to be *homogeneous* if it satisfies

H 0. $$\left(c_E^{\exp t}\right)_* \circ \mathcal{H} = \mathcal{H} \circ c_{\pi^*TM}^{\exp t} \quad \text{for all } t \in \mathbb{R}.$$

Lemma 2. *Let $\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E$ be a horizontal map, and let $\mathbf{h}$, K and $H\pi := \operatorname{Im}\mathcal{H}$ be the horizontal projector, the connector and the horizontal subbundle belonging to $\mathcal{H}$, respectively. The following conditions are equivalent:*

H 0. $\mathcal{H}$ *is homogeneous.*

H 1. $\mathbf{h} \circ (c_E^t)_* = (c_E^t)_* \circ \mathbf{h}$ *for all* $t \in \mathbb{R}_+^*$.

H 2. $K \circ (c_E^t)_* = c_E^t \circ K$ *for all* $t \in \mathbb{R}_+^*$.

H 3. $H_{tz}E = \left(c_E^t\right)_* H_z E$ *for all* $z \in E,\ t \in \mathbb{R}_+^*$.

Indication of proof. The equivalence of **H 0** and **H 2** is an easy consequence of Lemma 1. We show that **H 0** implies **H 1**; in the other cases the argument is similar.

Choose a point $z \in E$, a vector $w \in T_zE$, and let $t \in \mathbb{R}_+^*$. Then

$$\begin{aligned}\left(c_E^t\right)_* \circ \mathbf{h}(w) = \left(c_E^t\right)_* \circ \mathcal{H} \circ \mathbf{j}(w) = \left(c_E^t\right)_* \circ \mathcal{H}(z, (\pi_*)_z(w)) &\overset{\mathbf{H\,0}}{=} \mathcal{H}(tz, (\pi_*)_z(w)) \\ &= \mathcal{H}(tz, (\pi \circ c_E^t)_*(w)) = \mathcal{H}\Big(tz, (\pi_*)_{tz}\left[(c_E^t)_*(w)\right]\Big) \\ &= \mathcal{H} \circ \mathbf{j}\big((c_E^t)_*(w)\big) = \mathbf{h} \circ (c_E^t)_*(w),\end{aligned}$$

so **H 1** holds.

Remark 1. If a nonlinear connection is given by the assignment of a horizontal projector, a connector, or a horizontal subbundle, then, naturally, homogeneity is defined by **H 1**, **H 2** and **H 3**, respectively. Due to the next observation, this definition is consistent in the case of horizontal projectors.

Lemma 3 and definition. *A horizontal projector $\mathbf{h} \in \mathcal{B}^1(E)$ satisfies* **H 1** *if, and only if, it is homogeneous of degree 1 as a vector-valued form, i.e., if*

H 4. $$d_{C_E}\mathbf{h} = 0$$

holds. The vector-valued one-form

$$\mathbf{t} := -d_{C_E}\mathbf{h} \in \mathcal{B}^1(E)$$

is said to be the tension *of the nonlinear connection given by* $\mathbf{h}$.

We omit the simple proof.

Remark 2. *The tension of a nonlinear connection is a semibasic vector-valued form.* Indeed, for any vector field ξ on E we have

$$\mathbf{t}(\xi) := -(d_{C_E}\mathbf{h})(\xi) = -[C_E, \mathbf{h}\xi] + \mathbf{h}[C_E, \xi] = [\mathbf{h}\xi, C_E] - \mathbf{h}[\mathbf{h}\xi, C_E] = \mathbf{v}[\mathbf{h}\xi, C_E].$$

Thus $\mathbf{t}$ is vertical-valued and $\mathbf{t}(\xi) = 0$ whenever ξ is vertical.

Corollary. *A nonlinear connection is homogeneous if, and only if, the horizontal lift of any vector field is homogeneous of degree* 1.

Proof. Since $d_{C_E}\mathbf{h}$ is semibasic and $\mathfrak{X}^h(E)$ is locally generated by the horizontal lifts of vector fields on M, $d_{C_E}\mathbf{h}$ is determined by its action on vector fields of the form X^h $(X \in \mathfrak{X}(M))$. Using the result of the preceding calculation,

$$(d_{C_E}\mathbf{h})(X^h) = -\mathbf{v}[X^h, C_E] = [C_E, X^h],$$

since $X^h \underset{\pi}{\sim} X$ and $C_E \underset{\pi}{\sim} 0$ imply that $[C_E, X^h]$ is vertical. Thus we have the following chain of equivalent statements:

$$\mathbf{h} \text{ is homogeneous} \overset{\mathbf{H\,4}}{\iff} d_{C_E}\mathbf{h} = 0 \iff \forall X \in \mathfrak{X}(M) : (d_{C_E}\mathbf{h})(X^h) = 0$$

$$\iff [C_E, X^h] = 0 \iff: X^h \text{ is homogeneous of degree } 1.$$

□

2.15. Nonlinear covariant derivatives.

Lemma. *Let σ and s be sections of π. Suppose that f is a smooth function on M that vanishes at a point p. If $\widetilde{\sigma} = \sigma + fs$, then*

$$vpr_E \circ \big((\widetilde{\sigma}_*)_p - (\sigma_*)_p\big) = (df)_p \otimes s(p).$$

Proof. Choose a (small enough) neighbourhood $\mathcal{U}$ of p in M, and suppose that $(\pi^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^j)_{j=1}^k))$ is an adapted chart on E. If, over $\mathcal{U}$, $\sigma = \sum_{j=1}^k \sigma^j \varepsilon_j$, $s = \sum_{j=1}^k s^j \varepsilon_j$, $\widetilde{\sigma} = \sum_{j=1}^k \widetilde{\sigma}^j \varepsilon_j$ (cf. 1.20), then

$$\frac{\partial \widetilde{\sigma}^j}{\partial u^i}(p) - \frac{\partial \sigma^j}{\partial u^i}(p) = \frac{\partial f}{\partial u^i}(p) s^j(p) \qquad (1 \leqq i \leqq n,\ 1 \leqq j \leqq k).$$

For any vector $v = \sum_{i=1}^n v^i \left(\frac{\partial}{\partial u^i}\right)_p \in T_pM$,

$$(\sigma_*)_p(v) = \sum_{i=1}^n v^i \left(\frac{\partial}{\partial x^i}\right)_{\sigma(p)} + \sum_{i=1}^n \sum_{j=1}^k v^i \frac{\partial \sigma^j}{\partial u^i}(p) \left(\frac{\partial}{\partial y^j}\right)_{\sigma(p)},$$

and we have a completely analogous expression for $(\widetilde{\sigma}_*)_p(v)$ (notice that $\widetilde{\sigma}(p) = \sigma(p)$). Hence

$$\begin{aligned}(\widetilde{\sigma}_*)_p(v) - (\sigma_*)_p(v) &= \sum_{i=1}^n \sum_{j=1}^k v^i \frac{\partial f}{\partial u^i}(p) s^j(p) \left(\frac{\partial}{\partial y^j}\right)_{\sigma(p)} \\ &= (df)_p(v) \sum_{j=1}^k s^j(p) \left(\frac{\partial}{\partial y^j}\right)_{\sigma(p)},\end{aligned}$$

and therefore

$$vpr_E\big[(\widetilde{\sigma}_*)_p(v) - (\sigma_*)_p(v)\big] = (df)_p(v) s(p) = \big[(df)_p \otimes s(p)\big](v). \qquad \square$$

Definition. A *nonlinear covariant derivative operator* or simply a *nonlinear covariant derivative* in the vector bundle (E, π, M) is a map

$$\nabla : \Gamma(\pi) \longrightarrow \mathcal{A}^1(M, \pi) \cong \mathcal{A}^1(M) \otimes \Gamma(\pi)$$

which satisfies the following conditions:

NCD 1. If o is the zero section of π, then $\nabla o = 0$.

NCD 2. If $\sigma \in \Gamma(\pi)$ and, at a point $p \in M$, $\sigma(p) \neq 0$, then $\nabla\sigma$ is smooth in a neighbourhood of p.

NCD 3. For any sections σ_1, σ_2 in $\Gamma(\pi)$ such that $\sigma_1(p) = \sigma_2(p)$, we have

$$(\nabla\sigma_1)(p) - (\nabla\sigma_2)(p) = vpr_E \circ \big((\sigma_1)_*(p) - (\sigma_2)_*(p)\big).$$

If, in addition,

NCD 4. $$\nabla(t\sigma) = t\nabla\sigma \quad \text{for all } \sigma \in \Gamma(\pi), \quad t \in \mathbb{R}$$

then ∇ is called a *homogeneous* nonlinear covariant derivative operator.

Example 1. *If ∇ is a covariant derivative operator in π, then ∇ is a homogeneous nonlinear covariant derivative as well.*

Indeed, **NCD 1** is satisfied according to Remark (2) in 1.41. **NCD 2** holds automatically, while **NCD 4** is an immediate consequence of formula $(*)$ in 1.41. To check **NCD 3**, observe that σ_1 may be written in the form

$$\sigma_1 = \sigma_2 + fs; \quad f \in C^\infty(M),\ f(p) = 0; \quad s \in \Gamma(\pi).$$

Now, using the cited formula $(*)$ and the preceding Lemma, we obtain

$$\begin{aligned}(\nabla\sigma_1)(p) - (\nabla\sigma_2)(p) &= \big[\nabla(fs)\big](p) = (df)_p \otimes s(p) + f(p)(\nabla\sigma)(p) \\ &= (df)_p \otimes s(p) = vpr_E \circ \big((\sigma_1)_*(p) - (\sigma_2)_*(p)\big),\end{aligned}$$

as was to be proved.

Proposition. *If $\nabla : \Gamma(\pi) \longrightarrow \mathcal{A}^1(M,\pi)$ is a homogeneous nonlinear covariant derivative operator in π, then*

$$\nabla(f\sigma) = df \otimes \sigma + f\,\nabla\sigma \quad \textit{for all } f \in C^\infty(M),\ \sigma \in \Gamma(\pi).$$

Proof. Let p be a point of M. Using the homogeneity of ∇ and **NCD 3**, it follows at once that

$$\begin{aligned}\big[\nabla(f\sigma)\big](p) - f(p)(\nabla\sigma)(p) &= \big[\nabla(f\sigma)\big](p) - \big[\nabla(f(p)\sigma)\big](p) \overset{\textbf{NCD 3}}{=} \\ &= vpr_E \circ \big[(f\sigma)_*(p) - (f(p)\sigma)_*(p)\big].\end{aligned}$$

Since $f\sigma = f(p)\sigma + (f - f(p))\sigma$ and $(f - f(p))(p) = 0$, the above lemma yields the relation

$$vpr_E \circ \big[(f\sigma)_*(p) - (f(p)\sigma)_*(p)\big] = \big[d(f - f(p))\big]_p \otimes \sigma(p) = (df)_p \otimes \sigma(p),$$

thereby proving the proposition.

Example 2. Let $\mathcal{H} : \pi^*\tau_M \longrightarrow \tau_E$ be a horizontal map for π, and K the connector belonging to $\mathcal{H}$. Then

$$\nabla^{\mathcal{H}} : \Gamma(\pi) \longrightarrow \mathcal{A}^1(M, \pi), \quad \sigma \mapsto \nabla^{\mathcal{H}}\sigma := K \circ \sigma_*$$

is a nonlinear covariant derivative operator for π.

Indeed, **NCD 2** is satisfied automatically. According to the convention declared in 2.13,

$$H_{o(p)}E := \mathcal{H}_0(\{o(p)\} \times T_pM) := (o_*)_p(T_pM) \quad \text{for all } p \in M.$$

Thus for every vector $v \in T_pM$ we have

$$\begin{aligned}(\nabla^{\mathcal{H}}o)_p(v) &= K_{o(p)}\big[(o_*)_p(v)\big] = vpr_E \circ \Big(1_{T_{o(p)}E} - \mathcal{H}_0 \circ \mathbf{j}\Big)(o_*)_p(v) \\ &= vpr_E\big((o_*)_p(v) - \mathcal{H}_0(o(p), v)\big) = vpr_E\big((o_*)_p(v) - (o_*)_p(v)\big) = 0.\end{aligned}$$

This means that $\nabla^{\mathcal{H}}o = 0$, therefore **NCD 1** also holds. Finally, let σ_1 and σ_2 be two sections in $\Gamma(\pi)$ such that $\sigma_1(p) = \sigma_2(p)$. Denoting by $\mathbf{v}$ the vertical projector belonging to $\mathcal{H}$, we have

$$\begin{aligned}(\nabla^{\mathcal{H}}\sigma_1)(p) - (\nabla^{\mathcal{H}}\sigma_2)(p) &= vpr_E \circ \mathbf{v} \circ \big((\sigma_1)_*(p) - (\sigma_2)_*(p)\big) \\ &= vpr_E \circ \big((\sigma_1)_*(p) - (\sigma_2)_*(p)\big),\end{aligned}$$

since, as can be seen from the proof of the lemma, $(\sigma_1)_*(p) - (\sigma_2)_*(p)$ is vertical-valued. Thus **NCD 3** is also valid for $\nabla^{\mathcal{H}}$.

If, in addition,

$$(*) \qquad (c_E^t)_* \circ \mathcal{H} = \mathcal{H} \circ c_{\pi^*TM}^t \quad \text{for all } t \in \mathbb{R},$$

then

$$c_E^t \circ K = K \circ (c_E^t)_* \qquad \text{for all } t \in \mathbb{R}$$

also holds (and conversely). Therefore we get

$$\begin{aligned}\nabla^{\mathcal{H}}(t\sigma) &= K \circ (t\sigma)_* = K \circ (c_E^t \circ \sigma)_* = K \circ (c_E^t)_* \circ \sigma_* \\ &= c_E^t \circ K \circ \sigma_* = t\nabla^{\mathcal{H}}\sigma\end{aligned}$$

for any section σ in $\Gamma(\pi)$ and real number t. So we conclude that *under the homogeneity condition* $(*)$ $\nabla^{\mathcal{H}}$ *becomes a homogeneous nonlinear covariant derivative.*

Example 3. If ∇ is a covariant derivative operator in π, then the map

$$\begin{aligned}\mathcal{H}^{\nabla} : (\sigma(p), v) \in E \times_M TM \mapsto \mathcal{H}^{\nabla}(\sigma(p), v) \\ := (\sigma_*)_p(v) - \big(vpr_E \restriction V_{\sigma(p)}E\big)^{-1}\big((\nabla\sigma)_p(v)\big) \in TE\end{aligned}$$

$(\sigma \in \Gamma(\pi))$ is a horizontal map for π and $\nabla^{\mathcal{H}^{\nabla}} = \nabla$. Then

$$\mathcal{H}^{\nabla}, \quad \mathbf{h}^{\nabla} := \mathcal{H}^{\nabla} \circ \mathbf{j}, \quad K^{\nabla} := vpr_E \circ (1_{TE} - \mathbf{h}^{\nabla}), \quad H^{\nabla}E := \operatorname{Im}\mathcal{H}^{\nabla}$$

are called the horizontal map, the horizontal projector, the connector, and the horizontal subbundle induced by ∇, respectively. These are clearly homogeneous in the sense discussed in 2.14, moreover they satisfy the 'strong homogeneity condition' given by formula $(*)$ in the preceding Example.

2.16. Berwald derivative in $V\pi$.

Proposition 1. *Let a horizontal map $\mathcal{H}$ be given in π. There is a unique covariant derivative operator $\nabla : \mathfrak{X}(E) \times \mathfrak{X}^{\mathrm{v}}(E) \longrightarrow \mathfrak{X}^{\mathrm{v}}(E)$ in $V\pi$ such that*

$$\boxed{\begin{array}{lll}\nabla_{\sigma^{\mathrm{v}}} s^{\mathrm{v}} = 0 & \text{for all} & \sigma, s \in \Gamma(\pi); \\ \nabla_{X^h} s^{\mathrm{v}} = [X^h, s^{\mathrm{v}}] & \text{for all} & X \in \mathfrak{X}(M),\ s \in \Gamma(\pi)\end{array}}$$

where X^h is the $\mathcal{H}$-horizontal lift of $X \in \mathfrak{X}(M)$.

Proof. (1) Since $\mathfrak{X}(E) = \mathfrak{X}^h(E) \oplus \mathfrak{X}^{\mathrm{v}}(E)$, and the $C^\infty(E)$-modules $\mathfrak{X}^h(E)$ and $\mathfrak{X}^{\mathrm{v}}(E)$ are locally generated by the $\mathcal{H}$-horizontal lifts and the vertical lifts of vector fields on M, ∇ is uniquely determined by the prescribed conditions.

(2) Next, we show that there is a unique map

$$\nabla^\circ : \mathfrak{X}^{\mathrm{v}}(E) \times \mathfrak{X}^{\mathrm{v}}(E) \longrightarrow \mathfrak{X}^{\mathrm{v}}(E)$$

satisfying formally the rules **COVD 1–COVD 4** (i.e., ∇° is $C^\infty(E)$-linear in its first variable, additive in its second variable, and obeys the Leibniz rule $\nabla^\circ_\xi f\eta = (\xi f)\eta + f\nabla^\circ_\xi\eta$ for all $\xi, \eta \in \mathfrak{X}^{\mathrm{v}}(E)$, $f \in C^\infty(E)$) such that

$$\nabla^\circ_{\sigma^{\mathrm{v}}} s^{\mathrm{v}} = 0 \quad \text{for all } \sigma, s \in \Gamma(\pi).$$

By the above argument, the uniqueness statement is clear again. The existence will be established by a local construction. Choose an adapted chart $(\pi^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^j)_{j=1}^k))$ for E. If

$$\eta \in \mathfrak{X}^{\mathrm{v}}(E), \quad \eta \restriction \pi^{-1}(\mathcal{U}) = \sum_{j=1}^k \eta^j \frac{\partial}{\partial y^j},$$

then let

$$\nabla^\circ_\xi(\eta \restriction \pi^{-1}(\mathcal{U})) := \Big(\sum_{j=1}^k \xi\eta^j\Big)\frac{\partial}{\partial y^j} \quad \text{for all } \xi \in \mathfrak{X}^{\mathrm{v}}(E).$$

It can be seen at once that the operator

$$\nabla^\circ : \mathfrak{X}^{\mathrm{v}}(E) \times \mathfrak{X}^{\mathrm{v}}(\pi^{-1}(\mathcal{U})) \longrightarrow \mathfrak{X}^{\mathrm{v}}(\pi^{-1}\mathcal{U}))$$

satisfies the conditions **COVD 1–COVD4**. In virtue of the definition and 2.4(2) it is also clear that

$$\nabla^\circ_\xi(\sigma^{\mathrm{v}} \restriction \pi^{-1}(\mathcal{U})) = 0 \quad \text{for all } \sigma \in \Gamma(\pi),\ \xi \in \mathfrak{X}^{\mathrm{v}}(E).$$

So it remains only to check that the construction of ∇° is consistent: if $(\mathcal{U}, (u^i)_{i=1}^n)$ and $(\widetilde{\mathcal{U}}, (\widetilde{u}^i)_{i=1}^n)$ are overlapping charts, then $\nabla^\circ_\xi(\eta \restriction \pi^{-1}(\mathcal{U}))$ and $\nabla^\circ_\xi(\eta \restriction \pi^{-1}(\widetilde{\mathcal{U}}))$ coincide on $\pi^{-1}(\mathcal{U}\cap\widetilde{\mathcal{U}})$. We may assume that $\widetilde{\mathcal{U}}$ is the domain of a vector bundle chart as well; let $((\widetilde{x}^i)_{i=1}^n, (\widetilde{y}^j)_{j=1}^k)$ be the corresponding adapted local coordinate system on $\pi^{-1}(\widetilde{\mathcal{U}})$. There exist unique smooth functions

$$A^\ell_j : \mathcal{U} \cap \widetilde{\mathcal{U}} \longrightarrow \mathbb{R}, \qquad (1 \leqq j, \ell \leqq k)$$

such that

$$\frac{\partial}{\partial \widetilde{y}^j} = \big(A^\ell_j \circ \pi\big)\frac{\partial}{\partial y^\ell} \qquad (1 \leqq j \leqq k).$$

(Here, and in the next short calculation, we use the summation convention: for any pair of indices, if one is up and the other is down, we mean summation from 1 to k.) The matrix (A^ℓ_j) is obviously invertible, let $(B^j_\ell) := (A^\ell_j)^{-1}$. Suppose that $\eta \restriction \pi^{-1}(\widetilde{\mathcal{U}}) = \widetilde{\eta}^\ell \frac{\partial}{\partial \widetilde{y}^\ell}$. Then $\widetilde{\eta}^\ell = \big(B^\ell_j \circ \pi\big)\eta^j$, and

$$\nabla^\circ_\xi(\eta \restriction \pi^{-1}(\widetilde{\mathcal{U}})) := \big(\xi\widetilde{\eta}^\ell\big)\frac{\partial}{\partial \widetilde{y}^\ell}.$$

Over $\pi^{-1}(\mathcal{U} \cap \widetilde{\mathcal{U}})$ we have

$$\big(\xi\widetilde{\eta}^\ell\big)\frac{\partial}{\partial \widetilde{y}^\ell} = \xi\Big[\big(B^\ell_j \circ \pi\big)\eta^j\Big]\big(A^m_\ell \circ \pi\big)\frac{\partial}{\partial y^m} = \big(B^\ell_j A^m_\ell \circ \pi\big)(\xi\eta^j)\frac{\partial}{\partial y^m} = (\xi\eta^j)\frac{\partial}{\partial y^j},$$

as was to be checked.

(3) Having constructed the map ∇°, let finally ∇ be defined as follows:

$$\nabla_\xi \eta := \nabla^\circ_{\mathbf{v}\xi}\eta + \mathbf{v}[\mathbf{h}\xi, \eta] \quad \text{for all } \xi \in \mathfrak{X}(E),\ \eta \in \mathfrak{X}^{\mathrm{v}}(E),$$

where $\mathbf{h}$ and $\mathbf{v}$ are the horizontal and the vertical projector belonging to $\mathcal{H}$. It may be seen at once that ∇ is additive in both its variables. For any smooth function $f \in C^\infty(E)$ we have

$$\nabla_{f\xi}\eta := \nabla^\circ_{\mathbf{v}(f\xi)}\eta + \mathbf{v}[\mathbf{h}(f\xi), \eta] = f\nabla^\circ_{\mathbf{v}\xi}\eta + f\mathbf{v}[\mathbf{h}\xi, \eta] - \mathbf{v}\big((\eta f)\mathbf{h}\xi\big) = f\nabla_\xi\eta;$$

$$\begin{aligned}\nabla_\xi f\eta &:= \nabla^\circ_{\mathbf{v}\xi} f\eta + \mathbf{v}[\mathbf{h}\xi, f\eta] = f\nabla^\circ_{\mathbf{v}\xi}\eta + f\mathbf{v}[\mathbf{h}\xi, \eta] + \big[(\mathbf{v}\xi)f\big]\eta + \mathbf{v}\big[(\mathbf{h}\xi)f\big]\eta \\ &= f\nabla_\xi\eta + \big[(\mathbf{v}\xi + \mathbf{h}\xi)f\big]\eta = (\xi f)\eta + f\nabla_\xi\eta,\end{aligned}$$

so ∇ is indeed a covariant derivative operator in $V\pi$. □

Remark. The covariant derivative operator described by Proposition 1 is said to be the *Berwald derivative induced by $\mathcal{H}$ in $V\pi$*. The operator ∇° constructed in the course of the proof is called the *canonical v-covariant derivative* in $V\pi$. In the case of the vertical bundle $V\tau_M$ to τ_M, the whole tangent bundle τ_{TM} and the pull-back bundle $\tau_M^*\tau_M$ we shall present a direct definition of ∇° later, see 2.33 and 2.42, Example 2.

Proposition 2. (Jaak Vilms) *Let $\mathcal{H}$ be a horizontal map for π, $\mathbf{v}$ the vertical projector belonging to $\mathcal{H}$, and Ω the curvature of $\mathcal{H}$. If ∇ is the Berwald derivative induced by $\mathcal{H}$ in $V\pi$ then*

$$d^\nabla \mathbf{v} = \Omega.$$

Proof. It is enough to check that the formula is true for pairs of vector fields of the form

$$(\sigma^{\mathrm{v}}, s^{\mathrm{v}}), (\sigma^{\mathrm{v}}, X^h), (X^h, Y^h); \quad \sigma, s \in \Gamma(\pi);\ X, Y \in \mathfrak{X}(M).$$

In the first two cases the right-hand side vanishes automatically. The left-hand side also gives zero, since, e.g.,

$$(d^\nabla \mathbf{v})(\sigma^{\mathrm{v}}, X^h) \overset{1.43}{:=} \nabla_{\sigma^{\mathrm{v}}}\mathbf{v}X^h - \nabla_{X^h}\mathbf{v}\sigma^{\mathrm{v}} - \mathbf{v}[\sigma^{\mathrm{v}}, X^h] = -[X^h, \sigma^{\mathrm{v}}] + [X^h, \sigma^{\mathrm{v}}] = 0.$$

In the third case

$$\begin{aligned}(d^\nabla \mathbf{v})(X^h, Y^h) &= \nabla_{X^h}\mathbf{v}Y^h - \nabla_{Y^h}\mathbf{v}X^h - \mathbf{v}[X^h, Y^h] \\ &= -\mathbf{v}[\mathbf{h}X^h, \mathbf{h}Y^h] = \Omega(X^h, Y^h),\end{aligned}$$

as was to be proved.

C. Tensors along the tangent bundle projection. Lifts

Basic conventions

(1) From now on *the tangent bundle of a fixed n-dimensional manifold M will be denoted by* (TM, τ, M), abbreviated as τ. This τ will also be mentioned as the *tangent bundle projection.*

(2) For coordinate calculations we choose a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M and employ the *induced chart*
$(\tau^{-1}(\mathcal{U}), (x^i, y^i)),\ x^i := u^i \circ \tau,\ y^i : v \in \tau^{-1}(\mathcal{U}) \mapsto y^i(v) := v(u^i)$
$(1 \leqq i \leqq n)$ on TM. *The Einstein summation convention* will be used: an index occurring twice in a product, once as a subscript and once as a superscript is to be summed from 1 to n (a superscript in a denominator acts as a subscript).

(3) The big vertical morphism

$$\tau^*\tau \longrightarrow V\tau, \quad TM \times_M TM \longrightarrow VTM$$

will be denoted by $\mathbf{i}$, the strong bundle surjection

$$\tau_{TM} \longrightarrow \tau^*\tau, \quad w \in TTM \mapsto (\tau_{TM}(w),\ \tau_*(w)) \in TM \times_M TM$$

will be denoted by $\mathbf{j}$, as in the general case discussed in 2.2 and 2.3. Thus we can form the canonical short exact sequence

$$0 \longrightarrow \tau^*\tau \xrightarrow{\mathbf{i}} \tau_{TM} \xrightarrow{\mathbf{j}} \tau^*\tau \longrightarrow 0,$$

or written otherwise,

$$0 \longrightarrow TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \longrightarrow 0$$

arising from τ.

(4) C stands for the Liouville vector field on TM, $f^{\mathrm{v}} := f \circ \tau$ is the *vertical lift of a function $f \in C^\infty(M)$ into $C^\infty(TM)$.*

2.17. Vector fields along τ. (**1**) A section of the transverse bundle $\tau^*\tau$ is called a *vector field along τ*. As we have learnt in 1.22 and 2.8, there is a canonical isomorphism

$$\Gamma(\tau^*\tau) \cong \Gamma_\tau(\tau).$$

We repeat how this isomorphism may be established.

The fibres of $\tau^*\tau$ are the n-dimensional real vector spaces

$$\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M, \qquad v \in TM;$$

therefore any section in $\Gamma(\tau^*\tau)$ is of the form

$$\widetilde{X} : v \in TM \mapsto \widetilde{X}(v) = (v, \underline{X}(v))$$

where $\underline{X} : TM \longrightarrow TM$ is a smooth map such that $\tau \circ \underline{X} = \tau$:

$$\begin{tikzcd}
TM \arrow[r, "\tau"] & M \\
TM \arrow[u, "\underline{X}"] \arrow[ur, "\tau = \underline{X} \circ \tau"'] &
\end{tikzcd}$$

In these terms, the map

$$\widetilde{X} \in \Gamma(\tau^*\tau) \mapsto \underline{X} \in \Gamma_\tau(\tau)$$

is an isomorphism of $C^\infty(TM)$-modules. We shall use the convenient, 'neutral' notation $\mathfrak{X}(\tau)$ for these isomorphic modules, so

$$\mathfrak{X}(\tau) := \Gamma(\tau^*\tau) \cong \Gamma_\tau(\tau).$$

(**2**) There is a distinguished section

$$\delta : v \in TM \mapsto \delta(v) := (v, v) \in TM \times_M TM$$

in $\Gamma(\tau^*\tau)$, called the *canonical vector field along* τ. δ corresponds to the identity map 1_{TM} under the isomorphism $\Gamma_\tau(\tau) \cong \Gamma(\tau^*\tau)$ and it is in a close relationship with the Liouville vector field, namely

$$\mathbf{i} \circ \delta = C.$$

(**3**) For any vector field X on M, the map

$$\widehat{X} : TM \longrightarrow TM \times_M TM, \quad v \mapsto \widehat{X}(v) := (v, X \circ \tau(v))$$

is a section of $\tau^*\tau$, called the *lift of* X *into* $\mathfrak{X}(\tau)$ or a *basic vector field* along the tangent bundle projection. $\widehat{X}$ may be identified with the map $X \circ \tau : TM \longrightarrow TM$. Obviously, $\{\widehat{X} \mid X \in \mathfrak{X}(M)\}$ generates the $C^\infty(TM)$-module $\mathfrak{X}(\tau)$ locally. In particular, the sections

$$\widehat{\frac{\partial}{\partial u^i}} \longleftrightarrow \frac{\partial}{\partial u^i} \circ \tau : v \in \tau^{-1}(\mathcal{U}) \mapsto \left(v, \left(\frac{\partial}{\partial u^i}\right)_{\tau(v)}\right) \qquad (1 \leqq i \leqq n)$$

form a local basis for $\mathfrak{X}(\tau)$.

(**4**) Let $X \in \mathfrak{X}(M)$. Notice that for any tangent vector $v \in TM$,

$$\mathbf{i} \circ \widehat{X}(v) := \mathbf{i}(v, X \circ \tau(v)) = \mathbf{i}_v\big(X(\tau(v))\big) \overset{2.4(4)}{:=} \big[X(\tau(v))\big]^{\uparrow}(v) \overset{2.4(5)}{:=} X^{\mathrm{v}}(v),$$

therefore

$$\mathbf{i} \circ \widehat{X} = X^{\mathrm{v}}.$$

(Less pedantically, instead of $\mathbf{i} \circ \widehat{X}$, $\mathbf{j} \circ \xi$ ($\xi \in \mathfrak{X}(TM)$), etc., we shall also write $\mathbf{i}\widehat{X}$, $\mathbf{j}\xi$, etc.)

2.18. The vertical endomorphism. Consider the canonical short exact sequence

$$0 \longrightarrow TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \longrightarrow 0.$$

Due to its peculiarity, we may form the composition

$$J := \mathbf{i} \circ \mathbf{j} : TTM \longrightarrow TTM$$

which is a particularly important canonical object. J is said to be the *vertical endomorphism of TTM or the canonical almost tangent structure on TM.* Obviously, J is a TM-morphism from τ_{TM} into τ_{TM}, i.e., $J \in \mathrm{End}(\tau_{TM})$. Since

$$\mathrm{End}(\tau_{TM}) \cong \mathcal{B}^1(TM) := \mathcal{A}^1(TM, \tau_{TM}) \cong \mathrm{End}_{C^\infty(TM)}(\mathfrak{X}(TM)) \cong \mathcal{T}^1_1(TM),$$

we have an abundance of possibilities for an appropriate interpretation of J. From the definition,

$$J^2 = \mathbf{i} \circ \mathbf{j} \circ \mathbf{i} \circ \mathbf{j} = \mathbf{i} \circ 0 \circ \mathbf{j} = 0,$$

i.e., *J is a nilpotent endomorphism.* Having a look at the diagrams

$$\begin{tikzcd} TTM \arrow[r, "\mathbf{j}"] \arrow[dr, "J"'] & TM \times_M TM \arrow[d, "\mathbf{i}"] \\ & VTM \subset TTM \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{X}(TM) \arrow[r, "\mathbf{j}"] \arrow[dr, "J"'] & \mathfrak{X}(\tau) \arrow[d, "\mathbf{i}"] \\ & \mathfrak{X}^{\mathrm{v}}(TM) \subset \mathfrak{X}(TM) \end{tikzcd}$$

we see at once that

$$\mathrm{Im}\, J = \mathrm{Im}\, \mathbf{i} = \mathrm{Ker}\, \mathbf{j} = \mathrm{Ker}\, J = VTM$$

and

$$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^{\mathrm{v}}(TM).$$

(As for the abuse of notation, see 2.8.) It follows that a *vector field* ξ *on* TM *is vertical if, and only if,* $J\xi = 0$; in particular

$$JX^{\mathrm{v}} = 0 \quad \text{for all} \quad X \in \mathfrak{X}(M).$$

Coordinate description. Consider an induced chart $(\tau^{-1}(\mathcal{U}),((x^i)_{i=1}^n,(y^i)_{i=1}^n))$ on TM. If $v \in \tau^{-1}(\mathcal{U})$ and $w = w^i \left(\frac{\partial}{\partial x^i}\right)_v + w^{n+i}\left(\frac{\partial}{\partial y^i}\right)_v \in T_vTM$, then

$$J(w) = \mathbf{i} \circ \mathbf{j}(w) = \mathbf{i}\left(v, w^i \left(\frac{\partial}{\partial u^i}\right)_{\tau(v)}\right) = \left(w^i \left(\frac{\partial}{\partial u^i}\right)_{\tau(v)}\right)^{\uparrow}(v) = w^i \left(\frac{\partial}{\partial y^i}\right)_v,$$

therefore

$$\boxed{J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = 0 \qquad (1 \leqq i \leqq n)}.$$

Remark. With the help of the vertical endomorphism J the notion of a semibasic form (see 2.7) on TM may be reformulated as follows:

$\alpha \in \mathcal{A}(TM)$ is *semibasic* if $i_{J\xi}\alpha = 0$ for all $\xi \in \mathfrak{X}(TM)$;
$A \in \mathcal{B}(TM)$ is *semibasic* if $J \circ A = 0$ and $i_{J\xi}A = 0$ for all $\xi \in \mathfrak{X}(TM)$.

2.19. Fundamental relations. Suppose that $\mathcal{H}$ is a horizontal map on M and let $\mathcal{V}$ be the vertical map belonging to $\mathcal{H}$ (2.10). Then we have the 'double exact sequence'

$$0 \rightleftarrows TM \times_M TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftarrows}} TTM \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftarrows}} TM \times_M TM \rightleftarrows 0.$$

The defining relations are

$$\operatorname{Im} \mathbf{i} = \operatorname{Ker} \mathbf{j} \ \Rightarrow\ \mathbf{j} \circ \mathbf{i} = 0, \qquad \mathbf{j} \circ \mathcal{H} = 1_{TM \times_M TM},$$

$$\operatorname{Im} \mathcal{H} = \operatorname{Ker} \mathcal{V} \ \Rightarrow\ \mathcal{V} \circ \mathcal{H} = 0, \quad \mathcal{V} \circ \mathbf{i} = 1_{TM \times_M TM}.$$

We recall that

$$\mathbf{h} := \mathcal{H} \circ \mathbf{j} \quad \text{and} \quad \mathbf{v} = 1_{TTM} - \mathbf{h}$$

are the horizontal and the vertical projectors belonging to $\mathcal{H}$. We obtain immediately that

$$\boxed{\begin{array}{ll} \mathbf{v} = \mathbf{i} \circ \mathcal{V}, & \\ J \circ \mathbf{h} = J, & \mathbf{h} \circ J = 0 \\ J \circ \mathbf{v} = 0, & \mathbf{v} \circ J = J \end{array}}\,.$$

It is also clear from the definitions (see 2.11(4) and 2.17(3)) that the horizontal lift of a vector field X on M is just

$$X^h = \mathcal{H} \circ \widehat{X}.$$

For any vector fields X, Y on M we have

$$JX^h = X^{\mathrm{v}}, \quad J[X^h, Y^h] = [X, Y]^{\mathrm{v}}.$$

Indeed,

$$JX^h = \mathbf{i} \circ \mathbf{j} \circ \mathcal{H} \circ \widehat{X} = \mathbf{i} \circ \widehat{X} \overset{2.17(4)}{=} X^{\mathrm{v}}.$$

As for the second relation, in the course of the proof of Lemma 2 in 2.11 we have already shown that $\mathbf{h}[X^h, Y^h] = [X, Y]^h$. Hence

$$J[X^h, Y^h] = J \circ \mathbf{h}[X^h, Y^h] = J[X, Y]^h = [X, Y]^{\mathrm{v}}.$$

Lemma. *The map*

$$\mathbf{F} := \mathcal{H} \circ \mathcal{V} - \mathbf{i} \circ \mathbf{j} = \mathcal{H} \circ \mathcal{V} - J$$

is an almost complex structure on the manifold TM, i.e,

$$\mathbf{F} \in \mathcal{B}^1(TM) \cong \mathcal{T}^1_1(TM) \quad \textit{and} \quad \mathbf{F}^2 = -1_{TTM}.$$

Proof. $\mathbf{F}^2 = (\mathcal{H} \circ \mathcal{V} - \mathbf{i} \circ \mathbf{j}) \circ (\mathcal{H} \circ \mathcal{V} - \mathbf{i} \circ \mathbf{j}) = \mathcal{H} \circ (\mathcal{V} \circ \mathcal{H}) \circ \mathcal{V} - \mathbf{i} \circ (\mathbf{j} \circ \mathcal{H}) \circ \mathcal{V}$ $-\mathcal{H} \circ (\mathcal{V} \circ \mathbf{i}) \circ \mathbf{j} + J^2 = -\mathbf{i} \circ \mathcal{V} - \mathcal{H} \circ \mathbf{j} = -(\mathbf{v} + \mathbf{h}) = -1_{TTM}$, and the tensoriality of $\mathbf{F}$ is obvious. □

The almost complex structure $\mathbf{F}$ is called the *almost complex structure associated to* $\mathcal{H}$. We obtain immediately from the definition and the preceding relations the following formulae:

$$\boxed{\begin{array}{ll} \mathbf{F} \circ J = \mathbf{h}, & J \circ \mathbf{F} = \mathbf{v}; \\ \mathbf{F} \circ \mathbf{h} = -J, & \mathbf{h} \circ \mathbf{F} = \mathbf{F} \circ \mathbf{v} = J + \mathbf{F}, \\ \mathbf{v} \circ \mathbf{F} = -J & \end{array}}\,.$$

2.20. Complete lifts. The *complete lift* of a smooth function $f \in C^\infty(M)$ into $C^\infty(TM)$ is the smooth function

$$f^c : TM \longrightarrow \mathbb{R}, \quad v \mapsto f^c(v) := (df)_{\tau(v)}(v).$$

Then

$$(fh)^c = f^c h^{\mathrm{v}} + f^{\mathrm{v}} h^c \qquad \text{for all} \quad f, h \in C^\infty(M).$$

Local description. $f^c \restriction \tau^{-1}(\mathcal{U}) = y^i \left(\frac{\partial f}{\partial u^i} \circ \tau\right) = y^i \left(\frac{\partial f}{\partial u^i}\right)^{\mathrm{v}}$; in particular

$$(u^i)^c = y^i \qquad (1 \leqq i \leqq n).$$

Notice that (locally)

$$Cf^c = C\left(y^i\left(\frac{\partial f}{\partial u^i}\right)^{\mathrm{v}}\right) \overset{2.4(2)}{=} C(y^i)\left(\frac{\partial f}{\partial u^i}\right)^{\mathrm{v}} = y^i\left(\frac{\partial f}{\partial u^i}\right)^{\mathrm{v}} = f^c,$$

therefore the *complete lift of a function is positive-homogeneous of degree 1.* By the same calculation we obtain that

$$X^{\mathrm{v}} f^c = (Xf)^{\mathrm{v}} \quad \text{for all } f \in C^\infty(M) \text{ and } X \in \mathfrak{X}(M).$$

Lemma 1. *Any vector field on TM is uniquely determined by its action on the complete lifts of smooth functions on M.*

Proof. Let ξ be a vector field on TM. It is enough to check that *if $\xi f^c = 0$ for any function $f \in C^\infty(M)$, then $\xi = 0$.* Applying this condition we have

$$0 = \frac{1}{2}\xi\big((f^2)^c\big) = \frac{1}{2}\xi(2f^c f^{\mathrm{v}}) = (\xi f^c) f^{\mathrm{v}} + f^c(\xi f^{\mathrm{v}}) = f^c(\xi f^{\mathrm{v}}).$$

Since $f \in C^\infty(M)$ is arbitrary, we readily infer with the choices $f := u^i$, $1 \leqq i \leqq n$ that

$$\xi(x^i) = 0 \qquad \text{for all} \quad i \in \{1, \dots, n\}.$$

Hence ξ is vertical and, locally, we obtain that

$$\xi \restriction \tau^{-1}(\mathcal{U}) = \xi(y^i)\frac{\partial}{\partial y^i} = (\xi(u^i)^c)\frac{\partial}{\partial y^i} = 0.$$

This proves the lemma.

Corollary 1 and definition. *For any vector field X on M there is a unique vector field X^c on TM such that*

$$X^c f^c = (Xf)^c \qquad \textit{for all} \quad f \in C^\infty(M).$$

X^c is said to be the complete lift *of X.* $\square$

Corollary 2. *Let X be a vector field on M. Then*

$$X^c f^{\mathrm{v}} = (Xf)^{\mathrm{v}} \qquad \textit{for all} \quad f \in C^\infty(M).$$

Proof. Using the same trick as above, we have on the one hand

$$\frac{1}{2}X^c(f^2)^c = X^c(f^c f^{\mathrm{v}}) = (X^c f^c)f^{\mathrm{v}} + f^c(X^c f^{\mathrm{v}}) = (Xf)^c f^{\mathrm{v}} + f^c(X^c f^{\mathrm{v}}).$$

On the other hand,

$$\frac{1}{2}X^c(f^2)^c = \frac{1}{2}(Xf^2)^c = \big[f(Xf)\big]^c = f^c(Xf)^{\mathrm{v}} + f^{\mathrm{v}}(Xf)^c.$$

Comparing the right-hand sides of these relations, we obtain that $X^c f^{\mathrm{v}} = (Xf)^{\mathrm{v}}$. □

Coordinate expression. Let $X \restriction \mathcal{U} = X^i \frac{\partial}{\partial u^i}$, and suppose that

$$X^c \restriction \tau^{-1}(\mathcal{U}) = \xi^i \frac{\partial}{\partial x^i} + \widetilde{\xi}^i \frac{\partial}{\partial y^i}.$$

Then

$$\xi^i = X^c x^i = X^c(u^i)^{\mathrm{v}} \overset{\text{Cor. 2}}{=} (Xu^i)^{\mathrm{v}} = (X^i)^{\mathrm{v}} = X^i \circ \tau \quad (1 \leqq i \leqq n)$$

and

$$\widetilde{\xi}^i = X^c y^i = X^c(u^i)^c := (Xu^i)^c = (X^i)^c = y^j \left(\frac{\partial X^i}{\partial u^j} \circ \tau\right) \quad (1 \leqq i \leqq n),$$

therefore

$$\boxed{X^c \restriction \tau^{-1}(\mathcal{U}) = (X^i \circ \tau)\frac{\partial}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^j} \circ \tau\right)\frac{\partial}{\partial y^i}}.$$

In particular,

$$\left(\frac{\partial}{\partial u^i}\right)^c = \frac{\partial}{\partial x^i} \qquad (1 \leqq i \leqq n).$$

Corollary 3. (1) $JX^c = X^{\mathrm{v}}$, $\mathbf{j}X^c = \widehat{X}$ *for all* $X \in \mathfrak{X}(M)$.

(2) *If* $\mathcal{H}$ *is a horizontal map on* M *with horizontal projector* $\mathbf{h}$, *then*

$$\mathbf{h}X^c = X^h \quad \textit{for all} \quad X \in \mathfrak{X}(M).$$

Proof. The first relation is obvious from the coordinate expressions. Writing this in the form $\mathbf{i} \circ \mathbf{j} X^c = \mathbf{i}\widehat{X}$, it follows that $\mathbf{j} X^c = \widehat{X}$. Thus

$$X^h \overset{2.10}{=} \mathcal{H} \circ \widehat{X} = \mathcal{H} \circ \mathbf{j} X^c = \mathbf{h} X^c. \qquad \square$$

Corollary 4 (Local basis principles). *If $(X_i)_{i=1}^n$ is a frame field on a domain $\mathcal{U} \subset M$, then*

$$\big((X_i^{\mathrm{v}})_{i=1}^n,\ (X_i^c)_{i=1}^n\big) \quad \textit{and} \quad \big((X_i^{\mathrm{v}})_{i=1}^n,\ (X_i^h)_{i=1}^n\big)$$

are frame fields on TM with domain $\tau^{-1}(\mathcal{U})$. $\square$

This last observation provides a convenient tool for tensorial constructions on TM. Namely, it may be stated that in order to determine uniquely a tensor field on TM, it is sufficient to specify its action on the vertical and on the complete (or on the horizontal) lifts of vector fields on M.

Example. Let a k-form $\alpha \in \mathcal{A}^k(M)$ $(k \in \mathbb{N}^*)$ be given. The prescription

$$i_{X^{\mathrm{v}}}\alpha^{\mathrm{v}} := 0, \quad \alpha^{\mathrm{v}}(X_1^c, \dots, X_k^c) := \big[\alpha(X_1, \dots, X_k)\big]^{\mathrm{v}}$$
$$\text{for all } X, X_i \in \mathfrak{X}(M)\ (1 \leqq i \leqq n)$$

determines uniquely a k-form α^{v} on TM, called the *vertical lift* of α. The vertical lift of a vector-valued k-form on M may be defined in the same way. In particular, the vertical lift A^{v} of a vector-valued one-form $A \in \mathcal{B}^1(M) \cong \mathcal{T}_1^1(M)$ is given by

$$A^{\mathrm{v}}(X^{\mathrm{v}}) = 0, \quad A^{\mathrm{v}}(X^c) = \lfloor A(X)\rfloor^{\mathrm{v}} \qquad \text{for all} \quad X \in \mathfrak{X}(M).$$

More specifically, for the vertical lift of the unit tensor field $\iota_M \in \mathcal{T}_1^1(M)$ we obtain:

$$\iota_M^{\mathrm{v}}(X^{\mathrm{v}}) = 0, \quad \iota_M^{\mathrm{v}}(X^c) = X^{\mathrm{v}} \qquad \text{for all} \quad X \in \mathfrak{X}(M).$$

In view of 2.18 and Corollary 3 this means that

$$\boxed{J = \iota_M^{\mathrm{v}}}.$$

Lemma 2. *For any vector fields X, Y on M and function $f \in C^\infty(M)$ we have*

$$(X+Y)^c = X^c + Y^c, \qquad (fX)^c = f^{\mathrm{v}} X^c + f^c X^{\mathrm{v}};$$
$$[X,Y]^c = [X^c, Y^c], \qquad [X^{\mathrm{v}}, Y^c] = [X,Y]^{\mathrm{v}};$$
$$[C, X^{\mathrm{v}}] = -X^{\mathrm{v}}, \qquad [C, X^c] = 0.$$

The *proof* is a routine verification. To convey a possible pattern, we check the third and the last relation. Let $f \in C^\infty(M)$. Then

$$\begin{aligned}[X,Y]^c f^c &:= \big([X,Y]f\big)^c = \big[X(Yf)\big]^c - \big[Y(Xf)\big]^c \\ &= X^c(Y^c f^c) - Y^c(X^c f^c) = [X^c, Y^c]f^c,\end{aligned}$$

hence, by Lemma 1, $[X,Y]^c = [X^c, Y^c]$. Similarly,

$$[C, X^c]f^c = C(X^c f^c) - X^c(Cf^c) = C(Xf)^c - X^c f^c = (Xf)^c - X^c f^c = 0,$$

and so $[C, X^c] = 0$.

2.21. One-forms along τ. (**1**) To make the relationship between the investigated objects more transparent, first we have a look at the general situation. Let (E, π, M) be a vector bundle, and consider the pull-back of the cotangent bundle (T^*M, τ^*, M) by π. Then we obtain the vector bundle $\pi^*\tau^* = (E \times_M T^*M, p_1, E)$, displayed by the commutative diagram

$$\begin{tikzcd} E\times_M T^*M \arrow[r, "p_2"] \arrow[d, "p_1"'] & T^*M \arrow[d, "\tau^*"] \\ E \arrow[r, "\pi"'] & M \end{tikzcd}, \qquad \begin{aligned} p_1 &:= pr_1 \restriction E \times_M T^*M, \\ p_2 &:= pr_2 \restriction E \times_M T^*M \end{aligned} .$$

$\pi^*\tau^*$ is called the *transverse cotangent bundle to* π. This bundle is strongly isomorphic to the dual of the transverse bundle $\pi^*\tau$ to π (2.3), i.e.,

$$\boxed{\pi^*\tau^* \cong (\pi^*\tau)^*}.$$

Indeed, the fibres of $\pi^*\tau$ and $\pi^*\tau^*$ over a point $z \in E$ are the vector spaces

$$\{z\} \times T_{\pi(z)}M \quad \text{and} \quad \{z\} \times T^*_{\pi(z)}M$$

which are dual to each other. Thus 1.23(1) can be applied to conclude

$$\boxed{\Gamma(\pi^*\tau^*) \cong \big[\Gamma(\pi^*\tau)\big]^* \cong \big[\mathfrak{X}(\pi)\big]^*}.$$

(**2**) Now we turn to the case when the role of the vector bundle (E, π, M) is played by the tangent bundle (TM, τ, M). Then we obtain the *transverse cotangent bundle* $\tau^*\tau^* = (TM \times_M T^*M, \tau_1, TM)$, diagrammatically

$$\begin{tikzcd} TM \times_M T^*M \arrow[r, "\tau_2"] \arrow[d, "\tau_1"'] & T^*M \arrow[d, "\tau^*"] \\ TM \arrow[r, "\tau"'] & M \end{tikzcd} ; \qquad \begin{aligned} \tau_1 &:= pr_1 \restriction TM \times_M T^*M, \\ \tau_2 &:= pr_2 \restriction TM \times_M T^*M. \end{aligned}$$

In view of our preliminary remarks, $\tau^*\tau^* \cong (\tau^*\tau)^*$, therefore

$$\boxed{\Gamma(\tau^*\tau^*) \cong \big[\Gamma(\tau^*\tau)\big]^* \cong \lfloor\mathfrak{X}(\tau)]^* =: \mathfrak{X}^*(\tau)}.$$

For either of these isomorphic $C^\infty(TM)$-modules we use the convenient notation $\mathcal{A}^1(\tau)$; elements of $\mathcal{A}^1(\tau)$ are said to be *one-forms along* τ. As in the dual case of vector fields along τ, a section $\widetilde{\alpha} \in \Gamma(\tau^*\tau^*)$ acts as follows:

$$\begin{cases} \widetilde{\alpha} : v \in TM \mapsto \widetilde{\alpha}(v) := (v, \underline{\alpha}(v)) \in TM \times_M T^*M, \\ \underline{\alpha} \in C^\infty(TM, T^*M), \ \tau^* \circ \underline{\alpha} = \tau. \end{cases}$$

We shall freely use the identification

$$\widetilde{\alpha} \in \Gamma(\tau^*\tau^*) \longleftrightarrow \underline{\alpha} \in C^\infty(TM, T^*M), \ \tau^* \circ \underline{\alpha} = \tau.$$

Example. Let α be a one-form on M. Then the map

$$\widehat{\alpha} : v \in TM \mapsto \widehat{\alpha}(v) := \big(v, \alpha[\tau(v)]\big) \in TM \times_M T^*M$$

is a section of $\tau^*\tau^* \cong (\tau^*\tau)^*$, which corresponds to the map $\alpha \circ \tau \in C^\infty(TM, T^*M)$ under the above identification. $\widehat{\alpha}$ is called the *lift of* α *into* $\mathcal{A}^1(\tau)$ or a *basic one-form along* τ. Clearly, $\mathcal{A}^1(\tau)$ is locally generated by the set $\{\widehat{\alpha} \mid \alpha \in \mathcal{A}^1(M)\}$ of all basic one-forms along τ. In particular, the sections

$$\widehat{du^i} \longleftrightarrow du^i \circ \tau : v \in \tau^{-1}(\mathcal{U}) \mapsto (v, (du^i)_{\tau(v)}) \qquad (1 \leqq i \leqq n)$$

form a local basis for $\mathcal{A}^1(\tau)$.

Lemma. *The* $C^\infty(TM)$*-module of one-forms along* τ *is canonically isomorphic to the module of semibasic one-forms on* TM*, the canonical isomorphism being given by the map*

$$\begin{cases} \widetilde{\alpha} \in \mathcal{A}^1(\tau) \mapsto (\widetilde{\alpha})_0 \in \mathcal{A}^1(TM), \\ \forall \xi \in \mathfrak{X}(TM) : (\widetilde{\alpha})_0(\xi) := \widetilde{\alpha}(\mathbf{j}\xi). \end{cases}$$

Under this isomorphism the basic one-forms along τ and the vertical lifts of one-forms on M correspond to each other:

$$(\widehat{\alpha})_0 = \alpha^{\mathrm{v}} \qquad \textit{for all} \quad \alpha \in \mathcal{A}^1(M).$$

Proof. (1) For every vector field ξ on TM,

$$(\widetilde{\alpha})_0(J\xi) := \widetilde{\alpha}(\mathbf{j} \circ \mathbf{i} \circ \mathbf{j}\xi) = \widetilde{\alpha}(0\xi) = 0,$$

so $(\widetilde{\alpha})_0$ is indeed *semibasic.*

(2) The map $\widetilde{\alpha} \in \mathcal{A}^1(\tau) \mapsto (\widetilde{\alpha})_0 \in \mathcal{A}^1_0(TM)$ is *injective.* In fact, if $(\widetilde{\alpha})_0 = (\widetilde{\beta})_0$, then $\widetilde{\alpha}(\mathbf{j}\xi) = \widetilde{\beta}(\mathbf{j}\xi)$ for all $\xi \in \mathfrak{X}(TM)$. As $\mathbf{j} : \mathfrak{X}(TM) \longrightarrow \mathfrak{X}(\tau)$ is surjective, this implies that $\widetilde{\alpha} = \widetilde{\beta}$.

(3) We show that the map $\widetilde{\alpha} \mapsto (\widetilde{\alpha})_0$ is *surjective* as well. Let β be any semibasic one-form on TM. Consider the map $\widetilde{\alpha}$ given by

$$\widetilde{\alpha}(\widetilde{X}) := \beta(\mathcal{H}\widetilde{X}) \qquad \text{for all} \quad \widetilde{X} \in \mathfrak{X}(\tau),$$

where $\mathcal{H}$ is an arbitrary horizontal map on M. Then $\widetilde{\alpha}$ is well-defined. Indeed, if $\mathcal{H}_1$ is another horizontal map on M, then

$$J(\mathcal{H}_1\widetilde{X} - \mathcal{H}\widetilde{X}) = \mathbf{i} \circ \mathbf{j} \circ \mathcal{H}_1(\widetilde{X}) - \mathbf{i} \circ \mathbf{j} \circ \mathcal{H}(\widetilde{X}) = \mathbf{i}\widetilde{X} - \mathbf{i}\widetilde{X} = 0,$$

hence $\mathcal{H}_1\widetilde{X} - \mathcal{H}\widetilde{X}$ is vertical, and so $\beta(\mathcal{H}_1\widetilde{X}) = \beta(\mathcal{H}\widetilde{X})$, therefore $\widetilde{\alpha}(\widetilde{X})$ does not depend on the choice of $\mathcal{H}$. Since the $C^\infty(TM)$-linearity is obvious, $\widetilde{\alpha} \in \mathcal{A}^1(\tau)$. For every vector field ξ on TM we have

$$(\widetilde{\alpha})_0(\xi) := \widetilde{\alpha}(\mathbf{j}\xi) := \beta(\mathcal{H} \circ \mathbf{j}\xi) = \beta(\mathbf{h}\xi) = \beta(\xi - \mathbf{v}\xi) = \beta(\xi) - \beta(J\mathbf{F}\xi) = \beta(\xi)$$

($\mathbf{h}$ and $\mathbf{v}$ are the projectors belonging to $\mathcal{H}$, $\mathbf{F}$ is the almost complex structure associated to $\mathcal{H}$), therefore $(\widetilde{\alpha})_0 = \beta$, which proves the surjectivity.

(4) Let, finally, $\alpha \in \mathcal{A}^1(M)$. For every vector field X on M,

$$(\widehat{\alpha})_0(X^c) := \widehat{\alpha}(\mathbf{j}X^c) \overset{2.20\ \text{Cor. } 3}{=} \widehat{\alpha}(\widehat{X}) = \alpha(X) \circ \tau = \big(\alpha(X)\big)^{\mathrm{v}} = (\alpha^{\mathrm{v}})(X^c),$$

whence $(\widehat{\alpha})_0 = \alpha^{\mathrm{v}}$. □

2.22. Tensor fields and differential forms along τ. (**1**) In 1.23 we introduced the terminology 'π-tensor field' and the notation $\Gamma^r_s(\pi)$ for the module of type (r, s) π-tensor fields. Thus, in particular, we may also speak of '$\tau^*\tau$-tensor fields' and the $C^\infty(TM)$-module $\Gamma^r_s(\tau^*\tau)$. According to the general case, $\Gamma^r_s(\tau^*\tau)$ is constituted by the $C^\infty(TM)$-multilinear maps

$$\underbrace{[\Gamma(\tau^*\tau)]^* \times \cdots \times [\Gamma(\tau^*\tau)]^*}_{r \text{ times}} \times \underbrace{\Gamma(\tau^*\tau) \times \cdots \times \Gamma(\tau^*\tau)}_{s \text{ times}} \longrightarrow C^\infty(TM).$$

In view of our previous considerations,

$$[\Gamma(\tau^*\tau)]^* \cong \Gamma(\tau^*\tau^*) \cong [\mathfrak{X}(\tau)]^* =: \mathcal{A}^1(\tau), \quad \Gamma(\tau^*\tau) = \mathfrak{X}(\tau),$$

so $\Gamma^r_s(\tau^*\tau)$ may be regarded as the $C^\infty(TM)$-module of multilinear maps

$$\underbrace{\mathcal{A}^1(\tau) \times \cdots \times \mathcal{A}^1(\tau)}_{r \text{ times}} \times \underbrace{\mathfrak{X}(\tau) \times \cdots \times \mathfrak{X}(\tau)}_{s \text{ times}} \longrightarrow C^\infty(TM).$$

By abuse of notation, we shall write $\mathcal{T}^r_s(\tau)$ instead of $\Gamma^r_s(\tau^*\tau)$; $\mathcal{T}^\bullet_\bullet(\tau) := \bigoplus_{(r,s)\in\mathbb{N}\times\mathbb{N}} \mathcal{T}^r_s(\tau)$. Elements of $\mathcal{T}^r_s(\tau)$ will usually be mentioned as *tensor fields of type* (r,s) *along* τ. As in the special cases discussed in 2.17 and 2.21, any tensor in $\mathcal{T}^r_s(\tau)$ may be identified with a smooth map

$$\underline{A} : TM \longrightarrow \mathbf{T}^r_s M (:= \mathbf{T}^r_s TM) \text{ satisfying } \tau^r_s \circ \underline{A} = \tau$$

(τ^r_s is the canonical projection of $\mathbf{T}^r_s M$ onto M). In particular, *the basic tensor fields* along τ are of the form $A \circ \tau$, $A \in \mathcal{T}^\bullet_\bullet(M)$, denoted by $\widehat{A}$. $\widehat{A}$ is also called the *lift of* A into $\mathcal{T}^\bullet_\bullet(\tau)$.

(**2**) In view of 1.23(4b),

$$\begin{aligned}\mathcal{A}^k(\tau) &:= \mathcal{A}^k(\tau^*\tau) := \Gamma(\wedge^k(\tau^*\tau)^*) \cong \wedge^k_{C^\infty(TM)}\Gamma(\tau^*\tau) \\ &\cong \wedge^k_{C^\infty(TM)}(\Gamma(\tau^*\tau))^* = \wedge^k_{C^\infty(TM)}[\mathfrak{X}(\tau)]^* \cong \left(L^k_{\text{skew}}\right)_{C^\infty(TM)}\mathfrak{X}(\tau)\end{aligned}$$

($k \in \mathbb{N}$, $\mathcal{A}^0(\tau) := C^\infty(TM)$), so elements of $\mathcal{A}^k(\tau)$ may be regarded as skew-symmetric $C^\infty(TM)$-multilinear maps

$$\underbrace{\mathfrak{X}(\tau) \times \cdots \times \mathfrak{X}(\tau)}_{k \text{ times}} \longrightarrow C^\infty(TM).$$

$\mathcal{A}^k(\tau)$ is called the *module of k-forms along* τ, $\mathcal{A}(\tau) := \bigoplus_{k=1}^{n} \mathcal{A}^k(\tau)$ is the *Grassmann algebra of differential forms along* τ (with multiplication given by the wedge product). $\mathcal{B}(\tau) := \mathcal{A}(\tau) \otimes \mathfrak{X}(\tau)$ is the $C^\infty(TM)$-module of $\tau^*\tau$-valued forms, or simply *vector-valued forms along* τ. So a vector-valued k-form along τ may be interpreted as a skew-symmetric $C^\infty(TM)$-multilinear map

$$\underbrace{\mathfrak{X}(\tau) \times \cdots \times \mathfrak{X}(\tau)}_{k \text{ times}} \longrightarrow \mathfrak{X}(\tau),$$

if $k \in \mathbb{N}^*$; $\mathcal{B}^0(\tau) := \mathfrak{X}(\tau)$.

Lemma 1. *Let $k \in \mathbb{N}^*$. $\mathcal{A}^k(\tau)$ is canonically isomorphic to the $C^\infty(TM)$-module $\mathcal{A}_0^k(TM)$ of semibasic k-forms on TM, the canonical isomorphism being given by*

$$\begin{cases} \widetilde{\alpha} \in \mathcal{A}^k(\tau) \mapsto (\widetilde{\alpha})_0 \in \mathcal{A}_0^k(TM), \\ (\widetilde{\alpha})_0(\xi_1, \dots, \xi_k) := \widetilde{\alpha}(\mathbf{j}\xi_1, \dots, \mathbf{j}\xi_k) \quad (\xi_i \in \mathfrak{X}(TM),\ 1 \leqq i \leqq k). \end{cases}$$

In the same way, there is a canonical isomorphism

$$\boxed{\mathcal{B}^k(\tau) \cong \mathcal{B}_0^k(TM)},$$

given by

$$\begin{cases} \widetilde{A} \in \mathcal{B}^k(\tau) \mapsto (\widetilde{A})_0 \in \mathcal{B}_0^k(TM) \\ (\widetilde{A})_0(\xi_1, \dots, \xi_k) := \mathbf{i}\widetilde{A}(\mathbf{j}\xi_1, \dots, \mathbf{j}\xi_k) \quad (\xi_i \in \mathfrak{X}(TM),\ 1 \leqq i \leqq k). \end{cases}$$

Finally, the elements of $\mathcal{T}_s^0(\tau)$ and $\mathcal{T}_s^1(\tau)$ may be identified with the semibasic tensors in $\mathcal{T}_s^0(TM)$ and $\mathcal{T}_s^1(TM)$ respectively, by the same construction.

The proof is essentially the same as the proof of the analogous Lemma in 2.21 and is omitted. Notice, carrying on the analogy, that under the isomorphism $\mathcal{A}^k(\tau) \cong \mathcal{A}_0^k(TM)$ the basic k-forms along τ and the vertical lift of k-forms on M correspond to each other:

$$(\widehat{\alpha})_0 = \alpha^{\mathrm{v}} \quad \text{for all } \alpha \in \mathcal{A}^k(M).$$

Lemma 2. *Let $k \in \mathbb{N}^*$. For any $(k+1)$-form $\alpha \in \mathcal{A}^{k+1}(M)$, the map*

$$\begin{cases} \bar{\alpha} : TM \longrightarrow \mathbf{A}_k(TM),\ v \mapsto \bar{\alpha}_v \in \mathbf{A}_k(T_{\tau(v)}M) \\ \bar{\alpha}_v(v_1, \dots, v_k) := \alpha_{\tau(v)}(v, v_1, \dots, v_k) \ (v_i \in T_{\tau(v)}M,\ 1 \leqq i \leqq k) \end{cases}$$

is a k-form along τ. In particular, if $k = 1$ then

$$\bar{\alpha} : TM \longrightarrow \mathbb{R},\ v \mapsto \bar{\alpha}(v) := \alpha_{\tau(v)}(v)$$

is a smooth function on TM. More specifically,

$$\overline{df} = f^c \quad \text{for all } f \in C^\infty(M).$$

Similarly, for any vector-valued $(k+1)$-form $A \in \mathcal{B}^{k+1}(M)$, the map

$$\begin{cases} \bar{A} : v \in TM \mapsto \bar{A}_v \in L^k_{\mathrm{skew}}(T_{\tau(v)}M, T_{\tau(v)}M), \\ \bar{A}_v(v_1, \dots, v_k) := A_{\tau(v)}(v, v_1, \dots, v_k) \end{cases}$$

is a vector-valued k-form along τ.

Proof. Routine verification.

Note. Tensor fields along the tangent bundle projection are also called *Finsler tensor fields*, see e.g. Z. I. Szabó's paper [71]. This concept of Finsler tensor fields is essentially the same as M. Matsumoto's concept in the principal bundle framework.

2.23. Finsler metrics, Sasaki lift, Kähler lift.

Definition 1. A *Finsler metric* on a manifold M is a pseudo-Riemannian or Riemannian metric in the pull-back bundle $\tau^*\tau$, i.e., a type $(0,2)$ tensor field along τ, having the following properties:

F 1. g is *symmetric*, i.e., $g(\widetilde{X},\widetilde{Y}) = g(\widetilde{Y},\widetilde{X})$ for any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ.

F 2. g is *non-degenerate*, i.e., if $g(\widetilde{X},\widetilde{Y}) = 0$ for any vector field $\widetilde{Y}$ along τ, then $\widetilde{X} = 0$

or

F* 2. g is *positive definite*, i.e., for any $v \in TM$, $g_v : T_{\tau(v)}M \times T_{\tau(v)}M \longrightarrow \mathbb{R}$ is a positive definite bilinear form.

If g satisfies the stronger condition **F* 2**, we say that g is a *positive definite Finsler metric* on M. A manifold endowed with a Finsler metric is said to be a *generalized Finsler manifold*. If g is a positive definite Finsler metric on M, the (M,g) – or simply M – is called a *positive definite generalized Finsler manifold*. In a generalized Finsler manifold (M,g) the real number

$$\|w\|_v := \sqrt{|g_v(w,w)|}$$

is said to be the *norm* or *length* of the vector

$$w \in (TM \times_M TM)_v = \{v\} \times T_{\tau(v)}M$$

with respect to the 'supporting element' v; $\|w\| := \|w\|_w$ is also mentioned as the *absolute length* of w.

Example. If (M, g_M) is a pseudo-Riemannian manifold then $(M, \widehat{g}_M)$ is a generalized Finsler manifold. From another direction, if (M,g) is a generalized Finsler manifold and there is a pseudo-Riemannian metric g_M on M such that $g = \widehat{g}_M$, then we say that (M,g) *reduces to a pseudo-Riemannian manifold.*

Remark. In accordance with a more systematic view on generalized Finsler manifolds and Finsler manifolds in Chapter 3, *smoothness of a Finsler metric will be required, only over the deleted bundle* $\overset{\circ}{\tau}{}^*\tau$.

Definition 2 and lemma. Suppose that $\mathcal{H}$ is a horizontal map on the manifold M. Let $\mathcal{V}$ and $\mathbf{F}$ be the vertical map and the almost complex structure belonging to $\mathcal{H}$, respectively. Consider a symmetric tensor field $g \in \mathcal{T}_2^0(\tau)$ along τ.

(i) The map g^S given by

$$g^S(\xi,\eta) := g(\mathcal{V}\xi,\mathcal{V}\eta) + g(\mathbf{j}\xi,\mathbf{j}\eta) \quad \text{for all } \xi,\eta \in \mathfrak{X}(TM)$$

is a symmetric tensor field of type $(0,2)$ on TM, called the *Sasaki lift* of g. If g is a Finsler metric on M, then g^S is a pseudo-Riemannian metric on TM.

(ii) Let

$$g^K(\xi,\eta) := g^S(\xi,\mathbf{F}\eta) \quad \text{for all } \xi,\eta \in \mathfrak{X}(TM).$$

Then g^K is a two-form on TM, called the *Kähler lift* of g.

Concerning the Sasaki lift and the Kähler lift, we have the following relations:

(1) $g^S(X^{\mathrm{v}},Y^{\mathrm{v}}) = g^S(X^h,Y^h) = g(\widehat{X},\widehat{Y}), \quad g^S(X^{\mathrm{v}},Y^h) = 0$ for all $X,Y \in \mathfrak{X}(M)$.

(2) $g^S(\xi,\eta) = g^S(\mathbf{F}\xi,\mathbf{F}\eta)$ $(\xi,\eta \in \mathfrak{X}(TM))$, i.e., g^S is 'Hermitian' with respect to the almost complex structure $\mathbf{F}$.

(3) $g^K(X^{\mathrm{v}},Y^{\mathrm{v}}) = g^K(X^h,Y^h) = 0, \quad g^K(X^{\mathrm{v}},Y^h) = -g^K(Y^h,X^{\mathrm{v}})$ $(X,Y \in \mathfrak{X}(M))$, therefore g^K is indeed a two-form on TM.

Proof. It is enough to verify that the formulae (1)–(3) are valid. Applying the results collected in 2.19, we get

$$g^S(X^{\mathrm{v}},Y^{\mathrm{v}}) = g(\mathcal{V}\circ\mathbf{i}\widehat{X},\mathcal{V}\circ\mathbf{i}\widehat{Y}) + g(\mathbf{j}\circ\mathbf{i}\widehat{X},\mathbf{j}\circ\mathbf{i}\widehat{Y}) = g(\widehat{X},\widehat{Y}),$$

$$g^S(X^h,Y^h) = g(\mathcal{V}\circ\mathcal{H}\widehat{X},\mathcal{V}\circ\mathcal{H}\widehat{Y}) + g(\mathbf{j}\circ\mathcal{H}\widehat{X},\mathbf{j}\circ\mathcal{H}\widehat{Y}) = g(\widehat{X},\widehat{Y}),$$

$$g^S(X^{\mathrm{v}},Y^h) = g(\mathcal{V}\circ\mathbf{i}\widehat{X},\mathcal{V}\circ\mathcal{H}\widehat{Y}) + g(\mathbf{j}\circ\mathbf{i}\widehat{X},\mathbf{j}\circ\mathcal{H}\widehat{Y}) = 0,$$

whence (1). Taking into account that

$$\mathbf{F}X^{\mathrm{v}} = \mathbf{F}\circ JX^h = \mathbf{h}X^h = X^h \quad \text{and} \quad \mathbf{F}X^h = \mathbf{F}\circ\mathbf{h}X^h = -JX^h = -X^{\mathrm{v}},$$

we obtain

$$
\begin{aligned}
g^S(\mathbf{F}X^{\mathrm{v}}, \mathbf{F}Y^{\mathrm{v}}) &= g^S(X^h, Y^h) = g(X^{\mathrm{v}}, Y^{\mathrm{v}}),\\
g^S(\mathbf{F}X^h, \mathbf{F}Y^h) &= g^S(-X^{\mathrm{v}}, -Y^{\mathrm{v}}) = g^S(X^{\mathrm{v}}, Y^{\mathrm{v}}),\\
g^S(\mathbf{F}X^{\mathrm{v}}, \mathbf{F}Y^h) &= g^S(X^h, -Y^{\mathrm{v}}) = 0 = g^S(X^{\mathrm{v}}, Y^h),
\end{aligned}
$$

which proves (2). In a similarly way,

$$
g^K(X^{\mathrm{v}}, Y^{\mathrm{v}}) := g^S(X^{\mathrm{v}}, \mathbf{F}Y^{\mathrm{v}}) = g^S(X^{\mathrm{v}}, Y^h) = 0,\ \ g^K(X^h, Y^h) = 0.
$$

Finally,

$$
\begin{aligned}
g^K(X^{\mathrm{v}}, Y^h) := g^S(X^{\mathrm{v}}, \mathbf{F}Y^h) = -g^S(X^{\mathrm{v}}, Y^{\mathrm{v}}) = -g^S(Y^{\mathrm{v}}, X^{\mathrm{v}}) \overset{(1)}{=}&\\
= -g^S(Y^h, X^h) = -g^S(Y^h, \mathbf{F}X^{\mathrm{v}}) =: -g^K(Y^h, X^{\mathrm{v}}).&
\end{aligned}
$$

This concludes the proof of the lemma.

D. The theory of A. Frölicher and A. Nijenhuis

In this section E is an m-dimensional manifold satisfying the conditions fixed in 1.4.

2.24. Generalities on graded derivations of $\mathcal{A}(E)$. The purely algebraic concept of a graded derivation of a graded algebra is presented in the Appendix, see A.7. In the case of the Grassmann algebra $\mathcal{A}(E)$ the smooth structure of E involves some important analytical consequences to be compared with properties (1) and (2) of tensor derivations in 1.32.

Lemma 1. *Graded derivations of the Grassmann algebra $\mathcal{A}(E)$ are local operators: if $\mathcal{D} : \mathcal{A}(E) \longrightarrow \mathcal{A}(E)$ is a graded derivation, then for each open subset $\mathcal{U}$ of E and each differential form $\alpha \in \mathcal{A}(E)$ such that $\alpha \restriction \mathcal{U} = 0$, we have $(\mathcal{D}\alpha) \restriction \mathcal{U} = 0$.*

The proof is identical with the proof of the analogous statement for tensor derivations.

Lemma 2. *Graded derivations of the Grassmann algebra $\mathcal{A}(E)$ are natural with respect to restrictions in the sense of 1.32(2).*

Strategy of proof. Let $\mathcal{D} : \mathcal{A}(E) \longrightarrow \mathcal{A}(E)$ be a graded derivation. Suppose that $\mathcal{U}$ is a nonempty open subset of E and let α be a differential form on $\mathcal{U}$. Choose a point $p \in \mathcal{U}$. Let $f \in C^\infty(E)$ be a 'bump function' at p such that

$$\begin{cases} \operatorname{Im} f = [0,1], \ \operatorname{supp}(f) \subset \mathcal{U}, \\ f \text{ is identically 1 in a neighbourhood of } p. \end{cases}$$

Define the operator $\mathcal{D}_\mathcal{U}$ as follows:

$$\begin{cases} (\mathcal{D}_\mathcal{U}\alpha)(p) := (\mathcal{D}\beta)(p), \ \text{where} \\ \beta := (f \restriction \mathcal{U})\alpha \text{ over } \mathcal{U}, \text{ and } \beta := 0 \text{ outside } \mathcal{U}. \end{cases}$$

Then:

(1) $\mathcal{D}_\mathcal{U}$ is well-defined, i.e., it does not depend on the choice of the bump function f. (This is a consequence of Lemma 1.)

(2) $\mathcal{D}_\mathcal{U}$ is a graded derivation of $\mathcal{A}(\mathcal{U})$.

(3) $\mathcal{D}_\mathcal{U}$ has the desired naturality property.

(4) $\mathcal{D}_{\mathcal{U}}$ is uniquely determined by the naturality condition. □

The operator $\mathcal{D}_{\mathcal{U}}$ is called the *restriction* of $\mathcal{D}$ to $\mathcal{U}$. By a less confusing abuse of notation, instead of $\mathcal{D}_{\mathcal{U}}$ we shall also write $\mathcal{D}$.

Proposition. *Every graded derivation on $\mathcal{A}(E)$ is determined by its action on $\mathcal{A}^0(E) = C^\infty(E)$ and $\mathcal{A}^1(E) \cong [\mathfrak{X}(E)]^* = \mathfrak{X}^*(E)$.*

Proof. Let $\mathcal{D}$ be a graded derivation of degree r on $\mathcal{A}(E)$. Due to Lemma 2, we can restrict ourselves to a chart $(\mathcal{U}, (x^i)_{i=1}^m)$ on E. Since $\mathcal{D}$ is additive, it is sufficient to consider only k-forms of the form

$$\alpha = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad f \in C^\infty(\mathcal{U}).$$

Then, taking into account Appendix A.7, Lemma 1,

$$\mathcal{D}(f\alpha) = (\mathcal{D}f) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + f\mathcal{D}(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = (\mathcal{D}f) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$
$$+ f \sum_{j=1}^{k} (-1)^{r(j-1)} dx^{i_1} \wedge \cdots \wedge \mathcal{D}dx^{i_j} \wedge \cdots \wedge dx^{i_k},$$

which proves the Proposition.

Corollary. *Every map $\mathcal{A}^0(E) \oplus \mathcal{A}^1(E) \longrightarrow \mathcal{A}(E)$ satisfying the formal rules for a graded derivation can be uniquely extended to a graded derivation of $\mathcal{A}(E)$.* □

2.25. Algebraic derivations.

Definition. A graded derivation $\mathcal{D}$ of $\mathcal{A}(E)$ is said to be *algebraic* or of *type* i_* if it acts trivially on the smooth functions, i.e., $\mathcal{D} \restriction C^\infty(E) = 0$.

Note. Substitution operators i_ξ ($\xi \in \mathfrak{X}(E)$, see 1.38) are the prototypes of algebraic graded derivations, hence the term 'type i_*'.

Lemma. *Any algebraic derivation is determined by its action on the module of one-forms.*

Proof. Let $\mathcal{D} : \mathcal{A}(E) \longrightarrow \mathcal{A}(E)$ be an algebraic derivation of degree r. First observe that $\mathcal{D}$ is $C^\infty(E)$-linear: for every function $f \in C^\infty(E)$ and differential form $\alpha \in \mathcal{A}(E)$ we have

$$\mathcal{D}(f\alpha) = \mathcal{D}(f \wedge \alpha) = (\mathcal{D}f) \wedge \alpha + (-1)^{0 \cdot r} f \wedge (\mathcal{D}\alpha) = f \wedge (\mathcal{D}\alpha) = f(\mathcal{D}\alpha).$$

Since every differential form may be represented locally as a $C^\infty(E)$-linear combination of one-forms and $\mathcal{D}$ is natural with respect to restrictions, the assertion immediately follows. □

Note. We see that the algebraic derivations have a tensorial character, hence the attribute 'algebraic'.

Proposition. *If $L \in \mathcal{B}^\ell(E)$ is a vector-valued ℓ-from on E then the map*

$$i_L : \mathcal{A}(E) \longrightarrow \mathcal{A}(E), \;\; \alpha \mapsto i_L\alpha := \alpha \barwedge L$$

(see 1.37(4)) is an algebraic derivation of degree $\ell - 1$ of $\mathcal{A}(E)$. Conversely, if $\mathcal{D} : \mathcal{A}(E) \longrightarrow \mathcal{A}(E)$ is an algebraic derivation of degree $\ell - 1 \geqq -1$, then there is a unique vector-valued ℓ-from L on E such that $\mathcal{D} = i_L$.

Proof. Using the definition of the wedge-bar product introduced in 1.37(4), a somewhat lengthy but straightforward calculation shows that

$$i_L(\alpha \wedge \beta) := (\alpha \wedge \beta) \barwedge L = (\alpha \barwedge L) \wedge \beta + (-1)^{k(l-1)}\alpha \wedge (\beta \barwedge L)$$

for every $\alpha \in \mathcal{A}^k(E)$, $\beta \in \mathcal{A}(E)$; therefore i_L is a graded derivation of degree $\ell - 1$ of $\mathcal{A}(E)$.

Conversely, suppose that $\mathcal{D} : \mathcal{A}(E) \longrightarrow \mathcal{A}(E)$ is a graded derivation of degree $\ell - 1 \geqq -1$. Define a map $L : \mathcal{A}^1(E) \times [\mathfrak{X}(E)]^\ell \longrightarrow C^\infty(E)$ by the rule

$$L(\alpha, \xi_1, \dots, \xi_\ell) := (\mathcal{D}\alpha)(\xi_1, \dots, \xi_\ell); \quad \alpha \in \mathcal{A}^1(E), \; \xi_i \in \mathfrak{X}(E) \; (1 \leqq i \leqq \ell).$$

Then L is automatically $C^\infty(E)$-multilinear and skew-symmetric in its vector field variables. Since $\mathcal{D}$ is algebraic and hence tensorial, for any function $f \in C^\infty(E)$ we have

$$\begin{aligned} L(f\alpha, \xi_1, \dots, \xi_\ell) &:= [\mathcal{D}(f\alpha)](\xi_1, \dots, \xi_\ell) = f(\mathcal{D}\alpha)(\xi_1, \dots, \xi_\ell) =: \\ &= fL(\alpha, \xi_1, \dots, \xi_\ell), \end{aligned}$$

therefore $L \in \mathcal{A}^\ell(E, \tau_E) = \mathcal{B}^\ell(E)$.

Now, in view of 1.37(4), Example (2) for any one-form $\alpha \in \mathcal{A}^1(E)$ and vector fields $\xi_1, \dots, \xi_\ell$ on E we have

$$\begin{aligned} i_L\alpha(\xi_1, \dots, \xi_\ell) &= \alpha \barwedge L(\xi_1, \dots, \xi_\ell) = \alpha(L(\xi_1, \dots, \xi_\ell)) = L(\alpha, \xi_1, \dots, \xi_\ell) \\ &= (\mathcal{D}\alpha)(\xi_1, \dots, \xi_\ell); \end{aligned}$$

consequently $i_L\alpha = \mathcal{D}\alpha$, as was to be proved. The uniqueness of the desired vector-valued ℓ-form is clear. □

2.26. The main theorem.

Lemma and definition. (1) *Let K be a vector-valued k-form on E. Then*

$$d_K := [i_K, d] := i_K \circ d - (-1)^{k-1} d \circ i_K$$

is a graded derivation of degree k on $\mathcal{A}(E)$, called the Lie derivative with respect to K *or a* (graded) derivation of type d_* *of* $\mathcal{A}(E)$.

(2) *The map $K \in \mathcal{A}(E) \mapsto d_K \in \operatorname{Der} \mathcal{A}(E)$ is an injective map from the Grassmann algebra $\mathcal{A}(E)$ into the graded Lie algebra of graded derivations of $\mathcal{A}(E)$.*

Proof. The purely algebraic first assertion is a consequence of A.7, Lemma 2. As for the second assertion, for any function $f \in C^\infty(E)$ we have

$$d_K f = (i_K \circ d - (-1)^k d \circ i_K) f = i_K df := df \barwedge K = df \circ K$$

(see 1.37(4), Example (2)). Now it is readily seen that $d_{K_1} = d_{K_2}$, implies $K_1 = K_2$. □

Theorem. (A. Frölicher and A. Nijenhuis). (1) *Every graded derivation $\mathcal{D}$ of $\mathcal{A}(E)$ of degree k can be written uniquely as $\mathcal{D} = d_K + i_L$, where K is a vector-valued k-form, L is a vector-valued $k+1$-form on E.*

(2) *In a decomposition $\mathcal{D} = d_K + i_L$ L vanishes if, and only if, $[\mathcal{D}, d] = 0$; K vanishes if, and only if, $\mathcal{D}$ is an algebraic derivation.*

(3) *If $K \in \mathcal{B}^0(E)$, i.e., if K is a vector field ξ on E, then d_K coincides with the 'ordinary' Lie derivative d_ξ; if $L \in \mathcal{B}^0(E)$, i.e., if L is a vector field η on E, then i_L coincides with the substitution operator induced by L.*

(4) *If L is a vector-valued one-form on E, i.e.*
$L \in \mathcal{B}^1(E) := \mathcal{A}^1(E, \tau_E) \cong \mathcal{T}^1_1(E)$, then for every k-form α on E we have

$$i_L \alpha(\xi_1, \dots, \xi_k) = \sum_{j=1}^{k} \alpha(\xi_1, \dots, L\xi_j, \dots, \xi_k) \quad (\xi_i \in \mathfrak{X}(E),\ 1 \leqq i \leqq k).$$

In particular, $i_{\iota_E} \alpha = k\alpha$.

Proof. (1) Let $\xi_i \in \mathfrak{X}(E)$ $(1 \leqq i \leqq k)$ be fixed but arbitrarily chosen vector fields on E. The map

$$f \in C^\infty(E) \mapsto (\mathcal{D}f)(\xi_1, \dots, \xi_k) \in C^\infty(E)$$

is obviously a derivation of the real algebra $C^\infty(E)$, so by 1.26 it comes from a unique vector field on E. It is reasonable to denote this vector field by $K(\xi_1, \dots, \xi_k)$. Then

$$(\mathcal{D}f)(\xi_1, \dots, \xi_k) = K(\xi_1, \dots, \xi_k) f = df(K(\xi_1, \dots, \xi_k)) \quad \text{for all } f \in C^\infty(E).$$

Thus we obtain a map

$$K : [\mathfrak{X}(E)]^k \longrightarrow \mathfrak{X}(E),\ (\xi_1, \dots, \xi_k) \mapsto K(\xi_1, \dots, \xi_k),$$

which is clearly $C^\infty(E)$-multilinear and skew-symmetric; therefore K is a vector-valued k-form on E. By the construction of K, for any function $f \in C^\infty(E)$ we have $\mathcal{D}f = df \circ K = d_K f$ (see the proof of the Lemma). From this it follows that $\mathcal{D} - d_K$ is an algebraic derivation of degree k of $\mathcal{A}(E)$. Thus, in view of the Proposition in 2.25, there is a unique vector-valued $(k+1)$-form L on E such that $\mathcal{D} - d_K = i_L$. Hence $\mathcal{D} = d_K + i_L$; this proves that $\mathcal{D}$ has the desired decomposition. The uniqueness of the vector-valued forms K and L is clear from the argument.

(2) Now we turn to the second assertion. Using the graded Jacobi identity (see A.7 (**3**)), the fact $[d, d] = 0$ (1.40(6)), the definition of d_K, and the graded anticommutativity, we have

$$\begin{aligned} 0 &= (-1)^{k-1}[i_K, [d, d]] + (-1)^{k-1}[d, [d, i_K]] + (-1)^{1\cdot 1}[d, [i_K, d]] \\ &= (-1)^{k-1} \cdot (-1)^k [d, [i_K, d]] - [d, d_K] = -2[d, d_K] = 2(-1)^k [d_K, d]. \end{aligned}$$

Hence

$$[\mathcal{D}, d] = [d_K, d] + [i_L, d] = [i_L, d] =: d_L.$$

In view of the preparatory Lemma, the map $L \mapsto d_L$ is injective. So we may conclude that $[\mathcal{D}, d] = 0$ if, and only if, $L = 0$.

If $K = 0$, then $\mathcal{D} = d_K + i_L$ is obviously an algebraic derivation. Suppose, conversely, that $\mathcal{D}$ is algebraic. Then for any function $f \in C^\infty(E)$ we have

$$0 = \mathcal{D}f = d_K f + i_L f = d_K f = i_K df.$$

In virtue of the Proposition in 2.24, this implies that $i_K = 0$, whence $K = 0$. This concludes the proof of assertion (2).

(3) If $L := \eta \in \mathfrak{X}(E) = \mathcal{B}^0(E)$, then, in view of Example (2) of 1.37(4),

$$i_\eta \alpha := \alpha \,\bar{\wedge}\, \eta = \alpha(\eta) \quad \text{for any one-form } \alpha \text{ on } E,$$

so we obtain the substitution operator induced by η. If
$K := \xi \in \mathfrak{X}(E) = \mathcal{B}^0(E)$, then $d_\xi := [i_\xi, d]$ is the usual Lie derivative with respect to ξ, cf. 1.40(3).

(4) Our last assertion is merely a restatement of the result 1.37(4), Example (1). □

Remark. $d_{\iota_E} = d$. Indeed, for any k-form α on E we have

$$d_{\iota_E}\alpha := [i_{\iota_E}, d]\alpha = i_{\iota_E} d\alpha - d i_{\iota_E}\alpha = (k+1)d\alpha - kd\alpha = d\alpha.$$

2.27. Summary. For the reader's convenience and for easy reference, we summarize the main results of subsections 2.24–2.26.

1. The graded derivations of the Grassmann algebra $\mathcal{A}(E)$ are local and, with respest to restrictions, natural operators.

2. Every graded derivation is determined by its action on the ring of smooth functions and on their differentials.

3. Each graded derivation $\mathcal{D}$ of degree $\ell - 1 \geqq -1$ that acts trivially on $\mathcal{A}^0(E) = C^\infty(E)$ is uniquely determined by a vector-valued form L of degree ℓ over E, namely $\mathcal{D} = i_L$. If α is a one-form on E, then

$$i_L\alpha(\xi_1, \dots, \xi_\ell) = \alpha\big(L(\xi_1, \dots, \xi_\ell)\big) \quad (\xi_i \in \mathfrak{X}(E),\ 1 \leqq i \leqq \ell),$$

and i_L is determined by this rule.

4. Each graded derivation $\mathcal{D}$ of degree k which commutes with d, i.e. satisfies $\mathcal{D} \circ d = (-1)^k d \circ \mathcal{D}$, is uniquely determined by a vector-valued form K of degree k such that

$$\mathcal{D} = d_K = [i_K, d] = i_K \circ d - (-1)^{k-1} d \circ i_K.$$

For any function $f \in C^\infty(E)$ we have $d_K f = i_K \circ df$, and d_K is determined by this relation.

5. Every graded derivation of degree k of $\mathcal{A}(E)$ is the sum of two graded derivations of degree k, one of each of the kinds mentioned in **3** and **4**.

2.28. The Frölicher–Nijenhuis bracket.

Lemma. *The graded commutator of two algebraic (resp. Lie) derivations is also an algebraic (resp. Lie) derivation.*

Proof. Let $K \in \mathcal{B}^k(E)$, $L \in \mathcal{B}^\ell(E)$. In view of 2.27, **3**, **4** it is enough to show that

$$[i_K, i_L] \restriction C^\infty(E) = 0 \quad \text{and} \quad \big[[d_K, d_L], d\big] = 0.$$

The first of these relations is obvious. Applying the graded Jacobi identity and 2.27, **4**, we obtain that

$$\begin{aligned} 0 &= (-1)^\ell[d, [d_K, d_L]] + (-1)^k\big[d_K, [d_L, d]\big] + (-1)^{k\ell}\big[d_L, [d, d_K]\big] \\ &= (-1)^\ell\big[d, [d_K, d_L]\big]. \end{aligned}$$

This proves the second relation.

Corollary 1 and definition. *For any two vector-valued forms $K \in \mathcal{B}^k(E)$, $L \in \mathcal{B}^\ell(E)$ there exists a unique vector-valued $(k+\ell)$-form $[K, L]$ on E such that*

$$[d_K, d_L] = d_{[K,L]}.$$

The vector-valued form $[K, L]$ is said to be the Frölicher–Nijenhuis bracket *of K and L.*

Proof. In view of the previous Lemma, $[d_K, d_L]$ is a Lie derivation of degree $k+\ell$ of $\mathcal{A}(E)$. Then 2.27, **4** guarantees that there exists a unique vector-valued $(k+\ell)$-form $F \in \mathcal{B}^{k+\ell}(E)$ such that $[d_K, d_L] = d_F$. Now with the only possible choice $[K, L] := F$ our claim follows.

Proposition. (1) *The real vector space $\mathcal{B}(E) := \overset{m}{\underset{k=0}{\oplus}} \mathcal{B}^k(E)$ of vector-valued forms on E is a graded Lie algebra with the multiplication given by the Frölicher–Nijenhuis bracket. Thus for any vector-valued forms $K_i \in \mathcal{B}^{k_i}(E)$ $(1 \leqq i \leqq 3)$ we have the graded anticommutativity*

$$[K_1, K_2] = -(-1)^{k_1 k_2}[K_2, K_1]$$

and the graded Jacobi identity

$$(-1)^{k_1 k_3}[K_1, [K_2, K_3]] + (-1)^{k_2 k_1}[K_2, [K_3, K_1]] + (-1)^{k_3 k_2}[K_3, [K_1, K_2]] = 0.$$

(2) *The unit tensor field $\iota_E \in \mathcal{T}^1_1(E) \cong \mathcal{B}^1(E)$ is in the centre of the algebra $\mathcal{B}(E)$, i.e.,*

$$[K, \iota_E] = 0 \quad \textit{for all } K \in \mathcal{B}(E).$$

(3) *The map $K \in \mathcal{B}(E) \mapsto d_K \in \operatorname{Der} \mathcal{A}(E)$ is an injective homomorphism of graded Lie algebras.*

(4) *If $K, L \in \mathcal{B}^0(E) = \mathfrak{X}(E)$, then $[K, L]$ is the usual Lie bracket of vector fields.*

(5) *If $K \in \mathcal{B}^0(E) = \mathfrak{X}(E)$, $L \in \mathcal{B}^\ell(E)$, then $[K, L]$ is the Lie derivative of L with respect to K.*

Proof. (1) $d_{[K_1,K_2]} := [d_{K_1}, d_{K_2}] = d_{K_1} \circ d_{K_2} - (-1)^{k_1 k_2} d_{K_2} \circ d_{K_1} =$ $-(-1)^{k_1 k_2}(d_{K_2} \circ d_{K_1} - (-1)^{k_2 k_1} d_{K_1} \circ d_{K_2}) = -(-1)^{k_1 k_2}[d_{K_2}, d_{K_1}] =:$ $d_{-(-1)^{k_1 k_2}[K_2,K_1]}$, whence the graded anticommutativity. The graded Jacobi identity may be obtained in the same way.

(2) Taking into account the Remark in 2.26, we have

$$d_{[K,\iota_E]} = [d_K, d_{\iota_E}] = [d_K, d] \overset{2.27,\, \mathbf{4}}{=} 0.$$

(3) This property is an immediate consequence of the definitions.

(4) For the moment let us denote $[\ ,\ ]^{\mathrm{FN}}$ and $[\ ,\ ]^{\mathrm{Lie}}$ the Frölicher–Nijenhuis bracket and the Lie-bracket, respectively. Then for any vector-valued null-forms $K := \xi$, $L := \eta$ on E we have

$$d_K f \overset{2.27,\,\mathbf{4}}{=} i_K \circ df = df(K) = \xi f,$$

and similarly, $d_L f = \eta f$, $d_{[K,L]^{\mathrm{FN}}} f = [\xi,\eta]^{\mathrm{FN}} f$. On the other hand,

$$\begin{aligned} d_{[K,L]^{\mathrm{FN}}} f &= [d_K, d_L] f = [d_\xi, d_\eta] f = d_\xi(d_\eta f) - d_\eta(d_\xi f) \\ &= d_\xi(\eta f) - d_\eta(\xi f) = \xi(\eta f) - \eta(\xi f) =: [\xi,\eta]^{\mathrm{Lie}} f, \end{aligned}$$

therefore $[\xi,\eta]^{\mathrm{FN}} = [\xi,\eta]^{\mathrm{Lie}}$.

(5) Again, let $\xi \in \mathfrak{X}(E)$ and $K := \xi$. Then, on the one hand, for any function $f \in C^\infty(E)$ we have

$$d_{[\xi,L]} f = i_{[\xi,L]} df = df \circ [\xi, L].$$

Hence, for any ℓ-tuple $(\xi_1,\dots,\xi_\ell)$ ($\xi_i \in \mathfrak{X}(E)$, $1 \leqq i \leqq \ell$),

$$\big(d_{[\xi,L]} f\big)(\xi_1,\dots,\xi_\ell) = df([\xi,L](\xi_1,\dots,\xi_\ell)) = ([\xi,L](\xi_1,\dots,\xi_\ell))f.$$

On the other hand,

$$\begin{aligned} d_{[\xi,L]} f &= [d_\xi, d_L] f = d_\xi(i_L df) - d_L(\xi f) = d_\xi(i_L df) - i_L d(\xi f) \\ &= d_\xi(i_L df) - d(\xi f) \barwedge L. \end{aligned}$$

Applying 1.33, Corollary (2) and 1.37(4), Example (2), we evaluate the right-hand side on the ℓ-tuple $(\xi_1,\dots,\xi_\ell)$. We obtain:

$$\begin{aligned} &(d_\xi(i_L df) - d(\xi f) \barwedge L)(\xi_1,\dots,\xi_\ell) = \xi((i_L df)(\xi_1,\dots,\xi_\ell)) \\ &\quad - \sum_{j=1}^{\ell} (i_L df)(\xi_1,\dots,[\xi,\xi_j],\dots,\xi_\ell) - d(\xi f)(L(\xi_1,\dots,\xi_\ell)) \\ &= \xi[df(L(\xi_1,\dots,\xi_\ell))] - L(\xi_1,\dots,\xi_\ell)(\xi f) - df\left(\sum_{j=1}^{\ell} L(\xi_1,\dots,[\xi,\xi_j],\dots,\xi_\ell)\right) \\ &= \xi(L(\xi_1,\dots,\xi_\ell)f) - L(\xi_1,\dots,\xi_\ell)(\xi f) - \left(\sum_{j=1}^{\ell} L(\xi_1,\dots,[\xi,\xi_j],\dots,\xi_\ell)\right) f \\ &= \Big([\xi, L(\xi_1,\dots,\xi_\ell)] - \sum_{j=1}^{\ell} L(\xi_1,\dots,[\xi,\xi_j],\dots,\xi_\ell)\Big) f = (d_\xi L)(\xi_1,\dots,\xi_\ell) f. \end{aligned}$$

Comparing the two results we conclude that $[\xi, L] = d_\xi L$. $\square$

Corollary 2. *If ξ is a vector-valued 0-form, i.e., a vector field, and K is a vector-valued one-form on E, then for all $\eta \in \mathfrak{X}(E)$ we have*

$$[\xi, K]\eta = (d_\xi K)\eta = [\xi, K(\eta)] - K[\xi, \eta], \ [K, \xi]\eta = [K(\eta), \xi] - K[\eta, \xi]. \quad \square$$

2.29. Fundamental formulae.

Theorem. *Let K, L and Q be vector-valued forms on E, of degree k, ℓ and q respectively. Then the following relations hold:*

1	$[i_K, i_L] = i_{L \barwedge K} - (-1)^{(k-1)(\ell-1)} i_{K \barwedge L}$
2	$[i_K, d_L] = d_{L \barwedge K} + (-1)^\ell i_{[K,L]}$
3	$[K, L] \barwedge Q = [K \barwedge Q, L] - (-1)^{\ell(k-1)} K \barwedge [Q, L] + (-1)^{k\ell+1}([L \barwedge Q, K] - (-1)^{k(\ell-1)} L \barwedge [Q, K])$

Proof. (1) Since the graded commutator of two algebraic derivations is algebraic (2.28, Lemma), it is enough to check that both sides of **1** act in the same way on $\mathcal{A}^1(E)$. This may be seen as follows.

For any one-form α on E we have on the one hand

$$\begin{aligned}[i_K, i_L]\alpha = i_K(i_L\alpha) - (-1)^{(k-1)(\ell-1)} i_L(i_K\alpha) &\overset{2.27,\, \mathbf{3}}{=} i_K(\alpha \circ L) \\ &- (-1)^{(k-1)(\ell-1)} i_L(\alpha \circ K).\end{aligned}$$

On the other hand, taking into account 1.37, Lemma,

$$i_{L \barwedge K}\alpha := \alpha \barwedge (L \barwedge K) = (\alpha \barwedge L) \barwedge K = (\alpha \circ L) \barwedge K = i_K(\alpha \circ L).$$

Similarly, $i_{K \barwedge L}\alpha = i_L(\alpha \circ K)$, so we obtain the desired formula.

(2) To verify formula **2**, observe first that for any function $f \in C^\infty(E)$,

$$\begin{aligned}[i_K, d_L]f &= i_K(d_L f) - (-1)^{(k-1)\ell} d_L(i_K f) = i_K(i_L df) = i_K(df \barwedge L) \\ &= (df \barwedge L) \barwedge K \overset{1.37,\ \text{Lemma}}{=} df \barwedge (L \barwedge K) = i_{L \barwedge K} df = d_{L \barwedge K} f,\end{aligned}$$

therefore $[i_K, d_L] - d_{L \barwedge K}$ acts trivially on $C^\infty(E)$, hence this operator is an algebraic derivation. On the other hand, applying in the first step the graded Jacobi identity and graded anticommutativity, we have

$$\begin{aligned}\big[[i_K, d_L], d\big] &= \big[i_K, [d_L, d]\big] + (-1)^{k\ell-\ell-1}\big[d_L, [i_K, d]\big] \overset{2.27,\, \mathbf{4}}{=} (-1)^{k\ell-\ell-1}[d_L, d_K] \\ &= (-1)^\ell [d_K, d_L] = (-1)^\ell d_{[K,L]} = (-1)^\ell \big[i_{[K,L]}, d\big].\end{aligned}$$

In virtue of the main theorem in 2.26, $[i_K, d_L]$ can be written uniquely as

$$[i_K, d_L] = d_F + i_G; \quad F, G \in \mathcal{B}(E);$$

hence $\big[[i_K, d_L], d\big] = [i_G, d]$. Since the map

$$G \in \mathcal{B}(E) \mapsto d_G := [i_G, d] \in \operatorname{Der} \mathcal{A}(E)$$

is injective, we may conclude that the algebraic part of $[i_K, d_L]$ is $(-1)^\ell i_{[K,L]}$. This ends the proof of formula **2**.

(3) For a proof of relation **3** the reader is referred to our main source, the Frölicher–Nijenhuis paper [33]. □

Corollary. *Let ξ and η be vector fields, K and L vector-valued one-forms on the manifold E. Then we have:*

(i) $\qquad [i_\xi, i_L] = i_{L(\xi)},$

(ii) $\qquad [i_K, i_L] = i_{L\circ K} - i_{K\circ L},$

(iiii) $\qquad [i_\xi, d_L] = d_{L(\xi)} - i_{[\xi,L]},$

(iv) $\qquad [i_K, d_L] = d_{L\circ K} - i_{[K,L]}.$ □

2.30. The Nijenhuis torsion.

Lemma. *The Frölicher–Nijenhuis bracket of two vector-valued one-forms K, L on E acts by the rule*

$$\boxed{\begin{aligned}[K,L](\xi,\eta) &= [K\xi,L\eta]+[L\xi,K\eta]+(K\circ L+L\circ K)[\xi,\eta]\\ &\quad -K[L\xi,\eta]-K[\xi,L\eta]-L[K\xi,\eta]-L[\xi,K\eta]\end{aligned}}$$

$(\xi,\eta\in\mathfrak{X}(E))$.

Proof. We start with the definition

$$d_{[K,L]}:=[d_K,d_L]=d_K\circ d_L+d_L\circ d_K.$$

The graded derivation $d_K\circ d_L+d_L\circ d_K$ is of type d_*, so it is determined by its action on the smooth functions on E. For any function $f\in C^\infty(E)$ we have

$$\begin{aligned}&d_K\circ d_L(f)+d_L\circ d_K(f)=d_K(i_L df)+d_L(i_K df)=d_K(df\circ L)+d_L(df\circ K)\\ &\quad=d_K\alpha+d_L\beta:=[i_K,d]\alpha+[i_L,d]\alpha=i_K d\alpha-di_K\alpha+i_L d\beta-di_L\beta,\end{aligned}$$

where $\alpha:=df\circ L$ and $\beta:=df\circ K$ are one-forms on E. We evaluate the four terms on the right-hand side on a pair $(\xi,\eta)\in\mathfrak{X}(E)\times\mathfrak{X}(E)$.

$$\begin{aligned}(i_K d\alpha)(\xi,\eta)&:=(d\alpha\,\bar\wedge\,K)(\xi,\eta)=d\alpha(K\xi,\eta)+d\alpha(\xi,K\eta)\\ &=(K\xi)\alpha(\eta)-\eta\alpha(K\xi)-\alpha[K\xi,\eta]+\xi\alpha(K\eta)-(K\eta)\alpha(\xi)-\alpha[\xi,K\eta]\\ &=(K\xi)df(L\eta)-\eta df(L\circ K(\xi))-df(L[K\xi,\eta])+\xi df(L\circ K(\eta))\\ &\quad-(K\eta)df(L\xi)-df(L[\xi,K\eta])=K\xi[L\eta(f)]-\eta((L\circ K(\xi)f)\\ &\quad-L[K\xi,\eta]f+\xi(L\circ K(\eta)f)-K\eta[L\xi(f)]-L[\xi,K\eta]f\end{aligned}$$

(we used 1.37(4), Example (1) and the definition of the operator d, see 1.39). So we have obtained

(1) $$\begin{aligned}(i_K d\alpha)(\xi,\eta)&=(K\xi)[(L\eta)f]-(K\eta)[(L\xi)f]-(L[K\xi,\eta]+L[\xi,K\eta])f\\ &\quad-\eta(L\circ K(\xi)f)+\xi(L\circ K(\eta)f).\end{aligned}$$

A similar calculation yields the following relations:

(2) $$-d(i_K\alpha)(\xi,\eta)=(L\circ K[\xi,\eta])f-\xi(L\circ K(\eta)f)+\eta(L\circ K(\xi)f),$$

(3) $$\begin{aligned}(i_L d\beta)(\xi,\eta)&=(L\xi)[(K\eta)f]-(L\eta)[(K\xi)(f)]\\ &\quad-(K[L\xi,\eta]+K[\xi,L\eta])f-\eta(K\circ L(\xi)f)+\xi(K\circ L(\eta)f),\end{aligned}$$

(4) $$-d(i_L\beta)(\xi,\eta)=(K\circ L[\xi,\eta])f-\xi(K\circ L(\eta)f)+\eta(K\circ L(\xi)f).$$

Adding both sides of (1)–(4), the last two terms on the right-hand side cancel in pairs, and we obtain the right-hand side of the desired relation. As for the left-hand side, we have

$$\big(d_{[K,L]}f\big)(\xi,\eta) = \big(i_{[K,L]}df\big)(\xi,\eta) = df([K,L](\xi,\eta)) = ([K,L](\xi,\eta))f.$$

Comparing the two results, the assertion follows.

Note. Applying the fundamental formula 2.29, **3**, a shorter proof of the Lemma is also available. However, we did not prove that formula, so we felt more convincing to provide an independent deduction.

Definition. Let K be a vector-valued one-form on E. The vector-valued two-form

$$N_K := \frac{1}{2}[K,K]$$

is said to be the *Nijenhuis torsion* of K.

Remark. Let $K \in \mathcal{B}^1(E)$. $N_K = 0$ *if, and only if,* $d_K^2 = 0$.

Indeed, on the one hand, $[d_K, d_K] = d_K \circ d_K - (-1)^{1\cdot 1} d_K \circ d_K = 2d_K^2$, on the other hand $[d_K, d_K] =: d_{[K,K]}$, therefore

$$d_K^2 = d_{\frac{1}{2}[K,K]} = d_{N_K}.$$

Corollary. *For the Nijenhuis torsion N_K of a vector-valued one-form $K \in \mathcal{B}^1(E)$ we have*

$$\boxed{N_K(\xi,\eta) = [K\xi, K\eta] + K^2[\xi,\eta] - K[K\xi,\eta] - K[\xi,K\eta] \textit{ for all } \xi,\eta \in \mathfrak{X}(E)}.$$

Example. Suppose that (E,π,M) is a vector bundle and $\mathcal{H}$ is a horizontal map for π. Let $\mathbf{h}$, $\mathbf{v}$ and Ω be the horizontal and the vertical projector belonging to $\mathcal{H}$ and the curvature of $\mathcal{H}$, respectively (see (1), (2) and (5) in 2.11). Then

$$\boxed{\Omega = -N_{\mathbf{h}} = -\frac{1}{2}[\mathbf{h},\mathbf{h}]}.$$

Indeed, for any vector fields ξ, η on E we have

$$\begin{aligned} N_{\mathbf{h}}(\xi,\eta) &= [\mathbf{h}\xi,\mathbf{h}\eta] + \mathbf{h}[\xi,\eta] - \mathbf{h}[\mathbf{h}\xi,\eta] - \mathbf{h}[\xi,\mathbf{h}\eta] = [\mathbf{h}\xi,\mathbf{h}\eta] \\ &+ \mathbf{h}[\mathbf{v}\xi,\mathbf{h}\eta + \mathbf{v}\eta] - \mathbf{h}[\xi,\mathbf{h}\eta] = [\mathbf{h}\xi,\mathbf{h}\eta] + \mathbf{h}[\mathbf{v}\xi,\mathbf{h}\eta] - \mathbf{h}[\xi,\mathbf{h}\eta] \\ &= [\mathbf{h}\xi,\mathbf{h}\eta] - \mathbf{h}[\mathbf{h}\xi,\mathbf{h}\eta] = \mathbf{v}[\mathbf{h}\xi,\mathbf{h}\eta] = -\Omega(\xi,\eta). \end{aligned}$$

Due to the graded Jacobi identity for the Frölicher–Nijenhuis bracket, it follows immediately that

$$\boxed{[\mathbf{h}, \Omega] = 0}.$$

This relation is called the *general Bianchi identity* for the nonlinear connection given by $\mathcal{H}$. (It may be easily shown that if $\mathbf{h}$ arises from a covariant derivative operator in π according to 2.15, Example 3, then the general Bianchi identity reduces to the differential Bianchi identity deduced in 1.44.)

For the tension (see 2.14, Lemma 3) of $\mathcal{H}$, as an immediate consequence of 2.28, Proposition (5) and 2.28, Corollary 2, we obtain the relation

$$\boxed{\mathbf{t} = [\mathbf{h}, C_E]}.$$

2.31. Vertical apparatus.

In this subsection we work on the tangent bundle (TM, τ, M) of an n-dimensional manifold M.

(**1**) We begin by summarizing some further basic facts about the vertical endomorphism J of TTM which we shall need.

1. *The Nijenhuis torsion N_J of J vanishes.*

To prove this, it is enough to check that N_J vanishes when its arguments are a pair of complete lifts; a vertical lift and a complete lift; and a pair of vertical lifts, respectively. Taking into account that for any vector fields X, Y on M:

$$JX^c = X^{\mathrm{v}}, \quad JX^{\mathrm{v}} = 0, \quad J[X^{\mathrm{v}}, Y^c] = J[X, Y]^{\mathrm{v}} = 0, \quad \text{etc.}$$

(see 2.20), and applying 2.30, Corollary, we obtain

$$\begin{aligned} N_J(X^c, Y^c) &= [JX^c, JY^c] + J^2[X^c, Y^c] - J[JX^c, Y^c] - J[X^c, JY^c] \\ &= [X^{\mathrm{v}}, Y^{\mathrm{v}}] - J[X^{\mathrm{v}}, Y^c] - J[X^c, Y^{\mathrm{v}}] = 0. \end{aligned}$$

Similarly,

$$N_J(X^{\mathrm{v}}, Y^c) = [JX^{\mathrm{v}}, JY^c] - J[JX^{\mathrm{v}}, Y^c] - J[X^{\mathrm{v}}, JY^c] = -J[X^{\mathrm{v}}, Y^{\mathrm{v}}] = 0,$$

and finally $N_J(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0$, as was to be shown.

2. $$\boxed{d_J^2 = 0}$$

This is obvious, since $d_J^2 = d_{N_J}$ (see 2.30, Remark).

3. For any vector field X on M we have

$$\boxed{[J, X^c] = [J, X^{\mathrm{v}}] = 0}.$$

Indeed, in virtue of 2.28, Corollary 2 and 2.20, Lemma 2,

$$\begin{aligned}[J, X^c]Y^c &= [JY^c, X^c] - J[Y^c, X^c] = [Y^{\mathrm{v}}, X^c] - J[Y, X]^c \\ &= [Y, X]^{\mathrm{v}} - [Y, X]^{\mathrm{v}} = 0,\end{aligned}$$

$$[J, X^c]Y^{\mathrm{v}} = [JY^{\mathrm{v}}, X^c] - J[Y^{\mathrm{v}}, X^c] = 0, \quad \text{therefore } [J, X^c] = 0.$$

A similar calculation verifies the second relation.

4. $$\boxed{[J, C] = J}$$

The straightforward proof is left to the reader.

5. *If $\eta \in \mathfrak{X}(TM)$ and $J\eta = C$, then $J[J\xi, \eta] = J\xi$ for all $\xi \in \mathfrak{X}(TM)$.*

Indeed, the vanishing of N_J leads to

$$0 = \frac{1}{2}[J, J](\xi, \eta) = [J\xi, J\eta] - J[J\xi, \eta] - J[\xi, J\eta] = [J\xi, C] - J[J\xi, \eta] - J[\xi, C],$$

hence $J[J\xi, \eta] = [J\xi, C] - J[\xi, C]$.

On the other hand, applying property **4**, we get

$$J\xi = [J, C]\xi \overset{2.28,\,\text{Cor. 2}}{=} [J\xi, C] - J[\xi, C];$$

whence the desired relation.

(**2**) According to the main results of the Frölicher–Nijenhuis theory (see 2.27), two graded derivations are associated to the vertical endomorphism
$J \in \mathcal{B}^1(TM) \cong \mathcal{A}^1(M, \tau)$: the algebraic derivation i_J of degree 0 and the Lie derivation

$$d_J := [i_J, d] = i_J \circ d + d \circ i_J$$

of degree 1. i_J and d_J are said to be the *vertical derivation* and the *vertical differentiation* on TM, respectively. In view of 2.27, these operators have the following characteristic properties:

$i_J \restriction C^\infty(TM) = 0$, $(i_J\alpha)(\xi_1, \ldots, \xi_k) = \sum_{i=1}^{k} \alpha(\xi_1, \ldots, J\xi_i, \ldots, \xi_k)$ $(\alpha \in \mathcal{A}^k(TM),\ \xi_i \in \mathfrak{X}(TM),\ 1 \leqq i \leqq k)$
$d_J f = i_J df = df \circ J \quad$ for all $f \in C^\infty(TM)$

Corollary 1. $[i_C, i_J] = 0$, $[i_C, d_J] = i_J$, $[d_J, d_C] = d_J$, $[i_J, d_J] = 0$.

Proof. We apply relations 2.29, (i)–(iv), and the definition of the Frölicher–Nijenhuis bracket. Then we get

$$[i_C, i_J] = i_{JC} = 0, \quad [i_C, d_J] = d_{JC} - i_{[C,J]} \stackrel{\mathbf{4}}{=} i_J,$$

$$[d_J, d_C] =: d_{[J,C]} = d_J, \quad [i_J, d_J] = d_{J^2} - i_{[J,J]} = 0. \qquad \square$$

Coordinate expression. Let $(\tau^{-1}(\mathfrak{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ be an induced chart on TM. For any function $f \in C^\infty(TM)$ and for all $i \in \{1, \dots, n\}$ we have

$$d_J f\left(\frac{\partial}{\partial x^i}\right) = i_J df\left(\frac{\partial}{\partial x^i}\right) = df\left(J\frac{\partial}{\partial x^i}\right) = df\left(\frac{\partial}{\partial y^i}\right), \ d_J f\left(\frac{\partial}{\partial y^i}\right) = 0,$$

therefore

$$d_J f \restriction \tau^{-1}(\mathfrak{U}) = \sum_{i=1}^n \frac{\partial f}{\partial y^i} dx^i.$$

Lemma. (i) *A function $F \in C^\infty(TM)$ is the vertical lift of a smooth function on M if, and only if, $d_J F = 0$.*

(ii) *A vertical vector field $\xi \in \mathfrak{X}(TM)$ is the vertical lift of a vector field on M if, and only if, $[Y^{\mathrm{v}}, \xi] = 0$ for all $Y \in \mathfrak{X}(M)$.*

Proof. (i) The assertion is an immediate consequence of the above local formula.

(ii) Necessity is evident. For the converse implication suppose that $[Y^{\mathrm{v}}, \xi] = 0$ for any vector field Y on M. Since ξ is vertical, by 2.4(2) we find for any function $f \in C^\infty(M)$ that:

$$0 = [Y^{\mathrm{v}}, \xi]f^c = Y^{\mathrm{v}}(\xi f^c) - \xi(Y^{\mathrm{v}} f^c) \stackrel{2.20}{=} Y^{\mathrm{v}}(\xi f^c) - \xi(Yf)^{\mathrm{v}} = Y^{\mathrm{v}}(\xi f^c).$$

From this it follows that $d_J(\xi f^c) = 0$, hence, according to part (i), the function ξf^c is a vertical lift. Applying 2.20, Lemma 1 and the fact that $X^{\mathrm{v}} f^c$ $(X \in \mathfrak{X}(M))$ is always a vertical lift, we conclude that the vector field $\xi \in \mathfrak{X}^{\mathrm{v}}(TM)$ is indeed a vertical lift of a vector field on M. $\square$

Corollary 2. *A vertical vector field $\xi \in \mathfrak{X}^{\mathrm{v}}(TM)$ is the vertical lift of a vector field on M if, and only if, $[J, \xi] = 0$.*

Proof. In view of the above property **3** the necessity of the condition is obvious. Conversely, suppose that $[J, \xi] = 0$. Then for any vector field Y on M we have

$$0 = [J, \xi]Y^c \stackrel{2.28,\ \mathrm{Cor.}\ 2}{=} [JY^c, \xi] - J[Y^c, \xi] = [Y^{\mathrm{v}}, \xi] - J[Y^c, \xi].$$

Since $Y^c \underset{\tau}{\sim} Y$ (see e.g. the coordinate expression given in 2.20) and $\xi \underset{\tau}{\sim} 0$, it follows that $[Y^c, \xi] \underset{\tau}{\sim} 0$ (see 1.27), hence $J[Y^c, \xi] = 0$ and we conclude that

$$[Y^{\mathrm{v}}, \xi] = 0 \quad \text{for all } Y \in \mathfrak{X}(M).$$

By the preceding lemma this implies that ξ is the vertical lift of a vector field on M. □

Definition. (1) The map J^* given by

$$\begin{cases} J^*f := f, & f \in C^\infty(TM); \\ J^*\alpha(\xi_1, \dots, \xi_k) := \alpha(J\xi_1, \dots, J\xi_k); & \alpha \in \mathcal{A}^k(TM),\ \xi_i \in \mathfrak{X}(TM), \\ & (1 \leqq i \leqq k) \end{cases}$$

is said to be the *adjoint operator* of J.

(2) Let

$$\theta_J L := [J, L] \quad \text{for all } L \in \mathcal{B}^\ell(TM),\ \ell \geqq 1.$$

Then L is called *θ_J-closed* if $\theta_J L = 0$; *θ_J-exact* if there is a vector-valued $(\ell - 1)$-form $K \in \mathcal{B}^{\ell-1}(TM)$ such that $L = \theta_J K$.

Theorem (E. Ayassou)**.** *Let $L \in \mathcal{B}^\ell(TM)$ $(\ell \geqq 1)$ be a θ_J-closed vector-valued form. The necessary and sufficient condition for L to be locally θ_J-exact is that*

$$(\ell - 1)!J^*L + J \circ (i_J)^{\ell-1} L = 0.$$

For a proof the reader is referred to Ayassou's thesis [6].

2.32. Applications to nonlinear connections on a manifold.

Suppose that $\mathcal{H}$ is a horizontal map on a manifold M and consider

the *horizontal projector*	$\mathbf{h} := \mathcal{H} \circ \mathbf{j}$	(2.11/(1))
the *almost complex structure*	$\mathbf{F} := \mathcal{H} \circ \mathcal{V} - \mathbf{i} \circ \mathbf{j}$	(2.19, Lemma)
the *curvature*	$\Omega = -\frac{1}{2}[\mathbf{h}, \mathbf{h}]$	(2.11/(5); 2.30, Example)
the *tension*	$\mathbf{t} = [\mathbf{h}, C]$	(2.14 Lemma 3; 2.30, Example)

belonging to $\mathcal{H}$. First of all we attach a further geometric object to $\mathcal{H}$.

Definition. The vector-valued two-form

$$\boxed{\mathbf{T} := [J, \mathbf{h}] \in \mathcal{B}^2(TM)}$$

is said to be the *torsion* of $\mathcal{H}$ (or of the nonlinear connection 'represented' by $\mathcal{H}$).

Lemma. *The torsion of a nonlinear connection is a semibasic vector-valued two-form. More precisely, if* $\mathbf{T}$ *is the torsion of a horizontal map* $\mathcal{H}$ *on* M*, then for any vector fields* X*,* Y *on* M *we have*

$$\mathbf{T}(X^c, Y^c) = [X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}} \in \mathfrak{X}^{\mathrm{v}}(TM)$$

(X^h *and* Y^h *are the* $\mathcal{H}$*-horizontal lifts of* X *and* Y*) and* $\mathbf{T}$ *is completely determined by this formula.*

Proof. Applying the first box-formula from 2.30 and taking into account that $J \circ \mathbf{h} + \mathbf{h} \circ J = J$ (see the first box in 2.19), for any vector fields ξ, η on TM we have

$$\begin{aligned}\mathbf{T}(\xi, \eta) = {} & [J\xi, \mathbf{h}\eta] + [\mathbf{h}\xi, J\eta] + J[\xi, \eta] - J[\mathbf{h}\xi, \eta] - J[\xi, \mathbf{h}\eta] \\ & - \mathbf{h}[J\xi, \eta] - \mathbf{h}[\xi, J\eta].\end{aligned}$$

From this it follows at once that $\mathbf{T}$ vanishes if one of its arguments is vertical; therefore $\mathbf{T}$ is determined by its action on pairs of the form (X^c, Y^c), where $(X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M)$. Since

$$JX^c = X^{\mathrm{v}}, \quad \mathbf{h}X^c = X^h, \quad J[X^c, Y^c] = J[X, Y]^c = [X, Y]^{\mathrm{v}}, \text{ etc.}$$

(see 2.20), we obtain that

$$\mathbf{T}(X^c, Y^c) = [X^{\mathrm{v}}, Y^h] + [X^h, Y^{\mathrm{v}}] + [X, Y]^{\mathrm{v}} - J[X^h, Y^c] - J[X^c, Y^h] \\ - \mathbf{h}[X^{\mathrm{v}}, Y^c] - \mathbf{h}[X^c, Y^v].$$

The last two terms on the right-hand side vanish automatically because $[X^{\mathrm{v}}, Y^c]$ and $[X^c, Y^{\mathrm{v}}]$ are vertical vector fields. Observe that

$$0 \overset{2.31,\,\mathbf{3}}{=} [J, Y^c]X^h = [X^{\mathrm{v}}, Y^c] - J[X^h, Y^c] = [X, Y]^{\mathrm{v}} - J[X^h, Y^c].$$

Hence $J[X^h, Y^c] = [X, Y]^{\mathrm{v}}$, $J[X^c, Y^h] = -J[Y^h, X^c] = -[Y, X]^{\mathrm{v}} = [X, Y]^{\mathrm{v}}$, and we obtain the desired formula. □

Coordinate expressions. According to our basic conventions (see the beginning of section C), let $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ be an induced chart on TM. Consider the holonomic frame $\left(\left(\frac{\partial}{\partial x^i}\right)_{i=1}^n, \left(\frac{\partial}{\partial y^i}\right)_{i=1}^n\right)$ over $\tau^{-1}(\mathcal{U})$ and the local base $\left(\widehat{\frac{\partial}{\partial u^i}}\right)_{i=1}^n$ for $\mathfrak{X}(\tau)$.

1. For all $i \in \{1, \dots, n\}$ we have

$$\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = A_i^j \frac{\partial}{\partial x^j} + B_i^j \frac{\partial}{\partial y^j}, \quad A_i^j, B_i^j \in C^\infty(\overset{\circ}{\tau}{}^{-1}(\mathcal{U})) \qquad (1 \leqq i, j \leqq n).$$

(Recall that the smoothness of $\mathcal{H}$ is not required on the zero section; see the box at the end of 2.13.) Since $\mathbf{j} \circ \mathcal{H} = 1_{TTM}$, it follows that $(A_i^j) = (\delta_i^j)$. Let, as usual, $\Gamma_i^j := -B_i^j$ $(1 \leqq i, j \leqq n)$. Then we have

$$\boxed{\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = \left(\frac{\partial}{\partial u^i}\right)^h = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j} \quad (1 \leqq i \leqq n)}.$$

The functions Γ_i^j are called the *Christoffel symbols* of $\mathcal{H}$ with respect to the chart $(\mathcal{U}, (u^i)_{i=1}^n)$.

2. $$\mathbf{h}\left(\frac{\partial}{\partial x^i}\right) = \mathbf{h}\left(\left(\frac{\partial}{\partial u^i}\right)^h + \Gamma_i^j \frac{\partial}{\partial y^j}\right) = \left(\frac{\partial}{\partial u^i}\right)^h + \Gamma_i^j \mathbf{h} \circ J\left(\frac{\partial}{\partial x^j}\right) \overset{2.19}{=} \\ = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j},$$

$$\mathbf{h}\left(\frac{\partial}{\partial y^i}\right) = 0, \quad \mathbf{v}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \mathbf{h}\left(\frac{\partial}{\partial x^i}\right) = \Gamma_i^j \frac{\partial}{\partial y^j}, \ \mathbf{v}\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial y^i};$$

thus the projectors $\mathbf{h}$ and $\mathbf{v}$ have the following matrix representations:

$$\boxed{\mathbf{h} \longleftrightarrow \begin{pmatrix} (\delta^j_i) & 0 \\ -(\Gamma^j_i) & 0 \end{pmatrix}, \ \mathbf{v} \longleftrightarrow \begin{pmatrix} 0 & 0 \\ (\Gamma^j_i) & (\delta^j_i) \end{pmatrix}}.$$

3. $$\mathbf{F}\left(\frac{\partial}{\partial x^i}\right) = \mathbf{F}\circ\mathbf{h}\left(\frac{\partial}{\partial x^i}\right) + \mathbf{F}\circ\mathbf{v}\left(\frac{\partial}{\partial x^i}\right) = -J\left(\frac{\partial}{\partial x^i}\right) + \Gamma^j_i\mathbf{F}\circ J\left(\frac{\partial}{\partial x^j}\right)$$
$$= -\frac{\partial}{\partial y^i} + \Gamma^j_i\mathbf{h}\left(\frac{\partial}{\partial x^j}\right) = \Gamma^j_i\frac{\partial}{\partial x^j} - (\delta^j_i + \Gamma^k_i\Gamma^j_k)\frac{\partial}{\partial y^j},$$

$$\mathbf{F}\left(\frac{\partial}{\partial y^i}\right) = \mathbf{F}\circ J\left(\frac{\partial}{\partial x^i}\right) = \mathbf{h}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \Gamma^j_i\frac{\partial}{\partial y^j};$$

therefore $\mathbf{F}$ has the following matrix description:

$$\boxed{\mathbf{F} \longleftrightarrow \begin{pmatrix} (\Gamma^j_i) & (\delta^j_i) \\ (\Gamma^k_i\Gamma^j_k) - (\delta^j_i) & -(\Gamma^j_i) \end{pmatrix}}.$$

4. $$\mathbf{t}\left(\frac{\partial}{\partial u^i}\right)^h = \left[\frac{\partial}{\partial x^i} - \Gamma^j_i\frac{\partial}{\partial y^j}, y^k\frac{\partial}{\partial y^k}\right] = y^k\frac{\partial\Gamma^j_i}{\partial y^k}\frac{\partial}{\partial y^j} - \Gamma^j_i\frac{\partial}{\partial y^j},$$

thus, since $\mathbf{t}$ is semibasic, for the tension we obtain the coordinate representation

$$\boxed{\mathbf{t} = \left(y^k\frac{\partial\Gamma^j_i}{\partial y^k} - \Gamma^j_i\right) dx^i \otimes \frac{\partial}{\partial y^j}}.$$

5. In view of the previous lemma, for all $i, j \in \{1, \dots, n\}$ we have

$$\mathbf{T}\left(\left(\frac{\partial}{\partial u^i}\right)^c, \left(\frac{\partial}{\partial u^j}\right)^c\right) = \left[\left(\frac{\partial}{\partial u^i}\right)^h, \frac{\partial}{\partial y^j}\right] - \left[\left(\frac{\partial}{\partial u^j}\right)^h, \frac{\partial}{\partial y^i}\right]$$
$$= \left(\frac{\partial\Gamma^k_i}{\partial y^j} - \frac{\partial\Gamma^k_j}{\partial y^i}\right)\frac{\partial}{\partial y^k},$$

hence, locally,

$$\boxed{\mathbf{T} = \left(\frac{\partial\Gamma^k_i}{\partial y^j} - \frac{\partial\Gamma^k_j}{\partial y^i}\right) dx^i \wedge dx^j \otimes \frac{\partial}{\partial y^k}}.$$

6. The curvature vector-valued two-form Ω of $\mathcal{H}$ has the coordinate expression

$$\boxed{\Omega = \left(\frac{\partial \Gamma^k_j}{\partial x^i} - \frac{\partial \Gamma^k_i}{\partial x^j} + \Gamma^\ell_j \frac{\partial \Gamma^k_i}{\partial y^\ell} - \Gamma^\ell_i \frac{\partial \Gamma^k_j}{\partial y^\ell}\right) dx^i \wedge dx^j \otimes \frac{\partial}{\partial y^k}},$$

as may be obtained by an immediate calculation.

7. Let D be the nonlinear covariant derivative induced by $\mathcal{H}$. Recall (see 2.15, Example 2) that

$$D_X Y = K \circ Y_* \circ X \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

where K is the connector belonging to $\mathcal{H}$. If $X \upharpoonright \mathcal{U} = X^i \frac{\partial}{\partial u^i}$, $Y \upharpoonright \mathcal{U} = Y^i \frac{\partial}{\partial u^i}$, and (Γ^j_i) is the matrix of the Christoffel symbols of $\mathcal{H}$ with respect to $(\mathcal{U}, (u^i)_{i=1}^n)$, then we have:

$$\boxed{(D_X Y) \upharpoonright \mathcal{U} = X^i \left(\frac{\partial Y^j}{\partial u^i} + \Gamma^j_i \circ Y\right) \frac{\partial}{\partial u^j}}.$$

Indeed,

$$\begin{aligned}
(D_X Y) \upharpoonright \mathcal{U} &= K \circ Y_* \circ X^i \frac{\partial}{\partial u^i} = X^i \left(\mathrm{vpr}_{TM} \circ \mathbf{v} \circ Y_* \circ \frac{\partial}{\partial u^i}\right) \\
&= X^i \left(\mathrm{vpr}_{TM} \circ \mathbf{v} \left(\frac{\partial}{\partial x^i} \circ Y + \frac{\partial Y^j}{\partial u^i}\left(\frac{\partial}{\partial y^j} \circ Y\right)\right)\right) \\
&= X^i \mathrm{vpr}_{TM} \left(\left(\Gamma^j_i \frac{\partial}{\partial y^j}\right) \circ Y + \frac{\partial Y^j}{\partial u^i}\left(\frac{\partial}{\partial y^j} \circ Y\right)\right) = X^i \left(\frac{\partial Y^j}{\partial u^i} + \Gamma^j_i \circ Y\right) \frac{\partial}{\partial u^j}.
\end{aligned}$$

Proposition 1. *For the horizontal projector, the curvature two-form, the torsion two-form and the tension of a nonlinear connection the following relations hold:*

$$\left.\begin{aligned} &[\mathbf{h}, \Omega] = 0 \\ &[J, \Omega] = [\mathbf{h}, \mathbf{T}] \end{aligned}\right\} \quad \textit{general Bianchi identities}; \qquad [C, \Omega] = [\mathbf{h}, \mathbf{t}].$$

Proof. The first relation has been obtained in 2.30. To verify the remaining two formulae, we apply the graded Jacobi identity for the Frölicher–Nijenhuis bracket in both cases.

(1) $[J, [\mathbf{h}, \mathbf{h}]] + [\mathbf{h}, [\mathbf{h}, J]] + [\mathbf{h}, [J, \mathbf{h}]] = 0$, hence $-2[J, \Omega] = -2[\mathbf{h}, [\mathbf{h}, J]]$, which yields $[J, \Omega] = [\mathbf{h}, \mathbf{T}]$.

(2) $[C, [\mathbf{h}, \mathbf{h}]] + [\mathbf{h}, [\mathbf{h}, C]] - [\mathbf{h}, [C, \mathbf{h}]] = 0$, hence $[C, [\mathbf{h}, \mathbf{h}]] = -2[\mathbf{h}, [\mathbf{h}, C]]$, and $[C, \Omega] = [\mathbf{h}, \mathbf{t}]$. □

Corollary 1. *If a nonlinear connection is homogeneous then its curvature two-form is also homogeneous of degree 1.* □

Proposition 2. *For any vector fields X, Y on M we have*

$$[J, \mathbf{F}](X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0, \quad [J, \mathbf{F}](X^{\mathrm{v}}, Y^h) = \mathbf{T}(X^h, Y^h),$$

$$[J, \mathbf{F}](X^h, Y^h) = -(\Omega + \mathbf{F} \circ \mathbf{T})(X^h, Y^h).$$

Proof. We check only the third relation, the calculations in the other two cases are similar but shorter. Applying the last box in 2.19, the first box-formula in 2.30 yields the following:

$$\begin{aligned}
[J, \mathbf{F}](X^h, Y^h) &= [JX^h, \mathbf{F}Y^h] + [\mathbf{F}X^h, JY^h] + [X^h, Y^h] - J[\mathbf{F}X^h, Y^h] \\
&\quad - J[X^h, \mathbf{F}Y^h] - \mathbf{F}[JX^h, Y^h] - \mathbf{F}[X^h, JY^h] = -[X^{\mathrm{v}}, Y^{\mathrm{v}}] - [X^{\mathrm{v}}, Y^{\mathrm{v}}] \\
&\quad + [X^h, Y^h] + J[X^{\mathrm{v}}, Y^h] + J[X^h, Y^{\mathrm{v}}] - \mathbf{F}[X^{\mathrm{v}}, Y^h] - \mathbf{F}[X^h, Y^{\mathrm{v}}] \\
&= [X^h, Y^h] - \mathbf{F}([X^{\mathrm{v}}, Y^h] + [X^h, Y^{\mathrm{v}}]) = [X^h, Y^h] - [X, Y]^h \\
&\quad - \mathbf{F}\big([X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}}\big) = -(\Omega + \mathbf{F} \circ \mathbf{T})(X^h, Y^h)
\end{aligned}$$

(in the last step the preliminary lemma and 2.11, Lemma 2 were applied). □

Corollary 2. (1) *The following properties are equivalent for a nonlinear connection:*

(i) $$[J, \mathbf{F}] = 0,$$

(ii) $$\Omega = 0 \quad \textit{and} \quad \mathbf{T} = 0.$$

(2) *If the torsion of a nonlinear connection vanishes then its curvature two-form may be expressed as the Frölicher–Nijenhuis bracket of the associated almost complex structure and the vertical endomorphism. Formally,*

$$\Omega = [\mathbf{F}, J].$$ □

Proposition 3. *Suppose that the horizontal map $\mathcal{H}$ is of class C^1 on its whole domain. Let D be the nonlinear covariant derivative operator induced by $\mathcal{H}$. Then $\mathcal{H}$ is homogeneous, if and only if,*

$$(D_X Y)^{\mathrm{v}} = [X^h, Y^{\mathrm{v}}] \quad \textit{for all } X, Y \in \mathfrak{X}(M).$$

In this case D becomes a covariant derivative operator on M.

Proof. We apply a local argument. If $X \restriction \mathcal{U} = X^i \frac{\partial}{\partial u^i}$, $Y \restriction \mathcal{U} = Y^i \frac{\partial}{\partial u^i}$, and (Γ^j_i) is the matrix of the Christoffel symbols with respect to $(\mathcal{U}, (u^i)^n_{i=1})$, then we have the following chain of equivalent statements:

$$[X^h, Y^{\mathrm{v}}] = (D_X Y)^{\mathrm{v}} \iff \frac{\partial \Gamma^j_i}{\partial y^k}(y^k \circ Y \circ \tau) = \Gamma^j_i \circ Y \circ \tau \quad (1 \leqq i, j \leqq n)$$

$$\iff \forall\, v \in T_p\mathcal{U} : \ \frac{\partial \Gamma^j_i}{\partial y^k}(v) y^k(Y(p)) = \Gamma^j_i(Y(p)) \quad (1 \leqq i, j \leqq n)$$

$$\iff \Gamma^j_i \restriction T_p M = y^k \frac{\partial \Gamma^j_i}{\partial y^k}(v) \quad (p \in \mathcal{U};\ 1 \leqq i, j \leqq n)$$

$$\iff \Gamma^j_i \restriction T_p M \ \textit{are linear functions}\ (1 \leqq i, j \leqq n)$$

$$\overset{(*)}{\iff} \ \text{the functions } \Gamma^j_i \text{ are positive-homogeneous of degree 1}$$

$$\iff y^k \frac{\partial \Gamma^j_i}{\partial y^k} = \Gamma^j_i \ (1 \leqq i, j \leqq n) \overset{\mathbf{4}}{\iff} \mathbf{t} = 0 \iff \mathcal{H} \text{ is homogeneous.}$$

(At the step $(*)$ we used 2.6, Lemma 2 (2).)

Now, applying 2.11, Lemma 1, we infer immediately that D is a covariant derivative operator on M. $\square$

Corollary 3. *Under the conditions of Proposition 3, the torsion and the curvature of the covariant derivative operator D and the horizontal map $\mathcal{H}$ are related as follows:*

$$(T^D(X,Y))^{\mathrm{v}} = \mathbf{T}(X^c, Y^c),\ (R^D(X,Y))^{\mathrm{v}} = [\Omega(X^c, Y^c), Z^{\mathrm{v}}]; \quad X, Y \in \mathfrak{X}(M).$$

2.33. The induced Berwald derivative in τ_{TM}.

In this subsection we continue to assume that $\mathcal{H}$ is a horizontal map on a manifold M. **h** *and* **v** *denote the horizontal and the vertical projectors belonging to $\mathcal{H}$,* **F** *is the associated almost complex structure; Ω and* **T** *are the curvature and the torsion of $\mathcal{H}$, respectively;* **t** *is the tension of $\mathcal{H}$.*

(**1**) The Berwald derivative induced by $\mathcal{H}$ in $V\tau_M$ according to Proposition 1 in 2.16 may be explicitly given as follows:

$$\nabla_\xi J\eta = J[\mathbf{v}\xi, \eta] + \mathbf{v}[\mathbf{h}\xi, J\eta] \quad \text{for all } \xi, \eta \in \mathfrak{X}(TM).$$

In fact, the map $\nabla : \mathfrak{X}(TM) \times \mathfrak{X}^{\mathrm{v}}(TM) \longrightarrow \mathfrak{X}^{\mathrm{v}}(TM)$ is indeed a covariant derivative operator and for any vector fields X, Y on M we have

$$\nabla_{X^{\mathrm{v}}} Y^{\mathrm{v}} = \nabla_{X^{\mathrm{v}}} JY^h = J[X^{\mathrm{v}}, Y^h] + \mathbf{v}[\mathbf{h}X^{\mathrm{v}}, Y^{\mathrm{v}}] = 0,$$

$$\nabla_{X^h} Y^{\mathrm{v}} = \nabla_{X^h} JY^h = J[\mathbf{v}X^h, Y^h] + \mathbf{v}[X^h, JY^h] = [X^h, Y^{\mathrm{v}}];$$

therefore the requirements of 2.16, Proposition 1 are satisfied. Notice that the canonical v-covariant derivative ∇° in $V\tau_M$ is the map given by

$$\nabla^\circ_{J\xi} J\eta = J[J\xi, \eta] \quad \text{for all } \xi, \eta \in \mathfrak{X}(TM).$$

This is an immediate consequence of the general rule because $\mathbf{v} \circ J = J$, $\mathbf{h} \circ J = 0$ (see the first box in 2.19). Observe that we may also write

$$\nabla^\circ_{J\xi} J\eta = [J, J\eta]\xi \quad \text{for all } \xi, \eta \in \mathfrak{X}(TM).$$

Indeed, on the one hand

$$[J, J\eta]\xi \overset{2.28,\,\mathrm{Cor.}\,2}{=} [J\xi, J\eta] - J[\xi, J\eta],$$

on the other hand

$$0 \overset{2.31,\,\mathbf{1}}{=} \frac{1}{2}[J, J](\xi, \eta) = [J\xi, J\eta] - J[J\xi, \eta] - J[\xi, J\eta],$$

therefore $J[J\xi, \eta] = [J, J\eta]\xi$.

We also remark that in this case the well-definedness of ∇° may easily be deduced. Indeed, suppose that $J\widetilde{\eta} = J\eta$, $\widetilde{\eta} \in \mathfrak{X}(TM)$. Then $\widetilde{\eta} - \eta$ is vertical, therefore $\widetilde{\eta} = \eta + J\xi$, $\xi \in \mathfrak{X}(TM)$, and so

$$\nabla^\circ_{J\xi} J\widetilde{\eta} := J[J\xi, \widetilde{\eta}] = J[J\xi, \eta + J\xi] = J[J\xi, \eta] = \nabla^\circ_{J\xi} J\eta.$$

Lemma 1. *For any vector fields ξ, η on TM we have*

$$\nabla_\xi J\eta = -J[\mathbf{F}, J\eta]\xi = \mathbf{v}[\xi, J\eta] - J[\mathbf{F}\xi, J\eta].$$

Proof. $0 = \frac{1}{2}[J, J](\mathbf{F}\xi, \eta) = [J \circ \mathbf{F}\xi, J\eta] - J[J \circ \mathbf{F}\xi, \eta] - J[\mathbf{F}\xi, J\eta] =$ $[\mathbf{v}\xi, J\eta] - J[\mathbf{v}\xi, \eta] - J[\mathbf{F}\xi, J\eta]$, hence $\mathbf{v}[\xi, J\eta] - J[\mathbf{F}\xi, J\eta] = \mathbf{v}[\xi, J\eta] - [\mathbf{v}\xi, J\eta] + J[\mathbf{v}\xi, \eta] = J[\mathbf{v}\xi, \eta] + \mathbf{v}[\mathbf{h}\xi, J\eta]$.

We have obtained the defining relation for $\nabla_\xi J\eta$. $\square$

(**2**) Now we prolong the operator ∇ into a covariant derivative operator $\widetilde{\nabla}$ in τ_{TM}. For any two vector fields ξ, η on TM let

$$\widetilde{\nabla}_\xi \eta := \nabla_\xi \mathbf{v}\eta + \mathbf{F}\nabla_\xi J\eta.$$

Then $\widetilde{\nabla}$ is indeed a covariant derivative operator in τ_{TM} and it extends ∇ since $\widetilde{\nabla}_\xi J\eta := \nabla_\xi \mathbf{v} J\eta + \mathbf{F}\nabla_\xi J^2\eta = \nabla_\xi J\eta$. $\widetilde{\nabla}$ is said to be the *Berwald derivative in* τ_{TM} (or on the manifold TM) induced by $\mathcal{H}$. For simplicity, the extended operator will also be denoted by ∇ from now on.

Summing up, the Berwald covariant derivative ∇ induced by $\mathcal{H}$ in τ_{TM} operates by the following rules of calculation:

$$\boxed{\begin{array}{lll}
\nabla_{J\xi} J\eta = J[J\xi, \eta] & \Longrightarrow & \nabla_{X^{\mathrm{v}}} Y^{\mathrm{v}} = 0 \\
\nabla_{\mathbf{h}\xi} J\eta = \mathbf{v}[\mathbf{h}\xi, J\eta] & \Longrightarrow & \nabla_{X^h} Y^{\mathrm{v}} = [X^h, Y^{\mathrm{v}}] \\
\nabla_{J\xi} \mathbf{h}\eta = \mathbf{h}[J\xi, \eta] & \Longrightarrow & \nabla_{X^{\mathrm{v}}} Y^h = 0 \\
\nabla_{\mathbf{h}\xi} \mathbf{h}\eta = \mathbf{h}\circ\mathbf{F}[\mathbf{h}\xi, J\eta] & \Longrightarrow & \nabla_{X^h} Y^h = \mathbf{F}[X^h, Y^{\mathrm{v}}]
\end{array}}$$

$$(\xi, \eta \in \mathfrak{X}(TM); \quad X, Y \in \mathfrak{X}(M)).$$

Remark. Other equivalent formulations are also possible and may be useful. For example, the first box-formula yields

$$\nabla_{J\xi}\mathbf{v}\eta = J[J\xi, \mathbf{F}\eta] \quad \text{or} \quad \nabla_{\mathbf{v}\xi}\mathbf{v}\eta = J[\mathbf{v}\xi, \mathbf{F}\eta]$$

since $\mathbf{v} = J\circ\mathbf{F}$. From the second box-formula we obtain

$$\begin{aligned}\nabla_{\mathbf{h}\xi}\mathbf{v}\eta = \nabla_{\mathbf{h}\xi} J\mathbf{F}\eta = \mathbf{v}[\mathbf{h}\xi, \mathbf{v}\eta] = \mathbf{v}[\mathbf{h}\xi, \eta] - \mathbf{v}[\mathbf{h}\xi, \mathbf{h}\eta] &\overset{2.11(5)}{=} \\ = \mathbf{v}[\mathbf{h}\xi, \eta] + \Omega(\xi, \eta),&\end{aligned}$$

and so on.

Basic properties.

1. *The Berwald derivative ∇ induced by $\mathcal{H}$ on TM is 'almost tangent', i.e. $\nabla J = 0$.*

2. *∇ is an 'almost complex' derivative, i.e. $\nabla\mathbf{F} = 0$.*

3. *The horizontal and the vertical projector are parallel with respect to ∇, i.e., $\nabla\mathbf{h} = \nabla\mathbf{v} = 0$.*

4. $\nabla C = \mathbf{v} + \mathbf{t}$.

5. *The torsion tensor field T^∇ of ∇ may be represented in the form*

$$T^\nabla = \Omega + \mathbf{F} \circ \mathbf{T}.$$

In particular, if the horizontal map $\mathcal{H}$ has vanishing torsion then its curvature two-form coincides with the torsion tensor field of the Berwald derivative induced by $\mathcal{H}$ in τ_{TM}:

$$\boxed{\mathbf{T} = 0 \Longrightarrow \Omega = T^\nabla}.$$

Properties **1**–**3** may be verified without any difficulty. Now we prove relations **4** and **5**.

(i) Choose a vector field $\eta \in \mathfrak{X}(TM)$ such that $J\eta = C$. Then for any vector field ξ on TM we have

$$(\nabla C)(\xi) = \nabla_\xi J\eta {:=} J[\mathbf{v}\xi, \eta] + \mathbf{v}[\mathbf{h}\xi, C].$$

In view of property **5** in 2.31 and Remark 2 in 2.14, the right-hand side is just $\mathbf{v}\xi + \mathbf{t}\xi$, which proves formula **4**.

(ii) Let X and Y be any two vector fields on M. Using the above box-formulae and the box-formulae in 2.19, we obtain:

$$\begin{aligned}
&T^\nabla(X^\mathrm{v}, Y^\mathrm{v}) = \nabla_{X^\mathrm{v}} Y^\mathrm{v} - \nabla_{Y^\mathrm{v}} X^\mathrm{v} - [X^\mathrm{v}, Y^\mathrm{v}] = 0, \\
&T^\nabla(X^\mathrm{v}, Y^h) = \nabla_{X^\mathrm{v}} Y^h - \nabla_{Y^h} X^\mathrm{v} - [X^\mathrm{v}, Y^h] = -[Y^h, X^\mathrm{v}] - [X^\mathrm{v}, Y^h] = 0, \\
&T^\nabla(X^h, Y^h) = \nabla_{X^h} Y^h - \nabla_{Y^h} X^h - [X^h, Y^h] = \mathbf{F}\big([X^h, Y^\mathrm{v}] - [Y^h, X^\mathrm{v}]\big) \\
&\quad - \mathbf{h}[X^h, Y^h] - \mathbf{v}[X^h, Y^h] = \mathbf{F}\big([X^h, Y^\mathrm{v}] - [Y^h, X^\mathrm{v}] - J[X^h, Y^h]\big) \\
&\quad + \Omega(X^h, Y^h) = (\Omega + \mathbf{F} \circ \mathbf{T})(X^h, Y^h);
\end{aligned}$$

whence property **5**.

(**3**) We define the *vertical Berwald differential* (or briefly *v-Berwald differential*) $\nabla^J : \mathcal{T}^r_s(TM) \longrightarrow \mathcal{T}^r_{s+1}(TM)$ and the *horizontal* (briefly *h-*) *Berwald differential* $\nabla^h : \mathcal{T}^r_s(TM) \longrightarrow \mathcal{T}^r_{s+1}(TM)$ as follows:

$$i_\xi(\nabla^J A) := \nabla_{J\xi} A, \; i_\xi(\nabla^h A) := \nabla_{\mathbf{h}\xi} A \quad \text{for all } A \in \mathcal{T}^r_s(TM), \; \xi \in \mathfrak{X}(TM).$$

Notice that, in particular,

$$\nabla^J J\eta = [J, J\eta] \in \mathcal{T}^1_1(TM) \quad \text{for all } \eta \in \mathfrak{X}(TM),$$

so ∇^J is just the canonical v-covariant derivative in $V\tau_M$; cf. 2.16, Remark.

Example. $\nabla^J C = \mathbf{v}$, $\nabla^h C = \mathbf{t}$, as a consequence of property **4**. From this it follows that a nonlinear connection is homogeneous if, and only if, the induced h-Berwald differential ∇^h satisfies the condition $\nabla^h C = 0$.

Lemma 2. *Suppose that $A \in \mathcal{T}^1_s(TM)$ is a semibasic (symmetric or skew-symmetric) tensor field on TM. Then*

$$(\nabla^J A)(X^c) = d_{X^{\mathrm{v}}} A \quad \textit{for all } X \in \mathfrak{X}(M);$$

explicitly, for any vector fields $X_1, \dots, X_s$ on M we have

$$(\nabla_{X^{\mathrm{v}}} A)(X_1^c, \dots, X_s^c) = (d_{X^{\mathrm{v}}} A)(X_1^c, \dots, X_s^c) = [X^{\mathrm{v}}, A(X_1^c, \dots, X_s^c)].$$

Proof. Since the vector fields $[X^{\mathrm{v}}, X_i^c]$ $(1 \leqq i \leqq s)$ are vertical and A is semibasic, 1.22 Corollary (2) leads to

$$d_{X^{\mathrm{v}}} A = [X^{\mathrm{v}}, A(X_1^c, \dots, X_s^c)].$$

On the other hand, according to the above Remark,

$$\begin{aligned} \nabla_{X^{\mathrm{v}}} X_i^c &= \nabla_{X^{\mathrm{v}}} \mathbf{v} X_i^c + \nabla_{X^{\mathrm{v}}} \mathbf{h} X_i^c = J[X^{\mathrm{v}}, \mathbf{F} X_i^c] + \nabla_{X^{\mathrm{v}}} X_i^h \\ &= J[X^{\mathrm{v}}, \mathbf{F} X_i^c] \in \mathfrak{X}^{\mathrm{v}}(TM) \qquad (1 \leqq i \leqq s), \end{aligned}$$

therefore

$$\begin{aligned} &(\nabla_{X^{\mathrm{v}}} A)(X_1^c, \dots, X_s^c) \overset{1.32(3)}{=} \nabla_{X^{\mathrm{v}}}(A(X_1^c, \dots, X_s^c)) \\ &\quad - \sum_{i=1}^{s} A(X_1^c, \dots, \nabla_{X^{\mathrm{v}}} X_i^c, \dots, X_s^c) = \nabla_{X^{\mathrm{v}}}(A(X_1^c, \dots, X_s^c)) \\ &= \nabla_{X^{\mathrm{v}}}(J \circ \mathbf{F} A(X_1^c, \dots, X_s^c)) = J[X^{\mathrm{v}}, \mathbf{F} A(X_1^c, \dots, X_s^c)] \\ &= [J, X^{\mathrm{v}}]\mathbf{F} A(X_1^c, \dots, X_s^c) - [J \circ \mathbf{F} A(X_1^c, \dots, X_s^c), X^{\mathrm{v}}] = [X^{\mathrm{v}}, A(X_1^c, \dots, X_s^c)] \end{aligned}$$

(we applied in the last steps the final box-formula in 2.19; 2.28, Corollary 2; and 2.31, **3**). This concludes the proof.

(**4**) Now we turn to the curvature tensor R^∇ and the torsion tensor T^∇ of the Berwald derivative ∇ on TM. By the following table we define three 'partial curvatures' and five 'partial torsions'.

$\mathbf{R}(\xi,\eta)\zeta := R^{\nabla}(\mathbf{h}\xi,\mathbf{h}\eta)J\zeta$	horizontal or Riemann curvature
$\mathbf{P}(\xi,\eta)\zeta := R^{\nabla}(\mathbf{h}\xi,J\eta)J\zeta$	mixed or Berwald curvature
$\mathbf{Q}(\xi,\eta)\zeta := R^{\nabla}(J\xi,J\eta)J\zeta$	vertical curvature
$\mathcal{T}(\xi,\eta) := \mathbf{h}T^{\nabla}(\mathbf{h}\xi,\mathbf{h}\eta)$	h-horizontal torsion
$\mathcal{S}(\xi,\eta) := \mathbf{h}T^{\nabla}(\mathbf{h}\xi,J\eta)$	h-mixed torsion
$\mathbf{R}^1(\xi,\eta) := \mathbf{v}T^{\nabla}(\mathbf{h}\xi,\mathbf{h}\eta)$	v-horizontal torsion
$\mathbf{P}^1(\xi,\eta) := \mathbf{v}T^{\nabla}(\mathbf{h}\xi,J\eta)$	v-mixed torsion
$\mathbf{Q}^1(\xi,\eta) := \mathbf{v}T^{\nabla}(J\xi,J\eta)$	v-vertical torsion

($\xi,\eta,\zeta\in\mathfrak{X}(TM)$). Evidently, $\mathbf{R}$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}^1$, $\mathbf{P}^1$, $\mathbf{Q}^1$ are semibasic tensors; R^{∇} is uniquely determined by the three partial curvatures, T^{∇} is uniquely determined by the five partial torsions.

Basic properties (continued).

6. *The Riemann curvature of the Berwald derivative ∇ induced by $\mathcal{H}$ on TM is related to the curvature of $\mathcal{H}$ by the formula*

$$\boxed{\mathbf{R}(\xi,\eta)\zeta = (\nabla^J\Omega)(\zeta,\xi,\eta)}$$

($\xi,\eta,\zeta\in\mathfrak{X}(TM)$).

Proof. Applying the box-formulae for the Berwald derivative ∇, the second observation in the above Remark, Lemma 1, and some elementary properties of Ω and $\mathbf{F}$, we obtain:

$$\begin{aligned}\mathbf{R}(\xi,\eta)\zeta &:= \nabla_{\mathbf{h}\xi}\nabla_{\mathbf{h}\eta}J\zeta - \nabla_{\mathbf{h}\eta}\nabla_{\mathbf{h}\xi}J\zeta - \nabla_{[\mathbf{h}\xi,\mathbf{h}\eta]}J\zeta = \mathbf{v}[\mathbf{h}\xi,[\mathbf{h}\eta,J\zeta]]\\ &\quad + \Omega(\xi,[\mathbf{h}\eta,J\zeta]) - \mathbf{v}[\mathbf{h}\eta,[\mathbf{h}\xi,J\zeta]] - \Omega(\eta,[\mathbf{h}\xi,J\zeta])\\ &\quad - \mathbf{v}[[\mathbf{h}\xi,\mathbf{h}\eta],J\zeta] + J[\mathbf{F}[\mathbf{h}\xi,\mathbf{h}\eta],J\zeta] = \mathbf{v}([\mathbf{h}\xi,[\mathbf{h}\eta,J\zeta]]\\ &\quad + [\mathbf{h}\eta,[J\zeta,\mathbf{h}\xi]] + [J\zeta,[\mathbf{h}\xi,\mathbf{h}\eta]]) + J[\mathbf{F}\circ\mathbf{v}[\mathbf{h}\xi,\mathbf{h}\eta],J\zeta]\\ &\quad + J[\mathbf{F}\circ\mathbf{h}[\mathbf{h}\xi,\mathbf{h}\eta],J\zeta] - \Omega(\mathbf{h}[J\zeta,\xi],\eta) - \Omega(\xi,\mathbf{h}[J\zeta,\eta])\\ &= J[J\zeta,\mathbf{F}\Omega(\xi,\eta)] - \Omega(\nabla_{J\zeta}\xi,\eta) - \Omega(\xi,\nabla_{J\zeta}\eta) = (\nabla_{J\zeta}\Omega)(\xi,\eta);\end{aligned}$$

this verifies the assertion.

7. *Suppose that $\mathcal{H}$ is homogeneous. In this case*

(i) if $J\zeta = C$ then $\mathbf{R}(\xi,\eta)\zeta = \Omega(\xi,\eta)$ for all $\xi,\eta \in \mathfrak{X}(TM)$;

(ii) $\nabla_C\Omega = \Omega$;

(iii) *the vanishing of* $\mathbf{R}$ *is equivalent to the vanishing of* Ω.

Proof. From the preceding calculation we get

$$\mathbf{R}(\xi,\eta)\zeta = J[C, \mathbf{F}\Omega(\xi,\eta)] - \Omega([C,\xi],\eta) - \Omega(\xi,[C,\eta]).$$

Since $[J,C] = J$, $J \circ \mathbf{F} = \mathbf{v}$ and $\mathbf{v} \circ \Omega = \Omega$, the first term on the right-hand side is

$$[J,C]\mathbf{F}\Omega(\xi,\eta) - [\Omega(\xi,\eta),C] = [C,\Omega(\xi,\eta)] + \Omega(\xi,\eta),$$

therefore

$$\begin{aligned}\mathbf{R}(\xi,\eta)\zeta &= \Omega(\xi,\eta) + [C,\Omega(\xi,\eta)] - \Omega([C,\xi],\eta) - \Omega(\xi,[C,\eta]) \\ &= \Omega(\xi,\eta) + (d_C\Omega)(\xi,\eta).\end{aligned}$$

Due to the homogeneity of $\mathcal{H}$, Ω is also homogeneous of degree 1 (2.32, Corollary 1), so $d_C\Omega = 0$ and $\mathbf{R}(\xi,\eta)\zeta = \Omega(\xi,\eta)$. Then

$$(\nabla_C\Omega)(\xi,\eta) = (\nabla_{J\zeta}\Omega)(\xi,\eta) \overset{\mathbf{6}}{=} \mathbf{R}(\xi,\eta)\zeta = \Omega(\xi,\eta) \quad (\xi,\eta \in \mathfrak{X}(TM)),$$

whence $\nabla_C\Omega = \Omega$. Finally, we infer immediately that *in the homogeneous case* $\mathbf{R} = 0 \iff \Omega = 0$.

8. *The Berwald curvature of the Berwald derivative ∇ induced by $\mathcal{H}$ on TM acts by the rule*

$$\mathbf{P}(X^c,Y^c)Z^c = [[X^h,Y^{\mathrm{v}}],Z^{\mathrm{v}}] \quad \text{for all } X,Y,Z \in \mathfrak{X}(M).$$

$\mathbf{P}$ *is symmetric in its second and third argument. If, in addition, the torsion of $\mathcal{H}$ vanishes then* $\mathbf{P}$ *is (totally) symmetric.*

Proof. $\mathbf{P}(X^c,Y^c)Z^c := R^\nabla(\mathbf{h}X^c, JY^c)JZ^c = R^\nabla(X^h,Y^{\mathrm{v}})Z^{\mathrm{v}} = \nabla_{X^h}\nabla_{Y^{\mathrm{v}}}Z^{\mathrm{v}} - \nabla_{Y^{\mathrm{v}}}\nabla_{X^h}Z^{\mathrm{v}} - \nabla_{[X^h,Y^{\mathrm{v}}]}Z^{\mathrm{v}} = -\nabla_{Y^{\mathrm{v}}}[X^h,Z^{\mathrm{v}}] - \nabla_{[X^h,Y^{\mathrm{v}}]}Z^{\mathrm{v}}$.

The second term on the right-hand side vanishes, since the vector field $[X^h,Y^{\mathrm{v}}]$ is vertical and so it may be combined from vertically lifted vector fields. Applying Lemma 1 and the Jacobi identity, the first term may be formed as follows:

$$\begin{aligned}-\nabla_{Y^{\mathrm{v}}}[X^h, Z^{\mathrm{v}}] &= -\nabla_{Y^{\mathrm{v}}} J\mathbf{F}[X^h, Z^{\mathrm{v}}] = -\mathbf{v}[Y^{\mathrm{v}}, [X^h, Z^{\mathrm{v}}]] + J[\mathbf{F}Y^{\mathrm{v}}, [X^h, Z^{\mathrm{v}}]]\\ &= -[Y^{\mathrm{v}}, [X^h, Z^{\mathrm{v}}]] + J[Y^h, [X^h, Z^{\mathrm{v}}]] = -[Y^{\mathrm{v}}, [X^h, Z^{\mathrm{v}}]]\\ &= [X^h, [Z^{\mathrm{v}}, Y^{\mathrm{v}}]] + [Z^{\mathrm{v}}, [Y^{\mathrm{v}}, X^h]] = [[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}];\end{aligned}$$

hence $\mathbf{P}(X^c, Y^c)Z^c = [[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}]$, as we claimed. We see at the same time that

$$\mathbf{P}(X^c, Z^c)Y^c = [[X^h, Z^{\mathrm{v}}], Y^{\mathrm{v}}] = \mathbf{P}(X^c, Y^c)Z^c.$$

If the torsion vanishes, then $[X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}} = 0$ (see 2.32, Lemma) and we conclude that

$$\begin{aligned}\mathbf{P}(X^c, Y^c)Z^c &= [[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}] = [[Y^h, X^{\mathrm{v}}], Z^{\mathrm{v}}] + [[X, Y]^{\mathrm{v}}, Z^{\mathrm{v}}]\\ &= [[Y^h, X^{\mathrm{v}}], Z^{\mathrm{v}}] = \mathbf{P}(Y^c, X^c)Z^c,\end{aligned}$$

thus proving the (total) symmetry of $\mathbf{P}$.

9. *Suppose that $\mathcal{H}$ is homogeneous and has vanishing torsion, i.e.,*

$$[\mathbf{h}, C] = 0 \quad \text{and} \quad [J, \mathbf{h}] = 0.$$

Then the Berwald curvature of the Berwald derivative ∇ induced by $\mathcal{H}$ has the following further properties:

(i) $i_\eta \mathbf{P} = 0$ *for any vector field $\eta \in \mathfrak{X}(TM)$ satisfying $J\eta = C$;*

(ii) $d_C \mathbf{P} = -2\mathbf{P}$, *i.e. $\mathbf{P}$ is homogeneous of degree 2,*

(iii) $\nabla^J \mathbf{P}$ *is (totally) symmetric.*

Proof. Let X, Y, Z be arbitrary vector fields on M.

(i) Due to the symmetry of $\mathbf{P}$, to verify the first relation it is enough to check that $\mathbf{P}(X^c, \eta)Y^c = 0$. Applying the rules of calculation for ∇, the homogeneity of the horizontally lifted vector fields (2.14, Corollary) and Lemma 1, we obtain that

$$\begin{aligned}\mathbf{P}(X^c, \eta)Y^c &:= R^\nabla(\mathbf{h}X^c, J\eta)JY^c = R^\nabla(X^h, C)Y^{\mathrm{v}} = \nabla_{X^h}\nabla_C Y^{\mathrm{v}} - \nabla_C \nabla_{X^h} Y^{\mathrm{v}}\\ &- \nabla_{[X^h, C]} Y^{\mathrm{v}} = -\nabla_C J\mathbf{F}[X^h, Y^{\mathrm{v}}] = -\mathbf{v}[C, [X^h, Y^{\mathrm{v}}]]\\ &+ J[\mathbf{F} \circ C, [X^h, Y^{\mathrm{v}}]] = [[X^h, Y^{\mathrm{v}}], C] + J[\mathbf{F} \circ C, [X^h, Y^{\mathrm{v}}]].\end{aligned}$$

Since $\mathbf{F} := \mathcal{H} \circ \mathcal{V} - J$ and $C = \mathbf{i} \circ \delta$, we have

$$\mathbf{F} \circ C = \mathcal{H} \circ \mathcal{V} \circ \mathbf{i} \circ \delta - J \circ C = \mathcal{H} \circ \delta =: \widetilde{\eta}.$$

Clearly, the vector field $\widetilde{\eta}$ also has the property $J\widetilde{\eta} = C$, therefore, in view of 2.31, Property **5**, we conclude that

$$J[\mathbf{F} \circ C, [X^h, Y^{\mathrm{v}}]] = -J[J\mathbf{F}[X^h, Y^{\mathrm{v}}], \widetilde{\eta}] = -J\mathbf{F}[X^h, Y^{\mathrm{v}}] = -[X^h, Y^{\mathrm{v}}].$$

Hence

$$\begin{aligned}\mathbf{P}(X^c, \eta)Y^c &= [[X^h, Y^{\mathrm{v}}], C] - [X^h, Y^{\mathrm{v}}] = -[[Y^{\mathrm{v}}, C], X^h] - [[C, X^h], Y^{\mathrm{v}}] \\ &\quad - [X^h, Y^{\mathrm{v}}] \overset{2.20,\ \text{Lemma 2}}{=} -[Y^{\mathrm{v}}, X^h] + [Y^{\mathrm{v}}, X^h] = 0.\end{aligned}$$

This concludes the proof of the first relation.

(ii) Next we check the homogeneity property of $\mathbf{P}$. Since the Lie bracket of the Liouville vector field and a complete lift vanishes, we have the following simple expression for the Lie derivative $d_C\mathbf{P}$:

$$(d_C\mathbf{P})(X^c, Y^c, Z^c) = [C, \mathbf{P}(X^c, Y^c, Z^c)] \overset{\mathbf{8}}{=} [C, [[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}]].$$

By a repeated use of the Jacobi identity; 2.20, Lemma 2 and the homogeneity of $\mathcal{H}$, the right-hand side may be formed as follows:

$$\begin{aligned}&[C, [[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}]] = -[[X^h, Y^{\mathrm{v}}], [Z^{\mathrm{v}}, C]] - \big[Z^{\mathrm{v}}, [C, [X^h, Y^{\mathrm{v}}]]\big] \\ &\quad = -[[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}] + \big[Z^{\mathrm{v}}, [X^h, [Y^{\mathrm{v}}, C]]\big] + \big[Z^{\mathrm{v}}, [Y^{\mathrm{v}}, [C, X^h]]\big] \\ &\quad = -[[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}] - [[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}] = -2\mathbf{P}(X^c, Y^c)Z^c.\end{aligned}$$

Thus, $d_C\mathbf{P} = -2\mathbf{P}$, as we claimed.

(iii) The proof of the symmetry of $\nabla^J\mathbf{P}$ is very similar to the preceding argument and is left to the reader (or consult with [77]).

10. *The vertical curvature* $\mathbf{Q}$ *of* ∇ *vanishes.*

This is an immediate consequence of the definition.

11. *For the partial torsions of the Berwald derivative* ∇ *induced by* $\mathcal{H}$ *on* TM *we have*

$$\boxed{\mathcal{T} = \mathbf{F} \circ \mathbf{T}, \quad \mathcal{S} = 0, \quad \mathbf{R}^1 = \Omega, \quad \mathbf{P}^1 = 0, \quad \mathbf{Q}^1 = 0}.$$

Proof. In view of property **5**, $T^\nabla = \Omega + \mathbf{F} \circ \mathbf{T}$. Since Ω and $\mathbf{T}$ are semibasic and $\mathbf{h} \circ \mathbf{F} = \mathbf{F} \circ \mathbf{v}$, $\mathbf{v} \circ \mathbf{F} = -J$ (see the last box in 2.19), it follows that $\mathbf{h} \circ T^\nabla = \mathbf{F} \circ \mathbf{T}$ and $\mathbf{v} \circ T^\nabla = \Omega$, which imply the stated relations.

E. The theory of E. Martínez, J. F. Cariñena and W. Sarlet

In this section the main arena of our considerations is the pull-back vector bundle $\tau^*\tau$ of the tangent bundle (TM, τ, M) of an n-dimensional manifold M. According to 2.17(1) and 2.21(2), $\mathfrak{X}(\tau)$ and $\mathcal{A}^1(\tau)$ denote the $C^\infty(TM)$-modules of vector fields and one-forms along τ; $\mathcal{A}(\tau)$ is the Grassmann algebra of differential forms along τ, $\mathcal{B}(\tau) = \mathcal{A}(\tau) \otimes \mathfrak{X}(\tau)$ is the $C^\infty(TM)$-module of vector-valued forms along τ. All the basic conventions fixed at the beginning of section C will be preserved; in particular, the Einstein summation convention will be applied. For the convenience of the reader, we collect here some basic notational conventions concerning $\tau^*\tau$-tensor fields and vector fields on TM.

$\xi, \eta, \zeta, \dots$	vector fields on TM
X^{v}, X^c	the vertical and the complete lift of $X \in \mathfrak{X}(M)$
$\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \dots$	general sections in $\mathfrak{X}(\tau)$
$\widehat{X}, \widehat{Y}, \widehat{Z}, \dots$	basic vector fields in $\mathfrak{X}(\tau)$
$\delta : v \mapsto (v, v)$	the canonical vector field along τ
$\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \dots$	general sections in $\mathcal{A}^1(\tau)$
$\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \dots$	basic one-forms along τ
$\widetilde{K}, \widetilde{L}, \dots$	vector-valued forms along τ
$(\widetilde{\alpha})_0, (\widetilde{K})_0$	the semibasic form and the vector-valued form on TM associated to $\widetilde{\alpha}$ and $\widetilde{K}$, respectively
$\bar{\alpha} \in \mathcal{A}^k(\tau)$	the k-form along τ associated to $\alpha \in \mathcal{A}^{k+1}(M)$
$\bar{A} \in \mathcal{B}^k(\tau)$	the vector-valued k-form along τ associated to $A \in \mathcal{B}^{k+1}(M)$

2.34. The vertical exterior derivative on $\mathcal{A}(\tau)$. A Frölicher-Nijenhuis type theory of graded derivations of $\mathcal{A}(\tau)$ was elaborated by E. Martínez, J. F. Cariñena and W. Sarlet in the early nineties of the last century, and our brief account is based on their fundamental paper [50]. There are strict analogies to the classical theory, but interesting new phenomena also appear.

The first steps are the same as in 2.24.

Lemma 1. *The graded derivations of* $\mathcal{A}(\tau)$

(1) *are local operators in the sense of* 2.24, *Lemma 1;*

(2) *are natural with respect to restrictions (cf.* 2.24, *Lemma 2).*

Every graded derivation of $\mathcal{A}(\tau)$ *is determined by its action on* $\mathcal{A}^0(\tau) = C^\infty(TM)$ *and on the basic one-forms along* τ*.*

The proof parallels that of Lemmas 1, 2 and the Proposition in 2.24.

Proposition. *The action of a graded derivation of* $\mathcal{A}(\tau)$ *over* $C^\infty(TM)$ *is completely determined by its action on the vertical and the complete lifts of smooth functions on* M*.*

Proof. Let $\mathcal{D} : \mathcal{A}(\tau) \longrightarrow \mathcal{A}(\tau)$ be a graded derivation. Choose a point $v \in TM$. Let $(\mathcal{U}, (u^i)_{i=1}^n)$ be a chart around $\tau(v)$, and consider at the same time the induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n), (y^i)_{i=1}^n))$. We may suppose that $x^i(v) = y^i(v) = 0$ $(1 \leqq i \leqq n)$. Shrinking $\mathcal{U}$ if necessary, any function $f \in C^\infty(TM)$ may be represented locally in the form

$$f \restriction \tau^{-1}(\mathcal{U}) = f(v)\mathbf{1} + f_i x^i + f_{n+i} y^i,$$

where $\mathbf{1} : M \longrightarrow \mathbb{R}$, $q \mapsto \mathbf{1}(q) := 1$; $f_i, f_{n+i} \in C^\infty(\tau^{-1}(\mathcal{U}))$, $f_i(v) = \frac{\partial f}{\partial x^i}(v)$, $f_{n+i}(v) = \frac{\partial f}{\partial y^i}(v)$ $(1 \leqq i \leqq n)$. (For a standard proof of this basic result the reader is referred to [61], p. 8.) Since $x^i = (u^i)^{\mathrm{v}}$, $y^i = (u^i)^c$ $(1 \leqq i \leqq n)$, we may also write

$$f \restriction \tau^{-1}(\mathcal{U}) = f(v)\mathbf{1} + f_i (u^i)^{\mathrm{v}} + f_{n+i} (u^i)^c.$$

Now

$$\begin{aligned} \mathcal{D}f \restriction \tau^{-1}(\mathcal{U}) &\overset{\text{Lemma 1}}{=} \mathcal{D}(f \restriction \tau^{-1}(\mathcal{U})) = (\mathcal{D}f_i)(u^i)^{\mathrm{v}} + f_i \mathcal{D}(u^i)^{\mathrm{v}} \\ &\quad + (\mathcal{D}f_{n+i})(u^i)^c + f_{n+i}\mathcal{D}(u^i)^c, \end{aligned}$$

hence

$$\mathcal{D}f(v) = \frac{\partial f}{\partial x^i}(v)\big(\mathcal{D}(u^i)^{\mathrm{v}}\big)(v) + \frac{\partial f}{\partial y^i}(v)\big(\mathcal{D}(u^i)^c\big)(v),$$

which proves our claim.

Lemma 2 and definition. *There is a unique graded derivation* $d^{\mathrm{v}} : \mathcal{A}(\tau) \longrightarrow \mathcal{A}(\tau)$ *of degree 1 such that*

$$\begin{cases} (d^{\mathrm{v}} f)(\widetilde{X}) := df(\mathbf{i}\widetilde{X}) & \text{for all } f \in C^\infty(TM),\ \widetilde{X} \in \mathfrak{X}(\tau); \\ d^{\mathrm{v}}\widehat{\alpha} := 0 & \text{for all } \alpha \in \mathcal{A}^1(M). \end{cases}$$

d^{v} *is said to be the* vertical exterior derivative *(or briefly* v-exterior derivative*) on* $\mathcal{A}(\tau)$ *(or in* $\tau^*\tau$*).*

Proof. It may be seen at once that $d^{\mathrm{v}} \restriction C^\infty(TM)$ is a graded derivation of degree 1. Thus the assertion is an immediate consequence of Lemma 1.

Coordinate expression. Let $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ be an induced chart on TM. For every function $f \in C^\infty(TM)$ we have $df \restriction \tau^{-1}(\mathcal{U}) = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i$, therefore

$$d^{\mathrm{v}} f \left(\widehat{\frac{\partial}{\partial u^i}} \right) := df \left(\mathbf{i}\widehat{\frac{\partial}{\partial u^i}} \right) = df \left(\frac{\partial}{\partial y^i} \right) = \frac{\partial f}{\partial y^i} \quad (1 \leqq i \leqq n),$$

hence

$$\boxed{d^{\mathrm{v}} f \restriction \tau^{-1}(\mathcal{U}) = \frac{\partial f}{\partial y^i} \widehat{du^i}}.$$

Properties.

1. $d^{\mathrm{v}}\overline{\alpha} = k\widehat{\alpha}$ *for all* $\alpha \in \mathcal{A}^k(M)$ $(k \in \mathbb{N}^*)$*; in particular, for every function* $f \in C^\infty(M)$ *we have* $d^{\mathrm{v}} f^c = \widehat{df}$.

Proof. (1) First we check the statement for one-forms. Since the question is local, it is sufficient to consider a differential df $(f \in C^\infty(M))$. According to 2.22, Lemma 2, $\overline{df} = f^c$, hence for any basic vector field $\widehat{X}$ along τ we have

$$\begin{aligned} d^{\mathrm{v}}\overline{df}(\widehat{X}) &= (d^{\mathrm{v}} f^c)(\widehat{X}) := df^c(\mathbf{i}\widehat{X}) = df^c(X^{\mathrm{v}}) = X^{\mathrm{v}} f^c \overset{2.20}{=} \\ &= (Xf)^{\mathrm{v}} = df(X) \circ \tau = \widehat{df}(\widehat{X}). \end{aligned}$$

This proves that $d^{\mathrm{v}} f^c = \widehat{df}$.

(2) Now we turn to the general case. Due to the local character of the problem, our reasoning may also be local. Suppose that $\alpha \in \mathcal{A}^k(M)$, $k \in \mathbb{N}^*$. Choose a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M. At any point $p \in \mathcal{U}$ and for any vectors

$v, v_1, \dots, v_{k-1} \in T_pM$ we have

$$
\begin{aligned}
&\overline{\alpha}_v(v_1, \dots, v_{k-1}) := \alpha_p(v, v_1, \dots, v_{k-1}) \\
&= \alpha_{i_1 \dots i_k}(p)(du^{i_1})_p \wedge \dots \wedge (du^{i_k})_p(v, v_1, \dots, v_{k-1}) \\
&= k!\alpha_{i_1 \dots i_k}(p) \operatorname{Alt}(du^{i_1})_p \otimes \dots \otimes (du^{i_k})_p(v, v_1, \dots, v_{k-1}) \\
&= (k-1)! \sum_{j=1}^{k} (-1)^{j-1} \alpha_{i_1 \dots i_k}(p) v(u^{i_j}) \operatorname{Alt}(du^{i_1})_p \otimes \dots \otimes (du^{i_{j-1}})_p \\
&\otimes (du^{i_{j+1}})_p \otimes \dots \otimes (du^{i_k})_p(v_1, \dots, v_{k-1}) \\
&= \sum_{j=1}^{k} (-1)^{j-1} \alpha_{i_1 \dots i_k}(p) \widehat{du^{i_1}} \wedge \dots \wedge \overline{du^{i_j}} \wedge \dots \wedge \widehat{du^{i_k}}(v, v_1, \dots v_k),
\end{aligned}
$$

therefore

$$
\overline{\alpha} \restriction \tau^{-1}(\mathcal{U}) = \sum_{j=1}^{k} (-1)^{j-1} (\alpha_{i_1 \dots i_k} \circ \tau) \widehat{du^{i_1}} \wedge \dots \wedge \overline{du^{i_j}} \wedge \dots \wedge \widehat{du^{i_k}}.
$$

From this, taking into account that $d^{\mathrm{v}}(\overline{du^{i_j}}) = \widehat{du^{i_j}}$, and applying A.7, Lemma 1 we conclude the desired result.

2. $\boxed{d^{\mathrm{v}} \circ d^{\mathrm{v}} = 0}$

Proof. $d^{\mathrm{v}} \circ d^{\mathrm{v}} = \frac{1}{2}[d^{\mathrm{v}}, d^{\mathrm{v}}]$ is a graded derivation of degree 2, which kills the basic one-forms by the definition of d^{v}. Since for any function f on M and vector field X in $\mathfrak{X}(M)$ we have

$$
d^{\mathrm{v}} \circ d^{\mathrm{v}}(f^c) \stackrel{\mathbf{1}}{=} d^{\mathrm{v}} \widehat{df} := 0, \quad (d^{\mathrm{v}} f^{\mathrm{v}})(\widehat{X}) = (\mathbf{i}\widehat{X}) f^{\mathrm{v}} \stackrel{2.17(4)}{=} X^{\mathrm{v}} f^{\mathrm{v}} \stackrel{2.4(2)}{=} 0,
$$

it follows from Lemma 1 and the above Proposition that $d^{\mathrm{v}} \circ d^{\mathrm{v}} = 0$. □

2.35. Algebraic derivations on $\mathcal{A}(\tau)$.

Definition. A graded derivation of $\mathcal{A}(\tau)$ is said to be *algebraic* or of *type* i_* if it vanishes on the smooth functions on TM.

Remark. To characterize the algebraic derivations of $\mathcal{A}(\tau)$ we need the concept of wedge-bar product of a form and a vector-valued form along τ. This may be introduced in the same way as in the classical case (cf. 1.37(4)). Namely, let $\widetilde{\alpha} \in \mathcal{A}^k(\tau)$, $\widetilde{L} \in \mathcal{B}^\ell(\tau)$.

(1) If $k = 0$, then $\widetilde{\alpha} \barwedge \widetilde{L} := 0$.

(2) If $k > 0$, then $\widetilde{\alpha} \barwedge \widetilde{L} \in \mathcal{A}^{k+\ell-1}$ is given by

$$\begin{aligned}&\widetilde{\alpha} \barwedge \widetilde{L}(\widetilde{X}_1, \dots, \widetilde{X}_{k+\ell-1})\\ &:= \frac{1}{\ell!(k-1)!} \sum_{\sigma \in \mathfrak{S}_{k+\ell-1}} \varepsilon(\sigma) \widetilde{\alpha}\big(\widetilde{L}(\widetilde{X}_{\sigma(1)}, \dots, \widetilde{X}_{\sigma(\ell)}), \widetilde{X}_{\sigma(\ell+1)}, \dots, \widetilde{X}_{\sigma(\ell+k-1)}\big)\end{aligned}$$

$(\widetilde{X}_i \in \mathfrak{X}(\tau),\ 1 \leqq i \leqq k + \ell - 1)$.

In particular,

(3) $\widetilde{\alpha} \barwedge \widetilde{L} = \widetilde{\alpha} \circ \widetilde{L}$, if $\widetilde{\alpha} \in \mathcal{A}^1(\tau)$;

(4) $\widetilde{\alpha} \barwedge \widetilde{L} = \widetilde{\alpha}(\widetilde{L})$, if $\widetilde{\alpha} \in \mathcal{A}^1(\tau)$, $\widetilde{L} \in \mathcal{B}^0(\tau) := \mathfrak{X}(\tau)$;

(5) $\displaystyle \widetilde{\alpha} \barwedge \widetilde{L}(\widetilde{X}_1, \dots, \widetilde{X}_k) = \sum_{i=1}^{k} \widetilde{\alpha}(\widetilde{X}_1, \dots, \widetilde{L}(\widetilde{X}_i), \dots, \widetilde{X}_k)$,
if $\widetilde{\alpha} \in \mathcal{A}^k(\tau)$ $(k > 0)$ and $\widetilde{L} \in \mathcal{B}^1(\tau)$.

Keeping these in mind, the next result may be proved by the same argument as the analogous Proposition in 2.25.

Proposition. *If $\widetilde{L} \in \mathcal{B}^\ell(\tau)$ is a vector-valued ℓ-form along τ then the map*

$$i_{\widetilde{L}} : \mathcal{A}(\tau) \longrightarrow \mathcal{A}(\tau), \quad \widetilde{\alpha} \mapsto i_{\widetilde{L}}\widetilde{\alpha} := \widetilde{\alpha} \barwedge \widetilde{L}$$

is an algebraic derivation of degree $\ell - 1$ of $\mathcal{A}(\tau)$. Conversely, for every algebraic derivation $\mathcal{D}$ of degree $\ell - 1 \geqq -1$ there exists a unique vector-valued ℓ-form $\widetilde{L}$ along τ such that $\mathcal{D} = i_{\widetilde{L}}$.

2.36. v-Lie derivations of $\mathcal{A}(\tau)$.

Lemma 1. *Suppose that $\mathcal{D}$ is a graded derivation of $\mathcal{A}(\tau)$ which vanishes on the vertical lifts of smooth functions on M. Then there exist unique graded derivations $\mathcal{D}_1$, $\mathcal{D}_2$ of $\mathcal{A}(\tau)$ such that $\mathcal{D}_1$ is algebraic, $[\mathcal{D}_2, d^{\mathrm{v}}] = 0$ and $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$.*

Proof. Let $\mathcal{D}$ be of degree k. In view of 2.34, Lemma 1 it follows immediately that the prescription

$$\begin{cases} \mathcal{D}_2 f := \mathcal{D} f & \text{for all } f \in C^\infty(TM) \\ \mathcal{D}_2 \widehat{\alpha} := (-1)^k d^{\mathrm{v}}(\mathcal{D}\bar{\alpha}) & \text{for all } \alpha \in \mathcal{A}^1(M) \end{cases}$$

determines a graded derivation $\mathcal{D}_2 : \mathcal{A}(\tau) \longrightarrow \mathcal{A}(\tau)$ of degree k. Let $\mathcal{D}_1 := \mathcal{D} - \mathcal{D}_2$. Then, evidently, $\mathcal{D}_1$ is an algebraic derivation of $\mathcal{A}(\tau)$. We show that $\mathcal{D}_2$ satisfies the condition $[\mathcal{D}_2, d^{\mathrm{v}}] = 0$. For this it is enough to check that $[\mathcal{D}_2, d^{\mathrm{v}}]$ kills the vertical and the complete lifts of smooth functions on M and the basic one-forms. For every smooth function f and one-form α on M we have

$$[\mathcal{D}_2, d^{\mathrm{v}}]f^{\mathrm{v}} = \mathcal{D}_2(d^{\mathrm{v}}f^{\mathrm{v}}) - (-1)^k d^{\mathrm{v}}(\mathcal{D}_2 f^{\mathrm{v}}) = -(-1)^k d^{\mathrm{v}}(\mathcal{D}f^{\mathrm{v}}) = 0,$$

$$\begin{aligned}[\mathcal{D}_2, d^{\mathrm{v}}]f^c &= \mathcal{D}_2(d^{\mathrm{v}}f^c) - (-1)^k d^{\mathrm{v}}(\mathcal{D}_2 f^c) \overset{2.34,\mathbf{1}}{=} \mathcal{D}_2\widehat{df} - (-1)^k d^{\mathrm{v}}\mathcal{D}f^c \\ &= (-1)^k d^{\mathrm{v}}\mathcal{D}\overline{df} - (-1)^k d^{\mathrm{v}}\mathcal{D}\overline{df} = 0,\end{aligned}$$

$$[\mathcal{D}_2, d^{\mathrm{v}}]\widehat{\alpha} = \mathcal{D}d^{\mathrm{v}}\widehat{\alpha} - (-1)^k d^{\mathrm{v}}\mathcal{D}_2\widehat{\alpha} = -(-1)^{2k} d^{\mathrm{v}} \circ d^{\mathrm{v}}(\mathcal{D}\overline{\alpha}) \overset{2.34,\mathbf{2}}{=} 0.$$

Thus the property $[\mathcal{D}_2, d^{\mathrm{v}}] = 0$ indeed holds, and the existence of the desired graded derivations is proved. It may easily be seen that the definition of $\mathcal{D}_2$ (and hence $\mathcal{D}_1$) is forced by the requirements, so the uniqueness is obvious. □

Lemma 2 and definition. Let $\widetilde{K}$ be a vector-valued k-form along τ. Then

$$d^{\mathrm{v}}_{\widetilde{K}} := [i_{\widetilde{K}}, d^{\mathrm{v}}] = i_{\widetilde{K}} \circ d^{\mathrm{v}} - (-1)^{k-1} d^{\mathrm{v}} \circ i_{\widetilde{K}}$$

is a graded derivation of degree k of $\mathcal{A}(\tau)$, called the *v-Lie derivation with respect to* $\widetilde{K}$. Then we also speak of a *(graded) derivation of type* d^{v}_* of $\mathcal{A}(\tau)$.

Lemma 3. *A graded derivation* $\mathcal{D}$ *of* $\mathcal{A}(\tau)$ *is a v-Lie derivation if, and only if, it vanishes on the vertical lifts of smooth functions on* M *and* $[\mathcal{D}, d^{\mathrm{v}}] = 0$.

Proof. (a) *Necessity.* Let $\widetilde{K} \in \mathcal{B}^k(\tau)$. For every function $f \in C^\infty(M)$ we have

$$\big[i_{\widetilde{K}}, d^{\mathrm{v}}\big]f^{\mathrm{v}} = i_{\widetilde{K}} d^{\mathrm{v}} f^{\mathrm{v}} - (-1)^{k-1} d^{\mathrm{v}} i_{\widetilde{K}} f^{\mathrm{v}} = 0 - 0 = 0.$$

The other condition is also satisfied: applying the graded Jacobi identity, property **2** in 2.34 and the graded anticommutativity, we obtain

$$\begin{aligned}0 &= (-1)^{k-1}\big[[i_{\widetilde{K}}, d^{\mathrm{v}}], d^{\mathrm{v}}\big] + (-1)^{k-1}\big[[d^{\mathrm{v}}, d^{\mathrm{v}}], i_{\widetilde{K}}\big] - \big[[d^{\mathrm{v}}, i_{\widetilde{K}}], d^{\mathrm{v}}\big] \\ &= 2(-1)^{k-1}\big[[i_{\widetilde{K}}, d^{\mathrm{v}}], d^{\mathrm{v}}\big],\end{aligned}$$

whence $[d_{\widetilde{K}}, d^{\mathrm{v}}] = 0$.

(b) *Sufficiency.* Suppose that $\mathcal{D} : \mathcal{A}(\tau) \longrightarrow \mathcal{A}(\tau)$ is a graded derivation of degree k which vanishes on the basic functions and obeys the condition $[\mathcal{D}, d^{\mathrm{v}}] = 0$. Then the requirements

$$\begin{cases} \overline{\mathcal{D}}F := 0 & \text{for all } F \in C^\infty(TM) \\ \overline{\mathcal{D}}\widehat{\alpha} := \mathcal{D}\overline{\alpha} & \text{for all } \alpha \in \mathcal{A}^1(TM) \end{cases}$$

define an algebraic derivation of degree $k-1$. Thus, according to the Proposition in 2.35, there is a unique vector-valued k-form $\widetilde{K}$ along τ such that $\overline{\mathcal{D}} = i_{\widetilde{K}}$. Next we show that $d_{\widetilde{K}} = \mathcal{D}$. We have to check that the two derivations coincide on the vertical and the complete lifts of smooth functions on M and on the basic one-forms. Let $f \in C^\infty(M)$. Evidently, $d_{\widetilde{K}} f^{\mathrm{v}} = 0 = \mathcal{D} f^{\mathrm{v}}$, while

$$d_{\widetilde{K}} f^c = i_{\widetilde{K}} d^{\mathrm{v}} f^c - (-1)^{k-1} d^{\mathrm{v}} i_{\widetilde{K}} f^c = i_{\widetilde{K}} \widehat{df} = \overline{\mathcal{D}}\widehat{df} := \mathcal{D} f^c.$$

Finally, for any one-form $\alpha \in \mathcal{A}^1(M)$ we have

$$\begin{aligned} d_{\widetilde{K}}\widehat{\alpha} &= i_{\widetilde{K}} d^{\mathrm{v}}\widehat{\alpha} - (-1)^{k-1} d^{\mathrm{v}} i_{\widetilde{K}}\widehat{\alpha} = -(-1)^{k-1} d^{\mathrm{v}}\overline{\mathcal{D}}\widehat{\alpha} = (-1)^k d^{\mathrm{v}}\mathcal{D}\overline{\alpha} \\ &= \mathcal{D} d^{\mathrm{v}}\overline{\alpha} = \mathcal{D}\widehat{\alpha}, \end{aligned}$$

because $(-1)^k d^{\mathrm{v}} \circ \mathcal{D} = \mathcal{D} \circ d^{\mathrm{v}}$ by the condition $[\mathcal{D}, d^{\mathrm{v}}] = 0$. This concludes the proof.

Theorem. (Preliminary classification). *Let $\mathcal{D}$ be a graded derivation of degree ℓ of $\mathcal{A}(\tau)$. If $\mathcal{D}$ vanishes on the basic functions, then there exist unique vector-valued forms $\widetilde{K} \in \mathcal{B}^{\ell+1}(\tau)$ and $\widetilde{L} \in \mathcal{B}^{\ell}(\tau)$ such that*

$$\boxed{\mathcal{D} = i_{\widetilde{K}} + d^{\mathrm{v}}_{\widetilde{L}}}\,.$$

Proof. (a) *Existence.* In view of Lemma 1, $\mathcal{D}$ may be decomposed as $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$, where $\mathcal{D}_1$ is an algebraic derivation, and $[\mathcal{D}_2, d^{\mathrm{v}}] = 0$. 2.35, Proposition, and Lemma 3 above guarantee that there exist vector-valued forms $\widetilde{K} \in \mathcal{B}^{\ell+1}(\tau)$, $\widetilde{L} \in \mathcal{B}^{\ell}(\tau)$ such that $\mathcal{D}_1 = i_{\widetilde{K}}$, $\mathcal{D}_2 = d^{\mathrm{v}}_{\widetilde{L}}$. Thus $\mathcal{D} = i_{\widetilde{K}} + d^{\mathrm{v}}_{\widetilde{L}}$.

(b) *Uniqueness.* Suppose that $\mathcal{D} = i_{\widetilde{K}_1} + d^{\mathrm{v}}_{\widetilde{L}_1} = i_{\widetilde{K}_2} + d^{\mathrm{v}}_{\widetilde{L}_2}$. From Lemma 1 and 2.35, Proposition it follows at once that $i_{\widetilde{K}_1} = i_{\widetilde{K}_2}$ and hence $\widetilde{K}_1 = \widetilde{K}_2$. The remaining relation $d^{\mathrm{v}}_{\widetilde{L}_1} = d^{\mathrm{v}}_{\widetilde{L}_2}$ may be written in the form $[i_{\widetilde{L}_1 - \widetilde{L}_2}, d^{\mathrm{v}}] = 0$, thus the derivation $\widetilde{\mathcal{D}} := i_{\widetilde{L}_1 - \widetilde{L}_2}$ is both an algebraic and a v-Lie derivation. Hence $\widetilde{\mathcal{D}} \restriction C^\infty(TM) = 0$, further

$$\widetilde{\mathcal{D}}\widehat{\alpha} = \widetilde{\mathcal{D}} d^{\mathrm{v}}\overline{\alpha} = (-1)^\ell d^{\mathrm{v}}\widetilde{\mathcal{D}}\overline{\alpha} = 0 \quad \text{for all } \alpha \in \mathcal{A}^1(M);$$

therefore $\widetilde{\mathcal{D}} = i_{\widetilde{L}_1 - \widetilde{L}_2} = 0$, $\widetilde{L}_1 = \widetilde{L}_2$. This finishes the proof.

2.37. h-Lie derivations of $\mathcal{A}(\tau)$.

In this subsection we assume that a horizontal map $\mathcal{H} : TM \times_M TM \longrightarrow TTM$ is given.

Lemma 1 and definition. (1) *There is a unique graded derivation $d^h : \mathcal{A}(\tau) \longrightarrow \mathcal{A}(\tau)$ of degree 1 such that*

$$\begin{cases} (d^h f)(\widetilde{X}) := df(\mathcal{H}\widetilde{X}) & \text{for all } f \in C^\infty(TM),\ \widetilde{X} \in \mathfrak{X}(\tau); \\ d^h \widehat{\alpha} = \widehat{d\alpha} & \text{for all } \alpha \in \mathcal{A}^1(M). \end{cases}$$

d^h *is said to be the* horizontal exterior (*or briefly h-*exterior) derivative *on $\mathcal{A}(\tau)$ (or in $\tau^*\tau$) with respect to the horizontal map $\mathcal{H}$.*

(2) *For every vector-valued k-form $\widetilde{K}$ along τ the map $d^h_{\widetilde{K}} := [i_{\widetilde{K}}, d^h]$ is a graded derivation of $\mathcal{A}(\tau)$ of degree k, called the h-*Lie derivation with respect to $\widetilde{K}$. *Then we also speak of a* (graded) derivation of type d^h_* *of* $\mathcal{A}(\tau)$.

Lemma 2. *$d^h f^{\mathrm{v}} = d^{\mathrm{v}} f^c$ for all $f \in C^\infty(M)$.*

Proof. It is enough to check that for any vector field X on M we have $(d^h f^{\mathrm{v}})(\widehat{X}) = (d^{\mathrm{v}} f^c)(\widehat{X})$. But this is immediate. On the one hand

$$(d^h f^{\mathrm{v}})(\widehat{X}) := (\mathcal{H}\widehat{X}) f^{\mathrm{v}} = X^h f^{\mathrm{v}} = X^c f^{\mathrm{v}} = (Xf)^{\mathrm{v}},$$

using in the last step 2.20, Corollary 2, and taking into account in the preceding step that $X^h - X^c$ is obviously vertical. On the other hand,

$$(d^{\mathrm{v}} f^c)(\widehat{X}) := (\mathbf{i}\widehat{X}) f^c = X^{\mathrm{v}} f^c \overset{2.20}{=} (Xf)^{\mathrm{v}}$$

which concludes the proof.

Lemma 3. *For every one-form $\widetilde{\alpha} \in \mathcal{A}^1(\tau)$ and vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have*

$$d^h \widetilde{\alpha}(\widetilde{X}, \widetilde{Y}) = (\mathcal{H}\widetilde{X})\widetilde{\alpha}(\widetilde{Y}) - (\mathcal{H}\widetilde{Y})\widetilde{\alpha}(\widetilde{X}) - \widetilde{\alpha}\big(\mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]\big).$$

Proof. (1) We shall use a local argument. Choose a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M and consider the induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ on TM. Then $\widetilde{\alpha}$, $\widetilde{X}$ and $\widetilde{Y}$ may be represented in the form

$$\widetilde{\alpha} \restriction \tau^{-1}(\mathcal{U}) = \widetilde{\alpha}_i \widehat{du^i},\ \widetilde{X} \restriction \tau^{-1}(\mathcal{U}) = \widetilde{X}^i \widehat{\frac{\partial}{\partial u^i}},\ \widetilde{Y} \restriction \tau^{-1}(\mathcal{U}) = \widetilde{Y}^i \widehat{\frac{\partial}{\partial u^i}}$$

respectively, where $\widetilde{\alpha}_i$, $\widetilde{X}^i$, $\widetilde{Y}^i$ $(1 \leqq i \leqq n)$ are smooth functions on $\tau^{-1}(\mathcal{U})$. Notice that

$$\mathbf{j}\frac{\partial}{\partial x^i} = \mathbf{j}\left(\frac{\partial}{\partial u^i}\right)^c = \widehat{\frac{\partial}{\partial u^i}} \quad (1 \leqq i \leqq n)$$

(see 2.20, Corollary 3).

(2) Observe that

$$(\mathcal{H}\widetilde{X})x^i = \widetilde{X}^i, \quad (\mathcal{H}\widetilde{Y})x^i = \widetilde{Y}^i \quad (1 \leqq i \leqq n).$$

Indeed, if X is a vector field on M and $X \restriction U = X^i \frac{\partial}{\partial u^i}$, then

$$(\mathcal{H}\widehat{X})x^i = X^h x^i = X^c(u^i)^{\mathrm{v}} = (Xu^i)^{\mathrm{v}} = X^i \circ \tau \quad (1 \leqq i \leqq n)$$

which implies the desired relations.

(3) $$[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \restriction \tau^{-1}(\mathcal{U}) = \left([\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]x^i\right)\frac{\partial}{\partial x^i} + ([\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]y^i)\frac{\partial}{\partial y^i},$$

therefore

$$\mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \restriction \tau^{-1}(\mathcal{U}) \overset{(1)}{=} \left([\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]x^i\right)\widehat{\frac{\partial}{\partial u^i}} \overset{(2)}{=} [(\mathcal{H}\widetilde{X})\widetilde{Y}^i - (\mathcal{H}\widetilde{Y})\widetilde{X}^i]\widehat{\frac{\partial}{\partial u^i}}.$$

(4) After these preparations, we now see that

$$d^h\widetilde{\alpha} \restriction \tau^{-1}(\mathcal{U}) = d^h\widetilde{\alpha}_i \wedge \widehat{du^i} + \widetilde{\alpha}_i d^h\widehat{du^i} = d^h\widetilde{\alpha}_i \wedge \widehat{du^i} + \widetilde{\alpha}_i\widehat{d^2u^i} = d^h\widetilde{\alpha}_i \wedge \widehat{du^i},$$

therefore, over $\tau^{-1}(\mathcal{U})$,

$$\begin{aligned} d^h\widetilde{\alpha}(\widetilde{X}, \widetilde{Y}) &= (d^h\widetilde{\alpha}_i)(\widetilde{X})\widehat{du^i}(\widetilde{Y}) - (d^h\widetilde{\alpha}_i)(\widetilde{Y})\widehat{du^i}(\widetilde{X}) = \left[(\mathcal{H}\widetilde{X})\widetilde{\alpha}_i\right]\widetilde{Y}^i \\ &\quad - \left[(\mathcal{H}\widetilde{Y})\widetilde{\alpha}_i\right]\widetilde{X}^i = (\mathcal{H}\widetilde{X})(\widetilde{\alpha}_i\widetilde{Y}^i) - \widetilde{\alpha}_i(\mathcal{H}\widetilde{X})\widetilde{Y}^i - (\mathcal{H}\widetilde{Y})(\widetilde{\alpha}_i\widetilde{X}^i) \\ &\quad + \widetilde{\alpha}_i(\mathcal{H}\widetilde{Y})\widetilde{X}^i \overset{(3)}{=} (\mathcal{H}\widetilde{X})\widetilde{\alpha}(\widetilde{Y}) - (\mathcal{H}\widetilde{Y})\widetilde{\alpha}(\widetilde{X}) - \alpha(\mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]). \end{aligned}$$

This concludes the proof.

Proposition. *Let $\widetilde{\alpha}$ be a k-form along τ. Then*

$$\boxed{\begin{aligned} (d^h\widetilde{\alpha})(\widetilde{X}_1, \dots, \widetilde{X}_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}(\mathcal{H}\widetilde{X}_i)\left[\widetilde{\alpha}(\widetilde{X}_1, \dots, \overset{*}{\widetilde{X}}_i, \dots, \widetilde{X}_{k+1})\right] \\ &+ \sum_{1 \leqq i < j \leqq k}(-1)^{i+j}\widetilde{\alpha}\left(\mathbf{j}[\mathcal{H}\widetilde{X}_i, \mathcal{H}\widetilde{X}_j], \widetilde{X}_1, \dots, \overset{*}{\widetilde{X}}_i, \dots, \overset{*}{\widetilde{X}}_j, \dots, \widetilde{X}_{k+1}\right) \end{aligned}}$$

($\widetilde{X}_i \in \mathfrak{X}(\tau)$, $1 \leqq i \leqq k+1$; $\overset{*}{\widetilde{X}}_i$ means that $\widetilde{X}_i$ has to be deleted.)

Strategy of proof. We define a "new" operator $\widetilde{d^h}$ by the box formula and check that $\widetilde{d^h}$ is a graded derivation of $\mathcal{A}(\tau)$ of degree 1. The operators $\widetilde{d^h}$ and d^h obviously coincide on $C^\infty(TM)$, moreover, due to Lemma 3, they are also indentical on $\mathcal{A}^1(\tau)$. Then 2.34, Lemma 1 guarantees that $\widetilde{d^h} = d^h$, thus proving the Proposition.

Theorem. (Fine classification). *Let $\mathcal{D}$ be a graded derivation of degree k of $\mathcal{A}(\tau)$. There exist unique vector-valued forms $\widetilde{K}_1 \in \mathcal{B}^{k+1}(\tau)$ and $\widetilde{K}_2, \widetilde{K}_3 \in \mathcal{B}^k(\tau)$ such that*

$$\boxed{\mathcal{D} = i_{\widetilde{K}_1} + d^{\mathrm{v}}_{\widetilde{K}_2} + d^h_{\widetilde{K}_3}}\,.$$

Proof. (a) *Existence.* Let $\widetilde{X}_1, \dots, \widetilde{X}_k$ be arbitrarily chosen and temporarily fixed vector fields along τ. In view of 2.20, Lemma 1 the map

$$f^c \in C^\infty(TM) \mapsto (\mathcal{D} f^{\mathrm{v}})(\widetilde{X}_1, \dots, \widetilde{X}_k) \in C^\infty(TM) \quad (f \in C^\infty(M))$$

determines a unique vector field on TM; we denote this vector field by $\varphi(\widetilde{X}_1, \dots, \widetilde{X}_k)$. Our first claim is that

$$\varphi(\widetilde{X}_1, \dots, \widetilde{X}_k) \in \mathfrak{X}^{\mathrm{v}}(TM).$$

To show this, let $\xi := \varphi(\widetilde{X}_1, \dots, \widetilde{X}_k)$ for brevity. Then for any function $f \in C^\infty(M)$ we have

$$\xi\big((f^2)^c\big) = 2\xi(f^c f^{\mathrm{v}}) = 2f^{\mathrm{v}}\xi(f^c) + 2f^c\xi(f^{\mathrm{v}})$$

(cf. 2.20, proof of Lemma 1). On the other hand,

$$\begin{aligned}\xi\big((f^2)^c\big) &:= [\mathcal{D}(f^2)^{\mathrm{v}}](\widetilde{X}_1, \dots, \widetilde{X}_k) = [\mathcal{D}(f^{\mathrm{v}})^2](\widetilde{X}_1, \dots, \widetilde{X}_k) \\ &= 2f^{\mathrm{v}}(\mathcal{D}f^{\mathrm{v}})(\widetilde{X}_1, \dots, \widetilde{X}_k) = 2f^{\mathrm{v}}\xi(f^c).\end{aligned}$$

Comparing the two results we conclude that ξ vanishes on the vertical lifts of smooth functions on M, therefore it is indeed vertical.

Now let the map $\widetilde{K}_3$ be defined by

$$\widetilde{K}_3(\widetilde{X}_1, \dots, \widetilde{X}_k) := \mathcal{V}\varphi(\widetilde{X}_1, \dots, \widetilde{X}_k),$$

where $\mathcal{V}$ is the vertical map belonging to $\mathcal{H}$. Then $\widetilde{K}_3$ is obviously $C^\infty(TM)$-multilinear and skew-symmetric, hence $\widetilde{K}_3 \in \mathcal{B}^k(\tau)$. We show that the operators $d^h_{\widetilde{K}_3}$ and $\mathcal{D}$ coincide on the basic functions. Indeed, for any function

$f \in C^\infty(M)$ we have

$$
\begin{aligned}
(d^h_{\widetilde{K}_3} f^{\mathrm{v}})(\widetilde{X}_1, \dots, \widetilde{X}_k) &:= \big([i_{\widetilde{K}_3}, d^h] f^{\mathrm{v}}\big)(\widetilde{X}_1, \dots, \widetilde{X}_k) = \big(i_{\widetilde{K}_3}(d^h f^{\mathrm{v}})\big)(\widetilde{X}_1, \dots, \widetilde{X}_k) \\
&\overset{2.35(3)}{=} (d^h f^{\mathrm{v}})\widetilde{K}_3(\widetilde{X}_1, \dots, \widetilde{X}_k) \overset{\text{Lemma 2}}{=} (d^{\mathrm{v}} f^c)\widetilde{K}_3(\widetilde{X}_1, \dots, \widetilde{X}_k) \\
&:= [\mathbf{i}\widetilde{K}_3(\widetilde{X}_1, \dots, \widetilde{X}_k)] f^c = [\mathbf{i} \circ \mathcal{V}\varphi(\widetilde{X}_1, \dots, \widetilde{X}_k)] f^c = [\mathbf{v}\varphi(\widetilde{X}_1, \dots, \widetilde{X}_k)] f^c \\
&= \varphi(\widetilde{X}_1, \dots, \widetilde{X}_k) f^c := (\mathcal{D} f^{\mathrm{v}})(\widetilde{X}_1, \dots, \widetilde{X}_k)
\end{aligned}
$$

(using the fact that $\varphi(\widetilde{X}_1, \dots, \widetilde{X}_k)$ is vertical). Since $\mathcal{D} - d^h_{\widetilde{K}_3}$ kills the basic functions, the theorem of preliminary classification in 2.36 guarantees the existence of unique vector-valued forms $\widetilde{K}_1 \in \mathcal{B}^{k+1}(\tau)$ and $\widetilde{K}_2 \in \mathcal{B}^k(\tau)$ such that $\mathcal{D} - d^h_{\widetilde{K}_3} = i_{\widetilde{K}_1} + d^{\mathrm{v}}_{\widetilde{K}_2}$, whence

$$\mathcal{D} = i_{\widetilde{K}_1} + d^{\mathrm{v}}_{\widetilde{K}_2} + d^h_{\widetilde{K}_3}.$$

Thus the desired decomposition of $\mathcal{D}$ indeed exists.

(b) *Uniqueness.* Suppose that we have two decompositions for $\mathcal{D}$:

$$\mathcal{D} = i_{\widetilde{K}_1} + d^{\mathrm{v}}_{\widetilde{K}_2} + d^h_{\widetilde{K}_3} = i_{\widetilde{L}_1} + d^{\mathrm{v}}_{\widetilde{L}_2} + d^h_{\widetilde{L}_3}.$$

Then

$$i_{\widetilde{K}_1 - \widetilde{L}_1} + d^{\mathrm{v}}_{\widetilde{K}_2 - \widetilde{L}_2} = d^h_{\widetilde{L}_3} - d^h_{\widetilde{K}_3}.$$

Since the left-hand side operator kills the basic functions, so does the right-hand side operator. Thus for any function $f \in C^\infty(M)$ and vector fields $\widetilde{X}_1, \dots, \widetilde{X}_k$ along τ we have

$$
\begin{aligned}
\big[(d^h_{\widetilde{L}_3} - d^h_{\widetilde{K}_3}) f^{\mathrm{v}}\big](\widetilde{X}_1, \dots, \widetilde{X}_k) &= \big[(i_{\widetilde{L}_3} - i_{\widetilde{K}_3}) d^h f^{\mathrm{v}}\big](\widetilde{X}_1, \dots, \widetilde{X}_k) \\
&= \big[\mathcal{H}(\widetilde{L}_3(\widetilde{X}_1, \dots, \widetilde{X}_k) - \widetilde{K}_3(\widetilde{X}_1, \dots, \widetilde{X}_k))\big] f^{\mathrm{v}} = 0.
\end{aligned}
$$

From this it follows that the horizontal vector field $\mathcal{H}(\widetilde{L}_3(\widetilde{X}_1, \dots, \widetilde{X}_k) - \widetilde{K}_3(\widetilde{X}_1, \dots, \widetilde{X}_k))$ is also vertical, therefore it is the zero vector field. Since $\mathcal{H}$ is injective, we conclude that $\widetilde{L}_3 = \widetilde{K}_3$. As we have already seen, the uniqueness of the other two vector-valued forms is assured by the theorem of preliminary classification. □

2.38. Relations with the classical theory.

We assume that a horizontal map $\mathcal{H}$ is specified in for τ.

Lemma. *If β is a semibasic form on TM, then $d_J\beta$ and $d_{\mathbf{h}}\beta$ are also semibasic.*

Proof. Since (locally) every semibasic form may be expressed as a $C^\infty(TM)$-linear combination of wedge products of semibasic one-forms, it is sufficient to verify the assertion for a semibasic one-form $\beta \in \mathcal{A}_0^1(TM)$. Due to the fact that β is semibasic, we have

$$i_J\beta = 0, \quad i_{\mathbf{h}}\beta = \beta,$$

therefore

$$d_J\beta = i_J d\beta, \quad d_{\mathbf{h}}\beta = i_{\mathbf{h}} d\beta - d\beta.$$

Now an immediate calculation shows that $i_J d\beta$ and $i_{\mathbf{h}} d\beta - d\beta$ both kill the pairs of the form

$$(X^{\mathrm{v}}, Y^{\mathrm{v}}),\ (X^{\mathrm{v}}, Y^h),\ (X^h, Y^{\mathrm{v}}) \quad (X, Y \in \mathfrak{X}(M))$$

thus proving the lemma.

Proposition 1. *For every form $\widetilde{\alpha}$ along τ we have*

$$\boxed{(d^{\mathrm{v}}\widetilde{\alpha})_0 = d_J(\widetilde{\alpha})_0}.$$

Proof. It is sufficient to show that the relation is true for 0-forms and one-forms along τ. If $\widetilde{\alpha} := f \in C^\infty(TM) =: \mathcal{A}^0(\tau)$, then

$$(\widetilde{\alpha})_0 := f \in C^\infty(TM) =: \mathcal{A}_0^0(TM),$$

while for any vector field ξ on TM we have

$$(d^{\mathrm{v}}f)_0(\xi) := (d^{\mathrm{v}}f)(\mathbf{j}\xi) := df(\mathbf{i} \circ \mathbf{j}\xi) = df(J\xi) = (d_J f)(\xi) = (d_J(f)_0)(\xi),$$

therefore $(d^{\mathrm{v}}f)_0 = d_J(f)_0$.

As for the second case, it is enough to check the relation for basic one-forms. If $\alpha \in \mathcal{A}^1(M)$, then $d^{\mathrm{v}}\widehat{\alpha} := 0$. On the other hand, $(\widehat{\alpha})_0 = \alpha^{\mathrm{v}}$ (see 2.22(2)), thus for any vector fields X, Y on M we have

$$\begin{aligned}&(d_J(\widehat{\alpha})_0)(X^c, Y^c) = (d_J\alpha^{\mathrm{v}})(X^c, Y^c) = (i_J d\alpha^{\mathrm{v}})(X^c, Y^c) = d\alpha^{\mathrm{v}}(X^{\mathrm{v}}, Y^c)\\ &+ d\alpha^{\mathrm{v}}(X^c, Y^{\mathrm{v}}) = X^{\mathrm{v}}\alpha^{\mathrm{v}}(Y^c) - Y^c\alpha^{\mathrm{v}}(X^{\mathrm{v}}) - \alpha^{\mathrm{v}}([X^{\mathrm{v}}, Y^c]) + X^c\alpha^{\mathrm{v}}(Y^{\mathrm{v}}) - Y^{\mathrm{v}}\alpha^{\mathrm{v}}(X^c)\\ &- \alpha^{\mathrm{v}}([X^c, Y^{\mathrm{v}}]) = X^{\mathrm{v}}(\alpha(Y))^{\mathrm{v}} - \alpha^{\mathrm{v}}([X,Y]^{\mathrm{v}}) - Y^{\mathrm{v}}[\alpha(X)]^{\mathrm{v}} - \alpha([X,Y]^{\mathrm{v}}) = 0.\end{aligned}$$

(We applied some frequently used relations from 2.20.)

This concludes the proof of Proposition 1.

Corollary 1. *For any k-form $\widetilde{\alpha}$ along τ we have*

$$\boxed{\begin{aligned} d^{\mathrm{v}}\widetilde{\alpha}(\widetilde{X}_1,\dots,\widetilde{X}_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}(\mathbf{i}\widetilde{X}_i)\widetilde{\alpha}(\widetilde{X}_1,\dots,\overset{*}{\widetilde{X}}_i,\dots,\widetilde{X}_{k+1}) \\ &+ \sum_{1\leqq i<j\leqq k}(-1)^{i+j}\widetilde{\alpha}(\mathcal{V}[\mathbf{i}\widetilde{X}_i,\mathbf{i}\widetilde{X}_j],\dots,\overset{*}{\widetilde{X}}_i,\dots,\overset{*}{\widetilde{X}}_j,\dots,\widetilde{X}_{k+1}) \end{aligned}}$$

where $\mathcal{V}$ is the vertical map belonging to an arbitrarily chosen horizontal map; $\widetilde{X}_i \in \mathfrak{X}(\tau)$, $1 \leqq i \leqq k+1$; $\overset{*}{\widetilde{X}}_i$ means that $\widetilde{X}_i$ has to be deleted.

Proof. Any section $\widetilde{X}_i$ may be represented in the form $\widetilde{X}_i = \mathbf{j}\xi_i$, $\xi_i \in \mathfrak{X}(TM)$; then $J\xi_i = \mathbf{i}\widetilde{X}_i$ $(1 \leqq i \leqq k+1)$. Applying repeatedly that $(\widetilde{\alpha})_0$ is semibasic, we obtain:

$$\begin{aligned} &(d^{\mathrm{v}}\widetilde{\alpha})(\widetilde{X}_1,\dots,\widetilde{X}_{k+1}) = (d^{\mathrm{v}}\widetilde{\alpha})(\mathbf{j}\xi_1,\dots,\mathbf{j}\xi_{k+1}) = (d^{\mathrm{v}}\widetilde{\alpha})_0(\xi_1,\dots,\xi_{k+1}) \overset{\text{Prop. 1}}{=} \\ &= (d_J(\widetilde{\alpha})_0)(\xi_1,\dots,\xi_{k+1}) = (i_J d(\widetilde{\alpha})_0)(\xi_1,\dots,\xi_{k+1}) \\ &= \sum_{i=1}^{k+1} d(\widetilde{\alpha})_0(\xi_1,\dots,J\xi_i,\dots,\xi_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}(J\xi_i)[(\widetilde{\alpha})_0(\xi_1,\dots,\overset{*}{\xi}_i,\dots,\xi_{k+1}) \\ &+ \sum_{1\leqq i<j\leqq k+1}(-1)^{i+j}(\widetilde{\alpha})_0([J\xi_i,\xi_j]+[\xi_i,J\xi_j],\xi_1,\dots,\overset{*}{\xi}_i,\dots,\overset{*}{\xi}_j,\dots,\xi_{k+1}) \\ &= \sum_{i=1}^{k+1}(-1)^{i+1}(\mathbf{i}\widetilde{X}_i)\widetilde{\alpha}(\widetilde{X}_1,\dots,\overset{*}{\widetilde{X}}_i,\dots,\widetilde{X}_{k+1}) \\ &+ \sum_{1\leqq i<j\leqq k+1}(-1)^{i+j}\widetilde{\alpha}(\mathbf{j}[J\xi_i,\xi_j]+\mathbf{j}[\xi_i,J\xi_j],\widetilde{X}_1,\dots,\overset{*}{\widetilde{X}}_i,\dots,\overset{*}{\widetilde{X}}_j,\dots\widetilde{X}_{k+1}). \end{aligned}$$

Since $N_J = 0$, we have

$$0 = [J\xi_i, J\xi_j] - J[J\xi_i,\xi_j] - J[\xi_i,J\xi_j] = \mathbf{i}(\mathcal{V}[\mathbf{i}\widetilde{X}_i,\mathbf{i}\widetilde{X}_j] - \mathbf{j}[J\xi_i,\xi_j] - \mathbf{j}[\xi_i,J\xi_j],$$

hence $\mathbf{j}[J\xi_i,\xi_j] + \mathbf{j}[\xi_i,J\xi_j] = \mathcal{V}[\mathbf{i}\widetilde{X}_i,\mathbf{i}\widetilde{X}_j]$. This concludes the proof.

Proposition 2. *For any k-form $\widetilde{\alpha}$ and vector-valued form $\widetilde{L}$ along τ we have*

$$\boxed{i_{\mathbf{F}\circ\widetilde{L}_0}(\widetilde{\alpha})_0 = (i_{\widetilde{L}}\,\widetilde{\alpha})_0}$$

where $\mathbf{F}$ is the almost complex structure associated to an arbitrarily chosen horizontal map.

Proof. The equality holds trivially if $\widetilde{\alpha}$ is a 0-form, i.e. a smooth function on TM. Consider a one-form $\widetilde{\alpha}$ along τ, and suppose that $\widetilde{L}$ is of degree ℓ. Let $\mathbf{F}$ be associated to a horizontal map $\mathcal{H}$; then by 2.19, Lemma

$$\mathbf{F} \circ \mathbf{i} = \mathcal{H} \circ \mathcal{V} \circ \mathbf{i} - \mathbf{i} \circ \mathbf{j} \circ \mathbf{i} = \mathcal{H}.$$

For any vector fields $\xi_1, \dots \xi_\ell$ on TM we have

$$\begin{aligned}\big(i_{\mathbf{F}\circ\widetilde{L}_0}(\widetilde{\alpha})_0\big)(\xi_1,\dots,\xi_\ell) &= (\widetilde{\alpha})_0\big(\mathbf{F}\widetilde{L}_0(\xi_1,\dots,\xi_\ell)\big) = (\widetilde{\alpha})_0\big(\mathbf{F}\circ\mathbf{i}\widetilde{L}(\mathbf{j}\xi_1,\dots,\mathbf{j}\xi_\ell)\big)\\ &= (\widetilde{\alpha})_0\big(\mathcal{H}\widetilde{L}(\mathbf{j}\xi_1,\dots,\mathbf{j}\xi_\ell)\big) = \widetilde{\alpha}\big(\mathbf{j}\circ\mathcal{H}\widetilde{L}(\mathbf{j}\xi_1,\dots,\mathbf{j}\xi_\ell)\big)\\ &= \widetilde{\alpha}\big(\widetilde{L}(\mathbf{j}\xi_1,\dots,\mathbf{j}\xi_\ell)\big) = (i_{\widetilde{L}}\widetilde{\alpha})(\mathbf{j}\xi_1,\dots,\mathbf{j}\xi_\ell) = (i_{\widetilde{L}}\,\widetilde{\alpha})_0(\xi_1,\dots,\xi_\ell),\end{aligned}$$

whence the assertion.

Corollary 2. *Under the preceding condition,*

$$\boxed{\big(d^{\mathrm{v}}_{\widetilde{L}}\,\widetilde{\alpha}\big)_0 = \big[i_{\mathbf{F}\circ\widetilde{L}_0}, d_J\big](\widetilde{\alpha})_0}\,.$$

Proof. $(d^{\mathrm{v}}_{\widetilde{L}}\,\widetilde{\alpha})_0 = \big((i_{\widetilde{L}} \circ d^{\mathrm{v}} - (-1)^{\ell-1} d^{\mathrm{v}} \circ i_{\widetilde{L}})\widetilde{\alpha}\big)_0 = (i_{\widetilde{L}}\, d^{\mathrm{v}}\widetilde{\alpha})_0 -$ $(-1)^{\ell-1}\big(d^{\mathrm{v}}(i_{\widetilde{L}}\,\widetilde{\alpha})\big)_0 = i_{\mathbf{F}\circ\widetilde{L}_0}(d^{\mathrm{v}}\widetilde{\alpha})_0 - (-1)^{\ell-1} d_J(i_{\widetilde{L}}\,\widetilde{\alpha})_0 =$ $i_{\mathbf{F}\circ\widetilde{L}_0} \circ d_J(\widetilde{\alpha})_0 - (-1)^{\ell-1} d_J \circ i_{\mathbf{F}\circ\widetilde{L}_0}(\widetilde{\alpha})_0 = [i_{\mathbf{F}\circ\widetilde{L}_0}, d_J](\widetilde{\alpha})_0.$ □

Proposition 3. *Let* $\mathbf{h}$ *be the horizontal projector belonging to the horizontal map* $\mathcal{H}$*. For every form* $\widetilde{\alpha}$ *along* τ *we have*

$$\boxed{(d^h\widetilde{\alpha})_0 = d_{\mathbf{h}}(\widetilde{\alpha})_0}\,.$$

Proof. Due to the preparatory Lemma, the asserted relation makes sense. All we need to check is that for every 0-form $f \in C^\infty(TM)$ and basic one-form $\widehat{\alpha} \in \mathcal{A}^1(\tau)$ $(\alpha \in \mathcal{A}^1(M))$ the relation is true. The first case is immediate: for any vector field ξ on TM we have

$$\begin{aligned}(d^{\mathbf{h}} f)_0\xi &:= d^h f(\mathbf{j}\xi) := df(\mathcal{H}\circ\mathbf{j}\xi) = df(\mathbf{h}\xi) = (i_{\mathbf{h}} df)(\xi)\\ &= (d_{\mathbf{h}} f)(\xi) = (d_{\mathbf{h}}(f)_0)(\xi).\end{aligned}$$

To verify the relation for a basic one-form $\widehat{\alpha}$, let X and Y be any two vector fields on M. On the one hand, we have

$$(d^h\widehat{\alpha})_0(X^h, Y^h) = d^h\widehat{\alpha}(\mathbf{j}X^h, \mathbf{j}Y^h) = \widehat{d\alpha}(\widehat{X}, \widehat{Y}) = (d\alpha(X,Y))^{\mathrm{v}}.$$

On the other hand,

$$\begin{aligned}\big(d_{\mathbf{h}}(\widehat{\alpha})_0\big)(X^h,Y^h) &= (d_{\mathbf{h}}\alpha^{\mathrm{v}})(X^h,Y^h) = (i_{\mathbf{h}}d\alpha^{\mathrm{v}})(X^h,Y^h) - (d\alpha^{\mathrm{v}})(X^h,Y^h)\\ &= (d\alpha^{\mathrm{v}})(X^h,Y^h) = X^h\alpha^{\mathrm{v}}(Y^h) - Y^h\alpha^{\mathrm{v}}(X^h) - \alpha^{\mathrm{v}}\big([X^h,Y^h]\big)\\ &= X^h\big(\alpha(Y)\big)^{\mathrm{v}} - Y^h\big(\alpha(X)\big)^{\mathrm{v}} - \alpha^{\mathrm{v}}\big([X,Y]^h\big)\\ &= \big(X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])\big)^{\mathrm{v}} = \big((d\alpha)(X,Y)\big)^{\mathrm{v}}\end{aligned}$$

(taking into account some frequently used relations).

We conclude that $(d^h\widehat{\alpha})_0 = (d_{\mathbf{h}}\widehat{\alpha})_0$, which finishes the proof.

Corollary 3. *Keeping the hypothesis of Proposition 3, let a vector-valued form $\widetilde{L} \in \mathcal{B}^\ell(\tau)$ be given. Then*

$$\boxed{\big(d^h_{\widetilde{L}}\,\widetilde{\alpha}\big)_0 = \big[i_{\mathbf{F}\circ\widetilde{L}_0}, d_{\mathbf{h}}\big](\widetilde{\alpha})_0}\,.$$

Proof. $(d^h_{\widetilde{L}}\widetilde{\alpha})_0 = \big((i_{\widetilde{L}} \circ d^h - (-1)^{\ell-1}d^h \circ i_{\widetilde{L}})\widetilde{\alpha}\big)_0 = (i_{\widetilde{L}}\, d^h\widetilde{\alpha})_0 -$ $(-1)^{\ell-1}\big(d^h(i_{\widetilde{L}}\,\widetilde{\alpha})\big)_0 = i_{\mathbf{F}\circ\widetilde{L}_0}(d^h\widetilde{\alpha})_0 - (-1)^{\ell-1}d_{\mathbf{h}}(i_{\widetilde{L}}\,\widetilde{\alpha})_0 =$ $\big(i_{\mathbf{F}\circ\widetilde{L}_0} \circ d_{\mathbf{h}} - (-1)^{\ell-1}d_{\mathbf{h}} \circ i_{\mathbf{F}\circ\widetilde{L}_0}\big)(\widetilde{\alpha})_0 = [i_{\mathbf{F}\circ\widetilde{L}_0}, d_{\mathbf{h}}](\widetilde{\alpha})_0.$ □

2.39. Lie derivatives on the mixed tensor algebra along τ.

We continue to assume that a horizontal map $\mathcal{H}$ is specified in τ.

Let a vector field ξ on TM be given. Applying a 'reasonable' version of Willmore's theorem on tensor derivations (see 1.32) we may define three tensor derivations of 'Lie-type' on the mixed tensor algebra $\mathcal{T}^{\bullet}_{\bullet}(\tau)$:

(1) *Vertical Lie derivative* with respect to ξ:

$$\boxed{\begin{array}{lcl}\mathcal{L}^{\mathrm{v}}_\xi f &:=& \xi f \quad \text{for all } f \in C^\infty(TM),\\ \mathcal{L}^{\mathrm{v}}_\xi \widetilde{Y} &:=& \mathcal{V}[\xi, \mathbf{i}\widetilde{Y}] \quad \text{for all } \widetilde{Y} \in \mathfrak{X}(\tau)\end{array}}\,.$$

(2) *Horizontal Lie derivative* with respect to ξ:

$$\boxed{\begin{array}{lcl}\mathcal{L}^{h}_\xi f &:=& \xi f \quad \text{for all } f \in C^\infty(TM),\\ \mathcal{L}^{h}_\xi \widetilde{Y} &:=& \mathbf{j}[\xi, \mathcal{H}\widetilde{Y}] \quad \text{for all } \widetilde{Y} \in \mathfrak{X}(\tau)\end{array}}\,.$$

(3) *Total Lie derivative* with respect to ξ:

$$\boxed{\begin{array}{rcl} \mathcal{L}_\xi f &:=& \xi f \quad \text{for all } f \in C^\infty(TM), \\ \mathcal{L}_\xi \widetilde{Y} &:=& \mathcal{V}[\mathbf{v}\xi, \mathbf{i}\widetilde{Y}] + \mathbf{j}[\mathbf{h}\xi, \mathcal{H}\widetilde{Y}] \quad \text{for all } \widetilde{Y} \in \mathfrak{X}(\tau) \end{array}}.$$

Remark. (1) It is immediate to check that these operators indeed obey the condition of Willmore's theorem, i.e. $\mathcal{L}^{\mathrm{v}}_\xi(f\widetilde{Y}) = (\xi f)\widetilde{Y} + f\mathcal{L}^{\mathrm{v}}_\xi\widetilde{Y}$, etc.

(2) If, in particular, $\widetilde{\alpha} \in \mathcal{A}(\tau)$, i.e. $\widetilde{\alpha}$ is skew-symmetric, then so are $\mathcal{L}^{\mathrm{v}}_\xi \widetilde{\alpha}$, $\mathcal{L}^h_\xi \widetilde{\alpha}$ and $\mathcal{L}_\xi \widetilde{\alpha}$. Moreover, the Lie derivatives with respect to ξ are graded derivations of $\mathcal{A}(\tau)$ of degree zero. The operators $\mathcal{L}^{\mathrm{v}}$, $\mathcal{L}^h$ and $\mathcal{L}$ are related to the exterior derivatives d^{v} and d^h via 'H. Cartan's magic formulae'; namely, the following relations are true:

$$\boxed{\begin{array}{l} \mathcal{L}^{\mathrm{v}}_{\mathbf{i}\widetilde{X}} = i_{\widetilde{X}} \circ d^{\mathrm{v}} + d^{\mathrm{v}} \circ i_{\widetilde{X}} = \big[i_{\widetilde{X}}, d^{\mathrm{v}}\big] =: d^{\mathrm{v}}_{\widetilde{X}}, \\ \mathcal{L}^h_{\mathcal{H}\widetilde{X}} = i_{\widetilde{X}} \circ d^h + d^h \circ i_{\widetilde{X}} = \big[i_{\widetilde{X}}, d^h\big] =: d^h_{\widetilde{X}}, \\ \mathcal{L}_{\mathbf{i}\widetilde{X}} = i_{\widetilde{X}} \circ d^{\mathrm{v}} + d^{\mathrm{v}} \circ i_{\widetilde{X}} = d^{\mathrm{v}}_{\widetilde{X}}, \\ \mathcal{L}_{\mathcal{H}\widetilde{X}} = i_{\widetilde{X}} \circ d^h + d^h \circ i_{\widetilde{X}} = d^h_{\widetilde{X}} \end{array}}$$

($\widetilde{X} \in \mathfrak{X}(\tau)$); cf. 1.40(3).

(3) The operator $\mathcal{L}^h_\xi$ was introduced by H. Akbar-Zadeh in the context of Finsler geometry in terms of local coordinates; see his fundamental paper [3]. The operators $\mathcal{L}^{\mathrm{v}}_\xi$ and $\mathcal{L}_\xi$ were proposed by R. L. Lovas recently; see [47].

2.40. Torsion and curvature revisited.

Lemma. *Let a horizontal map $\mathcal{H} : \tau^*\tau \longrightarrow \tau_{TM}$ be given. There exist unique vector-valued forms $\widetilde{\mathbf{T}} \in \mathcal{B}^2(\tau)$, $\widetilde{\Omega} \in \mathcal{B}^2(\tau)$ and $\widetilde{\Psi} \in \mathcal{B}^3(\tau)$ such that*

$$\boxed{d^{\mathrm{v}}_{\widetilde{\mathbf{T}}} = [d^h, d^{\mathrm{v}}], \quad i_{\widetilde{\Psi}} + d^{\mathrm{v}}_{\widetilde{\Omega}} = -\frac{1}{2}[d^h, d^h]}.$$

Proof. (1) First we show that $[d^h, d^{\mathrm{v}}]$ satisfies the conditions of Lemma 3 in 2.36. Taking into account Lemma 2 in 2.37, for any function $f \in C^\infty(M)$ we have

$$[d^h, d^{\mathrm{v}}]f^{\mathrm{v}} = d^h(d^{\mathrm{v}} f^{\mathrm{v}}) + d^{\mathrm{v}}(d^h f^{\mathrm{v}}) = d^{\mathrm{v}}(d^{\mathrm{v}} f^c) \overset{2.34,\,\mathbf{2}}{=} 0.$$

Next, according to the graded Jacobi identity for graded derivations,

$$0 = \big[d^h, [d^{\mathrm{v}}, d^{\mathrm{v}}]\big] + \big[d^{\mathrm{v}}, [d^{\mathrm{v}}, d^h]\big] + \big[d^{\mathrm{v}}, [d^h, d^{\mathrm{v}}]\big] = 2\big[d^{\mathrm{v}}, [d^{\mathrm{v}}, d^h]\big],$$

hence $\big[[d^h, d^{\mathrm{v}}], d^{\mathrm{v}}\big] = 0$, as was to be checked. We conclude that $[d^h, d^{\mathrm{v}}]$ is a v-Lie derivation of degree 2 of $\mathcal{A}(\tau)$, and this proves the first assertion.

(2) For any smooth function f on M we have

$$[d^h, d^h]f^{\mathrm{v}} = 2d^h(d^h f^{\mathrm{v}}) = 2d^h(d^{\mathrm{v}} f^c) \overset{2.34,\,\mathbf{1}}{=} 2d^h\widehat{df} := 2\,\widehat{ddf} = 0,$$

i.e., $[d^h, d^h]$ vanishes on basic functions. Now the theorem on preliminary classification in 2.36 guarantees the existence and uniqueness of the desired vector-valued forms $\widetilde{\Omega}$ and $\widetilde{\Psi}$ along τ. $\square$

Corollary. *Keeping the hypothesis and notation of the Lemma, consider the torsion* $\mathbf{T} = [\mathbf{h}, J]$ *and the curvature* $\Omega = -\frac{1}{2}[\mathbf{h}, \mathbf{h}]$ *of* $\mathcal{H}$*. Then*

$$\boxed{\mathbf{T} = (\widetilde{\mathbf{T}})_0, \quad \Omega = (\widetilde{\Omega})_0}.$$

Proof. (1) Notice first that

$$\big(d^{\mathrm{v}}_{\widetilde{\mathbf{T}}}\,\widetilde{\alpha}\big)_0 = d_{\mathbf{T}}(\widetilde{\alpha})_0 \quad \text{for all } \widetilde{\alpha} \in \mathcal{A}(\tau).$$

Indeed, applying the Lemma and the results of 2.38 we get

$$\begin{aligned} \big(d^{\mathrm{v}}_{\widetilde{\mathbf{T}}}\,\widetilde{\alpha}\big)_0 &= \big(d^h(d^{\mathrm{v}}\widetilde{\alpha})\big)_0 + \big(d^{\mathrm{v}}(d^h\widetilde{\alpha})\big)_0 = d_{\mathbf{h}}(d^{\mathrm{v}}\widetilde{\alpha})_0 + d_J(d^h\widetilde{\alpha})_0 \\ &= d_{\mathbf{h}}d_J(\widetilde{\alpha})_0 + d_J d_{\mathbf{h}}(\widetilde{\alpha})_0 = [d_{\mathbf{h}}, d_J](\widetilde{\alpha})_0 = d_{[\mathbf{h},J]}(\widetilde{\alpha})_0 = d_{\mathbf{T}}(\widetilde{\alpha})_0. \end{aligned}$$

Now, on the one hand, for any smooth function f and vector fields ξ, η on TM we have

$$(d_{\mathbf{T}}(f)_0)(\xi, \eta) = (d_{\mathbf{T}} f)(\xi, \eta) = (i_{\mathbf{T}} df)(\xi, \eta) = df(\mathbf{T}(\xi, \eta)) = \mathbf{T}(\xi, \eta)(f).$$

On the other hand,

$$\begin{aligned} \big(d^{\mathrm{v}}_{\widetilde{\mathbf{T}}} f\big)_0(\xi, \eta) &= \big(d^{\mathrm{v}}_{\widetilde{\mathbf{T}}} f\big)(\mathbf{j}\xi, \mathbf{j}\eta) = \big(i_{\widetilde{\mathbf{T}}} d^{\mathrm{v}} f\big)(\mathbf{j}\xi, \mathbf{j}\eta) = d^{\mathrm{v}} f(\widetilde{\mathbf{T}}(\mathbf{j}\xi, \mathbf{j}\eta)) \\ &= \mathbf{i}\widetilde{\mathbf{T}}(\mathbf{j}\xi, \mathbf{j}\eta)(f) = \widetilde{\mathbf{T}}_0(\xi, \eta)(f). \end{aligned}$$

Comparing the two results, it follows that $\mathbf{T} = (\widetilde{\mathbf{T}})_0$.

(2) Arguing as above, observe first that

$$\big(i_{\widetilde{\Psi}}\,\widetilde{\alpha}\big)_0 + \big(d^{\mathrm{v}}_{\widetilde{\Omega}}\,\widetilde{\alpha}\big)_0 = d_{\Omega}(\widetilde{\alpha})_0 \quad \text{for all } \widetilde{\alpha} \in \mathcal{A}^1(\tau).$$

Indeed, $(i_{\widetilde{\Psi}}\,\widetilde{\alpha})_0 + (d^{\mathrm{v}}_{\widetilde{\Omega}}\,\widetilde{\alpha})_0 = -\frac{1}{2}\big([d^h, d^h]\widetilde{\alpha}\big)_0 = -(d^h(d^h\widetilde{\alpha}))_0 = -d_{\mathbf{h}}(d^h\widetilde{\alpha})_0 =$

$-d_{\mathbf{h}} \circ d_{\mathbf{h}}(\widetilde{\alpha})_0 = -\frac{1}{2}[d_{\mathbf{h}}, d_{\mathbf{h}}](\widetilde{\alpha})_0 = d_{-\frac{1}{2}[\mathbf{h},\mathbf{h}]}(\widetilde{\alpha})_0 = d_{\Omega}(\widetilde{\alpha})_0$. Next we operate with both sides of the relation on a smooth function $f \in C^{\infty}(TM) =: \mathcal{A}^0(\tau) = \mathcal{A}^0_0(TM)$, and evaluate the resulting two-forms on a pair of vector fields $(\xi, \eta) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM)$. We have on the one hand

$$(d_{\Omega}(f)_0)(\xi, \eta) = (d_{\Omega} f)(\xi, \eta) = (i_{\Omega} df)(\xi, \eta) = df(\Omega(\xi, \eta)) = \Omega(\xi, \eta)(f).$$

On the other hand,

$$\begin{aligned}\big((i_{\widetilde{\Psi}} f)_0 + (d^{\mathrm{v}}_{\widetilde{\Omega}} f)_0\big)(\xi, \eta) &= (d^{\mathrm{v}}_{\widetilde{\Omega}} f)(\mathbf{j}\xi, \mathbf{j}\eta) = (i_{\widetilde{\Omega}} d^{\mathrm{v}} f)(\mathbf{j}\xi, \mathbf{j}\eta) \\ &= d^{\mathrm{v}} f(\widetilde{\Omega}(\mathbf{j}\xi, \mathbf{j}\eta)) = [\mathbf{i}\widetilde{\Omega}(\mathbf{j}\xi, \mathbf{j}\eta)](f);\end{aligned}$$

therefore $\Omega(\xi, \eta) = \mathbf{i}\widetilde{\Omega}(\mathbf{j}\xi, \mathbf{j}\eta)$ for any vector fields ξ, η on TM. Hence $\Omega = \widetilde{\Omega}_0$, which concludes the proof of the corollary.

Remark. In the light of the last result, the $\tau^*\tau$-tensors $\widetilde{\mathbf{T}}$ and $\widetilde{\Omega}$ defined by

$$d^{\mathrm{v}}_{\widetilde{\mathbf{T}}} = [d^h, d^{\mathrm{v}}] \quad \text{and} \quad i_{\widetilde{\Psi}} + d^{\mathrm{v}}_{\widetilde{\Omega}} = -\frac{1}{2}[d^h, d^h]$$

respectively, may also be regarded as the torsion and the curvature of the nonlinear connection defined by the horizontal map $\mathcal{H} : \tau^*\tau \longrightarrow \tau_{TM}$.

2.41. Summary. We survey the main results of this section.

1. The graded derivations of the Grassmann algebra $\mathcal{A}(\tau)$ of forms along τ are local and, with respect to restrictions, natural operators.
2. Every graded derivation of $\mathcal{A}(\tau)$ is determined by its action on the vertical and the complete lifts of smooth functions on M, and on the basic one-forms along τ.
3. Every graded derivation of degree $\ell - 1 \geqq -1$ that acts trivially on $C^{\infty}(TM)$ is uniquely determined by a vector-valued form $\widetilde{L}$ of degree ℓ along τ, namely $\mathcal{D} = i_{\widetilde{L}}$. If $\widetilde{\alpha}$ is a one-form along τ, then $i_{\widetilde{L}}\widetilde{\alpha} = \widetilde{\alpha} \circ \widetilde{L}$, and $i_{\widetilde{L}}$ is determined by this rule.
4. Specifying a horizontal map $\mathcal{H}$ for τ, we have two exterior derivative operators on $\mathcal{A}(\tau)$, the v-exterior derivative d^{v} and the h-exterior derivative d^h. These are graded derivations of degree 1 of $\mathcal{A}(\tau)$ defined by

$$d^{\mathrm{v}} f = df \circ \mathbf{i}, \quad d^{\mathrm{v}}\widehat{\alpha} = 0 \qquad (f \in C^{\infty}(TM),\ \alpha \in \mathcal{A}^1(M))$$

and

$$d^h f = df \circ \mathcal{H},\ d^h\widehat{\alpha} = \widehat{d\alpha} \qquad (f \in C^{\infty}(TM),\ \alpha \in \mathcal{A}^1(M)),$$

respectively. Their action on an arbitrary k-form $\widetilde{\alpha}$ along τ is given by the second box-formula in 2.38 and the box-formula in 2.37, respectively.

5. Each graded derivation $\mathcal{D}$ of degree k of $\mathcal{A}(\tau)$ which vanishes on the basic functions and commutes with d^{v}, i.e., satisfies $\mathcal{D} \circ d^{\mathrm{v}} = (-1)^k d^{\mathrm{v}} \circ \mathcal{D}$, is uniquely determined by a vector-valued form $\widetilde{K} \in \mathcal{B}^k(\tau)$ such that

$$\mathcal{D} = d^{\mathrm{v}}_{\widetilde{K}} := \left[i_{\widetilde{K}}, d^{\mathrm{v}}\right] = i_{\widetilde{K}} \circ d^{\mathrm{v}} - (-1)^{k-1} d^{\mathrm{v}} \circ i_{\widetilde{K}}.$$

6. Each graded derivation $\mathcal{D}$ of $\mathcal{A}(\tau)$ which acts trivially on the basic functions may be uniquely represented in the form $\mathcal{D} = i_{\widetilde{K}} + d^{\mathrm{v}}_{\widetilde{L}}$; $\widetilde{K} \in \mathcal{B}^{\ell+1}(\tau)$, $\widetilde{L} \in \mathcal{B}^{\ell}(\tau)$, ℓ is the degree of $\mathcal{D}$. (*Preliminary classification theorem* of Martínez, Cariñena and Sarlet.)

7. Every graded derivation $\mathcal{D}$ of degree k of $\mathcal{A}(\tau)$ may be uniquely represented in the form

$$\mathcal{D} = i_{\widetilde{K}_1} + d^{\mathrm{v}}_{\widetilde{K}_2} + d^{h}_{\widetilde{K}_3},$$

where $\widetilde{K}_1 \in \mathcal{B}^{k+1}(\tau)$; $\widetilde{K}_2, \widetilde{K}_3 \in \mathcal{B}^k(\tau)$ and $d^h_{\widetilde{K}_3} := \left[i_{\widetilde{K}_3}, d^h\right]$.
(*Fine classification theorem* of Martínez, Cariñena and Sarlet.)

F. Covariant derivative operators along the tangent bundle projection

In this section we assume that a horizontal map $\mathcal{H}$ is specified for τ and consider the 'double exact' sequence

$$0 \to TM \times_M TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftarrows}} TTM \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftarrows}} TM \times_M TM \to 0.$$

As in the foregoing, our *canonical objects* are

$\delta : v \in TM \mapsto \delta(v) := (v, v)$ – the canonical field along τ,

$C := \mathbf{i} \circ \delta$ – the Liouville vector field on TM,

$J := \mathbf{i} \circ \mathbf{j}$ – the vertical endomorphism on TTM.

We shall need the *basic geometric data* associated to $\mathcal{H}$:

$$\mathbf{h} := \mathcal{H} \circ \mathbf{j}, \quad \mathbf{v} := 1_{TTM} - \mathbf{h}, \quad \mathbf{F} := \mathcal{H} \circ \mathcal{V} - \mathbf{i} \circ \mathbf{j} = \mathcal{H} \circ \mathcal{V} - J;$$

$$\mathbf{t} = [\mathbf{h}, C] \text{ (tension)}, \quad \mathbf{T} = [J, \mathbf{h}] \text{ (torsion)}, \Omega = -\frac{1}{2}[\mathbf{h}, \mathbf{h}] \text{ (curvature)};$$

some of these will be re-interpeted later.

2.42. v**- and** h**-covariant derivatives.**

Definition 1. A *v-covariant derivative operator*, or briefly a *v-covariant derivative* in $\tau^*\tau$ is a map

$$D^{\mathrm{v}} : \mathfrak{X}(\tau) \times \mathfrak{X}(\tau) \longrightarrow \mathfrak{X}(\tau), \quad (\widetilde{X}, \widetilde{Y}) \mapsto D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y}$$

which, for any vector fields $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ along τ and any function f in $C^\infty(TM)$, satisfies

v-COVD1. $\quad D^{\mathrm{v}}_{\widetilde{X}+\widetilde{Y}}\widetilde{Z} = D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Z} + D^{\mathrm{v}}_{\widetilde{Y}}\widetilde{Z}.$

v-COVD2. $\quad D^{\mathrm{v}}_{f\widetilde{X}}\widetilde{Y} = f D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y}.$

v-COVD3. $\quad D^{\mathrm{v}}_{\widetilde{X}}(\widetilde{Y} + \widetilde{Z}) = D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y} + D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Z}.$

v-COVD4. $\quad D^{\mathrm{v}}_{\widetilde{X}} f\widetilde{Y} = \big[(\mathbf{i}\widetilde{X})f\big]\widetilde{Y} + f D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y} = \big[(d^{\mathrm{v}} f)(\widetilde{X})\big]\widetilde{Y} + f D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y}.$

Remark 1. Let a vector field $\widetilde{X}$ along τ be given. The map

$$\widetilde{Y} \in \mathfrak{X}(\tau) \mapsto D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y} \in \mathfrak{X}(\tau)$$

can be extended to a unique tensor derivation of $\mathcal{T}^{\bullet}_{\bullet}(\tau)$ such that

$$D^{\mathrm{v}}_{\widetilde{X}} f := (d^{\mathrm{v}} f)(\widetilde{X}) := (\mathbf{i}\widetilde{X})f \quad \text{for all } f \in C^\infty(TM).$$

As a matter of fact, axioms **v-COVD3** and **v-COVD4** are exatly what is needed to apply a 'reasonable' variant of Willmore's theorem (see 1.32, (4), (5)) to obtain the desired extension.

If $\widetilde{A} \in \mathcal{T}^r_s(\tau)$, then the map $\widetilde{X} \mapsto D^{\mathrm{v}}_{\widetilde{X}}\widetilde{A}$ is $C^\infty(TM)$-linear since the tensor derivations

$$D^{\mathrm{v}}_{f\widetilde{X}+h\widetilde{Y}} \quad \text{and} \quad fD^{\mathrm{v}}_{\widetilde{X}} + hD^{\mathrm{v}}_{\widetilde{Y}} \quad (f, h \in C^\infty(TM))$$

agree on $C^\infty(TM)$ and $\mathfrak{X}(\tau)$.

These observations justify the following

Definition 2. Let a v-covariant derivative operator D^{v} in $\tau^*\tau$ be given. The *v-covariant differential* of a tensor field $\widetilde{A} \in \mathcal{T}^r_s(\tau)$ is the tensor field $D^{\mathrm{v}}\widetilde{A} \in \mathcal{T}^r_{s+1}(\tau)$ given by

$$(D^{\mathrm{v}}\widetilde{A})(\widetilde{X}, \widetilde{\alpha}^1, \dots, \widetilde{\alpha}^r, \widetilde{X}_1, \dots, \widetilde{X}_s) := \big(D^{\mathrm{v}}_{\widetilde{X}}\widetilde{A}\big)(\widetilde{\alpha}^1, \dots, \widetilde{\alpha}^r, \widetilde{X}_1, \dots, \widetilde{X}_s)$$

for all $\widetilde{X}, \widetilde{X}_i \in \mathfrak{X}(\tau)$ and $\widetilde{\alpha}^j \in \mathcal{A}^1(\tau)$; $1 \leqq i \leqq s$, $1 \leqq j \leqq r$. $\widetilde{A}$ is said to be *v-parallel* if $D^{\mathrm{v}}\widetilde{A} = 0$.

Definition 3. The maps $\mathrm{Tor}(D^{\mathrm{v}})$ and $\mathrm{Curv}(D^{\mathrm{v}})$ given by

$$\mathrm{Tor}(D^{\mathrm{v}})(\widetilde{X}, \widetilde{Y}) := D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y} - D^{\mathrm{v}}_{\widetilde{Y}}\widetilde{X} - \mathcal{V}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}] \quad \text{and}$$
$$\mathrm{Curv}(D^{\mathrm{v}})(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) := D^{\mathrm{v}}_{\widetilde{X}}D^{\mathrm{v}}_{\widetilde{Y}}\widetilde{Z} - D^{\mathrm{v}}_{\widetilde{Y}}D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Z} - D^{\mathrm{v}}_{\mathcal{V}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}]}\widetilde{Z} \quad (\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\tau)),$$

where $\mathcal{V}$ is the vertical map belonging to an arbitrarily chosen horizontal map, are called the *torsion* and the *curvature* of the v-covariant derivative operator D^{v}.

Remark 2. It may immediately be seen that $\mathrm{Tor}(D^{\mathrm{v}}) \in \mathcal{T}^1_2(\tau)$, $\mathrm{Curv}(D^{\mathrm{v}}) \in \mathcal{T}^1_3(\tau)$. Since $[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}]$ is vertical, $\mathbf{i}$ is injective, and $\mathbf{i}\mathcal{V}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}] = \mathbf{v}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}] = [\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}]$, $\mathrm{Tor}(D^{\mathrm{v}})$ and $\mathrm{Curv}(D^{\mathrm{v}})$ do not depend on the choice of the horizontal map.

Example 1. If D is a covariant derivative operator in $\tau^*\tau$ and

$$D^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y} := D_{\mathbf{i}\widetilde{X}}\widetilde{Y} \quad \text{for all } \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\tau),$$

then D^{v} is a v-covariant derivative operator in $\tau^*\tau$, called the *v-covariant derivative induced by D*.

Example 2. The map

$$\nabla^{\mathrm{v}} : (\widetilde{X}, \widetilde{Y}) \mapsto \nabla^{\mathrm{v}}_{\widetilde{X}} \widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}],$$

where $\mathcal{H}$ is an arbitrarily chosen horizontal map for τ, is a v-covariant derivative operator, called the *canonical v-covariant derivative in* $\tau^*\tau$. If ∇ is the Berwald derivative induced by $\mathcal{H}$ in τ_{TM} (see 2.33), then we have

$$\boxed{\nabla_{J\xi} J\eta = \mathbf{i}\nabla^{\mathrm{v}}_{\mathbf{j}\xi}\mathbf{j}\eta \text{ or, equivalently, } \nabla_{\mathbf{i}\widetilde{X}}\mathbf{i}\widetilde{Y} = \mathbf{i}\nabla^{\mathrm{v}}_{\widetilde{X}}\widetilde{Y}}$$

($\xi, \eta \in \mathfrak{X}(TM)$; $\widetilde{X} = \mathbf{j}\xi$, $\widetilde{Y} = \mathbf{j}\eta$). Indeed,

$$\mathbf{i}\nabla^{\mathrm{v}}_{\mathbf{j}\xi}\mathbf{j}\eta = \mathbf{i} \circ \mathbf{j}[J\xi, \mathcal{H} \circ \mathbf{j}\eta] = J[J\xi, \mathbf{h}\eta] = J[J\xi, \eta] = \nabla_{J\xi} J\eta$$

(see the first box in 2.33). An immediate consequence is that ∇^{v} *has a purely 'vertical character'*, i.e., ∇^{v} does not depend on the choice of the horizontal map. Moreover, for any vector fields ξ, η, ζ on TM we have

$$(\mathrm{Tor}(D^{\mathrm{v}}))_0(\xi, \eta) = \mathbf{Q}^1(\xi, \eta) := \mathbf{v}T^{\nabla}(J\xi, J\eta) \overset{2.33,\ \mathbf{11}}{=} 0,$$

$$(\mathrm{Curv}(D^{\mathrm{v}}))_0(\xi, \eta, \zeta) = \mathbf{Q}(\xi, \eta)\zeta := R^{\nabla}(J\xi, J\eta)J\zeta \overset{2.33,\ \mathbf{10}}{=} 0;$$

therefore *the canonical v-covariant derivative in* $\tau^*\tau$ *has vanishing torsion and curvature.*

Notice finally that

$$\boxed{\nabla^{\mathrm{v}}\delta = 1_{\mathfrak{X}(\tau)}}.$$

Indeed, for any vector field ξ on TM we have

$$(\nabla^{\mathrm{v}}\delta)(\mathbf{j}\xi) = \nabla^{\mathrm{v}}_{\mathbf{j}\xi}\delta = \mathbf{j}[J\xi, \mathcal{H}\delta] \overset{2.31,\ \mathbf{5}}{=} \mathbf{j}\xi,$$

since $J\mathcal{H}\delta = \mathbf{i} \circ \mathbf{j} \circ \mathcal{H}\delta = \mathbf{i} \circ \delta = C$.

Definition 4. An *h-covariant derivative operator*, or briefly an *h-covariant derivative* in $\tau^*\tau$ is a map

$$D^h : \mathfrak{X}(\tau) \times \mathfrak{X}(\tau) \longrightarrow \mathfrak{X}(\tau),\ (\widetilde{X}, \widetilde{Y}) \mapsto D^h_{\widetilde{X}}\widetilde{Y}$$

which, for any vector fields $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ along τ and any function f in $C^\infty(TM)$, satisfies

h-COVD1. $D^h_{\widetilde{X}+\widetilde{Y}}\widetilde{Z} = D^h_{\widetilde{X}}\widetilde{Z} + D^h_{\widetilde{Y}}\widetilde{Z}.$

h-COVD2. $D^h_{f\widetilde{X}}\widetilde{Y} = fD^h_{\widetilde{X}}\widetilde{Y}.$

h-COVD3. $D^h_{\widetilde{X}}(\widetilde{Y} + \widetilde{Z}) = D^h_{\widetilde{X}}\widetilde{Y} + D^h_{\widetilde{X}}\widetilde{Z}.$

h-COVD4. $D^h_{\widetilde{X}}f\widetilde{Y} = [(\mathcal{H}\widetilde{X})f]\widetilde{Y} + fD^h_{\widetilde{X}}\widetilde{Y} = [(d^h f)(\widetilde{X})]\widetilde{Y} + fD^h_{\widetilde{X}}\widetilde{Y}.$

Remark 3. We may copy the chain of ideas in Remark 1 to obtain tensor derivations $D^h_{\widetilde{X}} : \mathcal{T}^\bullet_\bullet(\tau) \longrightarrow \mathcal{T}^\bullet_\bullet(\tau)$ ($\widetilde{X} \in \mathfrak{X}(\tau)$), and define the *h-covariant differential* $D^h\widetilde{A} \in \mathcal{T}^r_{s+1}(\tau)$ of a tensor field $\widetilde{A} \in \mathcal{T}^r_s(\tau)$. Notice that $D^h \restriction C^\infty(TM) = d^h \restriction C^\infty(TM)$. A tensor field $\widetilde{A}$ along τ is called *h-parallel* if $D^h\widetilde{A} = 0$.

Example 3. If $D : \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \longrightarrow \mathfrak{X}(\tau)$ is a covariant derivative operator in $\tau^*\tau$ and

$$D^h_{\widetilde{X}}\widetilde{Y} := D_{\mathcal{H}\widetilde{X}}\widetilde{Y} \quad \text{for all } \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\tau)$$

then D^h is an h-covariant derivative operator in $\tau^*\tau$, called the *h-covariant derivative induced by D and* $\mathcal{H}$.

Example 4. The map

$$\nabla^h : (\widetilde{X}, \widetilde{Y}) \mapsto \nabla^h_{\widetilde{X}}\widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$$

is an h-covariant derivative operator in $\tau^*\tau$, called the *Berwald h-covariant derivative* in $\tau^*\tau$ induced by $\mathcal{H}$. If ∇ is the Berwald derivative in τ_{TM} determined by $\mathcal{H}$ then we have

$$\nabla_{\mathbf{h}\xi}J\eta = \mathbf{i}\nabla^h_{\mathbf{j}\xi}\mathbf{j}\eta \quad \text{for all } \xi, \eta \in \mathfrak{X}(TM).$$

In particular,

$$\boxed{(\nabla^h\delta)_0 = \mathbf{t} := \text{the tension of } \mathcal{H}}.$$

Indeed, for any vector field ξ on TM we have

$$(\nabla^h\delta)_0(\xi) := \mathbf{i}\big[(\nabla^h\delta)\mathbf{j}\xi\big] = \mathbf{i}\nabla^h_{\mathbf{j}\xi}\delta = \mathbf{v}[\mathbf{h}\xi, C] = \mathbf{t}(\xi);$$

see 2.22, Lemma 1, and cf. 2.14, Remark 2.

By reason of this relation, the $(1,1)$ tensor field

$$\boxed{\widetilde{\mathbf{t}} := \nabla^h \delta}$$

along τ will also be called the *tension* of the horizontal map $\mathcal{H}$. Notice that

$$\widetilde{\mathbf{t}}(\widetilde{X}) = \mathcal{V}[\mathcal{H}\widetilde{X}, C] \quad \text{for all } \widetilde{X} \in \mathfrak{X}(\tau).$$

2.43. Relations with the exterior derivatives d^{v} and d^h.

Proposition 1. *If ∇^{v} is the canonical v-covariant derivative in $\mathcal{T}^{\bullet}_{\bullet}(\tau)$, then for any k-form $\widetilde{\alpha}$ along τ we have*

$$\boxed{d^{\mathrm{v}}\widetilde{\alpha} = (k+1)\operatorname{Alt}\nabla^{\mathrm{v}}\widetilde{\alpha}}.$$

Proof. Let $\widetilde{X}_1, \dots, \widetilde{X}_{k+1}$ be arbitrary vector fields along τ. Then

$$(k+1)(\operatorname{Alt}\nabla^{\mathrm{v}}\widetilde{\alpha})(\widetilde{X}_1, \dots, \widetilde{X}_{k+1}) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)(\nabla^{\mathrm{v}}\widetilde{\alpha})(\widetilde{X}_{\sigma(1)}, \dots, \widetilde{X}_{\sigma(k+1)})$$

$$= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\big(\nabla^{\mathrm{v}}_{\widetilde{X}_{\sigma(1)}}\widetilde{\alpha}\big)(\widetilde{X}_{\sigma(2)}, \dots, \widetilde{X}_{\sigma(k+1)}) \overset{1.32(3)}{=}$$

$$= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\Big[(\mathbf{i}\widetilde{X}_{\sigma(1)})\widetilde{\alpha}(\widetilde{X}_{\sigma(2)}, \dots, \widetilde{X}_{\sigma(k+1)})$$

$$- \sum_{i=2}^{k+1}\widetilde{\alpha}(\widetilde{X}_{\sigma(2)}, \dots, \nabla^{\mathrm{v}}_{\widetilde{X}_{\sigma(1)}}\widetilde{X}_{\sigma(i)}, \dots, \widetilde{X}_{\sigma(k+1)})\Big]$$

$$= \sum_{i=1}^{k+1}(-1)^{i+1}(\mathbf{i}\widetilde{X}_i)\big[\widetilde{\alpha}(\widetilde{X}_1, \dots, \overset{*}{\widetilde{X}_i}, \dots, \widetilde{X}_{k+1})\big]$$

$$- \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_{k+1}}\sum_{i=2}^{k+1}\varepsilon(\sigma)\widetilde{\alpha}(\widetilde{X}_{\sigma(2)}, \dots, \mathbf{j}[\mathbf{i}\widetilde{X}_{\sigma(1)}, \mathcal{H}\widetilde{X}_{\sigma(i)}], \dots, \widetilde{X}_{\sigma(k+1)}).$$

Using elementary combinatorial tricks, the second expression on the right-hand side can be formed as follows:

2nd term

$$= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \sum_{i=2}^{k+1} (-1)^{\sigma(i)} \varepsilon(\sigma) \widetilde{\alpha}(\mathbf{j}[\mathbf{i}\widetilde{X}_{\sigma(1)}, \mathcal{H}\widetilde{X}_{\sigma(i)}], \widetilde{X}_{\sigma(2)}, \dots, \overset{*}{\widetilde{X}}_{\sigma(i)}, \dots, \widetilde{X}_{\sigma(k+1)})$$

$$= \sum_{1 \leqq i < j \leqq k+1} (-1)^{i-1}(-1)^{j} \widetilde{\alpha}(\mathbf{j}[\mathbf{i}\widetilde{X}_i, \mathcal{H}\widetilde{X}_j], \widetilde{X}_1, \dots, \overset{*}{\widetilde{X}}_i, \dots, \overset{*}{\widetilde{X}}_j, \dots, \widetilde{X}_{k+1})$$

$$+ \sum_{1 \leqq j < i \leqq k+1} (-1)^{i}(-1)^{j} \widetilde{\alpha}(\mathbf{j}[\mathbf{i}\widetilde{X}_i, \mathcal{H}\widetilde{X}_j], \widetilde{X}_1, \dots, \overset{*}{\widetilde{X}}_j, \dots, \overset{*}{\widetilde{X}}_i, \dots, \widetilde{X}_{k+1})$$

$$= - \sum_{1 \leqq i < j \leqq k} (-1)^{i+j} \widetilde{\alpha}(\mathbf{j}[\mathbf{i}\widetilde{X}_i, \mathcal{H}\widetilde{X}_j] + \mathbf{j}[\mathcal{H}\widetilde{X}_i, \mathbf{i}\widetilde{X}_j], \widetilde{X}_1, \dots, \overset{*}{\widetilde{X}}_i, \dots, \overset{*}{\widetilde{X}}_j, \dots, \widetilde{X}_{k+1}).$$

Here, as we have learnt in the proof of 2.38, Corollary 1,

$$\mathbf{j}[\mathbf{i}\widetilde{X}_i, \mathcal{H}\widetilde{X}_j] + \mathbf{j}[\mathcal{H}\widetilde{X}_i, \mathbf{i}\widetilde{X}_j] = \mathcal{V}[\mathbf{i}\widetilde{X}_i, \mathbf{i}\widetilde{X}_j];$$

so we obtain the desired formula. □

Proposition 2. *Suppose that the torsion of the horizontal map $\mathcal{H}$ vanishes, and let ∇^h be the Berwald h-covariant derivative induced by $\mathcal{H}$. Then for every k-form $\widetilde{\alpha}$ along τ we have*

$$\boxed{d^h \widetilde{\alpha} = (k+1) \operatorname{Alt} \nabla^h \widetilde{\alpha}}.$$

Proof. The proof is by induction on k. The case $k = 0$ is trivial, since $d^h \restriction C^\infty(TM) = D^h \restriction C^\infty(TM)$ is true for every h-covariant derivative operator D^h.

(1) Let $k = 1$. In view of 2.37, Lemma 3, for any vector fields X, Y on M we have

$$d^h \widetilde{\alpha}(\widehat{X}, \widehat{Y}) = X^h \widetilde{\alpha}(\widehat{Y}) - Y^h \widetilde{\alpha}(\widehat{X}) - \widetilde{\alpha}(\mathbf{j}[X^h, Y^h]).$$

Since $\mathbf{T} = 0$, $[X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}} = 0$. Hence

$$\mathbf{i}(\mathbf{j}[X^h, Y^h]) \overset{2.19}{=} [X, Y]^{\mathrm{v}} = [X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}] = \mathbf{i}(\mathcal{V}[X^h, Y^{\mathrm{v}}] - \mathcal{V}[Y^h, X^{\mathrm{v}}]),$$

therefore

$$\mathbf{j}[X^h, Y^h] = \mathcal{V}[X^h, Y^{\mathrm{v}}] - \mathcal{V}[Y^h, Y^{\mathrm{v}}] = \nabla^h_{\widehat{X}} \widehat{Y} - \nabla^h_{\widehat{Y}} \widehat{X}.$$

From this it follows that for any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have

$$(d^h\widetilde{\alpha})(\widetilde{X},\widetilde{Y}) = (\mathcal{H}\widetilde{X})\widetilde{\alpha}(\widetilde{Y}) - \widetilde{\alpha}\big(\nabla^h_{\widetilde{X}}\widetilde{Y}\big) - (\mathcal{H}\widetilde{Y})\widetilde{\alpha}(\widetilde{X}) + \widetilde{\alpha}\big(\nabla^h_{\widetilde{Y}}\widetilde{X}\big)$$
$$= (\nabla^h\widetilde{\alpha})(\widetilde{X},\widetilde{Y}) - (\nabla^h\widetilde{\alpha})(\widetilde{Y},\widetilde{X}) = [2(\operatorname{Alt}\nabla^h)\widetilde{\alpha}](\widetilde{X},\widetilde{Y}).$$

Thus the formula is valid for one-forms along τ.

(2) Next, let $k \geqq 2$ and suppose that the assertion holds for positive integers less than k. Then $\widetilde{\alpha}$, at least locally, can be written as the sum of terms of the form $\widetilde{\beta} \wedge \widetilde{\gamma}$ with $\deg\widetilde{\beta} < k$, $\deg\widetilde{\gamma} < k$. For simplicity we may assume that $\widetilde{\alpha} = \widetilde{\beta} \wedge \widetilde{\gamma}$. Let $\ell := \deg\widetilde{\beta}$. Then for any vector fields $\widetilde{X}_1, \dots, \widetilde{X}_{k+1}$ along τ we have

$$(d^h\widetilde{\alpha})(\widetilde{X}_1,\dots,\widetilde{X}_{k+1}) = [d^h(\widetilde{\beta}\wedge\widetilde{\gamma})](\widetilde{X}_1,\dots,\widetilde{X}_{k+1})$$
$$= \big((d^h\widetilde{\beta})\wedge\widetilde{\gamma} + (-1)^\ell\widetilde{\beta}\wedge d^h\widetilde{\gamma}\big)(\widetilde{X}_1,\dots,\widetilde{X}_{k+1})$$
$$= \frac{1}{(\ell+1)!(k-\ell)!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)(d^h\widetilde{\beta})(\widetilde{X}_{\sigma(1)},\dots,\widetilde{X}_{\sigma(\ell+1)})\widetilde{\gamma}(\widetilde{X}_{\sigma(\ell+2)},\dots,\widetilde{X}_{\sigma(k+1)})$$
$$+ \frac{1}{\ell!(k-\ell+1)!}(-1)^\ell\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\widetilde{\beta}(\widetilde{X}_{\sigma(1)},\dots,\widetilde{X}_{\sigma(\ell)})(d^h\widetilde{\gamma})(\widetilde{X}_{\sigma(\ell+1)},\dots,\widetilde{X}_{\sigma(k+1)})$$
$$= \frac{1}{\ell!(k-\ell)!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\operatorname{Alt}(\nabla^h\widetilde{\beta})(\widetilde{X}_{\sigma(1)},\dots,\widetilde{X}_{\sigma(\ell+1)})\widetilde{\gamma}(\widetilde{X}_{\sigma(\ell+2)},\dots,\widetilde{X}_{\sigma(k+1)})$$
$$+ \frac{1}{\ell!(k-\ell)!}(-1)^\ell\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\widetilde{\beta}(\widetilde{X}_{\sigma(1)},\dots,\widetilde{X}_{\sigma(\ell)})\operatorname{Alt}(\nabla^h\widetilde{\gamma})(\widetilde{X}_{\sigma(\ell+1)},\dots,\widetilde{X}_{\sigma(k+1)})$$
$$= \frac{1}{\ell!(k-\ell)!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)(\nabla^h(\widetilde{\beta}\otimes\widetilde{\gamma}))(\widetilde{X}_{\sigma(1)},\dots,\widetilde{X}_{\sigma(k+1)})$$
$$= \frac{(k+1)!}{\ell!(k-\ell)!}\operatorname{Alt}(\nabla^h(\widetilde{\beta}\otimes\widetilde{\gamma}))(\widetilde{X}_1,\dots,\widetilde{X}_{k+1}).$$

Thus

$$d^h\widetilde{\alpha} = d^h(\widetilde{\beta}\wedge\widetilde{\gamma}) = \frac{(k+1)!}{\ell!(k-\ell)!}\operatorname{Alt}(\nabla^h(\widetilde{\beta}\otimes\widetilde{\gamma})) = \frac{(k+1)!}{\ell!(k-\ell)!}\operatorname{Alt}(\nabla^h\operatorname{Alt}(\widetilde{\beta}\otimes\widetilde{\gamma}))$$
$$= (k+1)\operatorname{Alt}(\nabla^h(\widetilde{\beta}\wedge\widetilde{\gamma})) = (k+1)\operatorname{Alt}\nabla^h\widetilde{\alpha},$$

as was to be shown.

2.44. Covariant derivatives in $\tau^*\tau$.

Example 1. Let a v-covariant derivative operator D^{v} and an h-covariant derivative operator D^h be given in $\tau^*\tau$. Then the map

$$D : (\xi, \widetilde{X}) \in \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \mapsto D_\xi \widetilde{X} := D^{\mathrm{v}}_{\mathcal{V}\xi} \widetilde{X} + D^h_{\mathbf{j}\xi} \widetilde{X} \in \mathfrak{X}(\tau)$$

is a covariant derivative operator in $\tau^*\tau$ which induces the starting v- and h-covariant derivatives D^{v} and D^h, respectively (see 2.42, Examples 1,3). Suppose, in particular, that the canonical v-covariant derivative ∇^{v} and the Berwald h-covariant derivative ∇^h are given. Then the resulting covariant derivative operator

$$\widetilde{\nabla} : (\xi, \widetilde{X}) \in \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \mapsto \widetilde{\nabla}_\xi \widetilde{X} := \nabla^{\mathrm{v}}_{\mathcal{V}\xi} \widetilde{X} + \nabla^h_{\mathbf{j}\xi} \widetilde{X} \in \mathfrak{X}(\tau)$$

is called the *Berwald derivative in $\tau^*\tau$ induced by $\mathcal{H}$*. It is related with the Berwald derivative induced by $\mathcal{H}$ in τ_{TM} by the following formula:

$$\nabla_\xi J\eta = \mathbf{i}\widetilde{\nabla}_\xi \mathbf{j}\eta \quad \text{for all } \xi, \eta \in \mathfrak{X}(TM).$$

Indeed, taking into account 2.42, Examples 2,4, we get

$$\begin{aligned} \mathbf{i}\widetilde{\nabla}_\xi \mathbf{j}\eta &= \mathbf{i}\nabla^{\mathrm{v}}_{\mathcal{V}\xi} \mathbf{i}\eta + \mathbf{i}\nabla^h_{\mathbf{j}\xi} \mathbf{j}\eta = \mathbf{i}\nabla^{\mathrm{v}}_{\mathbf{j}\circ\mathcal{H}\circ\mathcal{V}\xi} \mathbf{j}\eta + \mathbf{i}\nabla^h_{\mathbf{j}\xi} \mathbf{j}\eta \\ &= \nabla_{\mathbf{v}\xi} J\eta + \nabla_{\mathbf{h}\xi} J\eta = \nabla_\xi J\eta. \end{aligned}$$

By a slight abuse of notation, the Berwald derivative in $\tau^*\tau$ will also be denoted by ∇ in what follows. For a convenient reference, we present the basic rules for calculation in the next box:

$$\boxed{\nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}], \quad \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]; \quad \widetilde{X},\ \widetilde{Y} \in \mathfrak{X}(\tau)}\,.$$

Another formulation: for any vector fields $\xi, \eta \in \mathfrak{X}(TM)$ we have

$$\nabla_{J\xi} \mathbf{j}\eta = \mathbf{j}[J\xi, \eta], \quad \nabla_{\mathbf{h}\xi} \mathbf{j}\eta = \mathcal{V}[\mathbf{h}\xi, J\eta].$$

If, in particular, $\xi = X^c$, $\eta = Y^c$ ($X, Y \in \mathfrak{X}(M)$), it follows that

$$\nabla_{X^{\mathrm{v}}} \widehat{Y} = 0, \quad \nabla_{X^h} \widehat{Y} = \mathcal{V}[X^h, Y^{\mathrm{v}}].$$

(Compare this list of formulae with the first box in 2.33 !)

Definition. A covariant derivative operator D in $\tau^*\tau$ is said to be *associated to the horizontal map $\mathcal{H}$* if $D\delta = \mathcal{V}$.

Remark. The type $(1,1)$ tensor field $D\delta$ along τ is said to be the *deflection* of D; $D^{\mathrm{v}}\delta$ and $D^h\delta$ are called the *v-deflection* and the *h-deflection*, respectively. It can be seen at once that *D is associated to $\mathcal{H}$ if, and only if, $D^h\delta = 0$ and $D^{\mathrm{v}}\delta = 1_{\mathfrak{X}(\tau)}$.*

Example 2. Let ∇ be the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$. Then

$$\boxed{\nabla\delta = \widetilde{\mathbf{t}} \circ \mathbf{j} + \mathcal{V}},$$

where $\widetilde{\mathbf{t}}$ is the tension of $\mathcal{H}$ (see 2.42, Example 4). From this it follows that *∇ is associated to $\mathcal{H}$ if, and only if, $\mathcal{H}$ is homogeneous.*

2.45. Torsions and partial curvatures.

Definition 1. Let D be a covariant derivative operator in $\tau^*\tau$.

(1) The $\tau^*\tau$-valued two-forms

$$T^h(D) := d^D\mathbf{j} \quad \text{and} \quad T^{\mathrm{v}}(D) := d^D\mathcal{V}$$

are said to be the *horizontal* and the *vertical torsion* of D, respectively.

(2) The maps $\mathcal{T}$ and $\mathcal{S}$ given by

$$\mathcal{T}(\widetilde{X},\widetilde{Y}) := T^h(D)(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}) \text{ and } \mathcal{S}(\widetilde{X},\widetilde{Y}) := T^h(\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}) \quad (\widetilde{X},\widetilde{Y} \in \mathfrak{X}(\tau))$$

are called the *h-horizontal* and the *h-mixed torsion* of D (with respect to $\mathcal{H}$), respectively. $\mathcal{T}$ will also be mentioned as *the torsion* of D, while for $\mathcal{S}$ we use the term *Finsler torsion* as well. D is said to be *symmetric* if $\mathcal{T} = 0$ and $\mathcal{S}$ is symmetric.

(3) The maps $\mathbf{R}^1$, $\mathbf{P}^1$ and $\mathbf{Q}^1$ given by

$$\mathbf{R}^1(\widetilde{X},\widetilde{Y}) := T^{\mathrm{v}}(D)(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}), \quad \mathbf{P}^1(\widetilde{X},\widetilde{Y}) := T^{\mathrm{v}}(D)(\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}) \quad \text{and}$$

$$\mathbf{Q}^1(\widetilde{X},\widetilde{Y}) := T^{\mathrm{v}}(D)(\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}) \quad \text{for all } \widetilde{X},\widetilde{Y} \in \mathfrak{X}(\tau)$$

are called the *v-horizontal*, the *v-mixed* and the *v-vertical torsion* of D, respectively.

Remark 1. We have learnt in 2.2 that $\mathbf{j}$ can be interpreted as a $\tau^*\tau$-valued one-form on TM, i.e., $\mathbf{j} \in \mathcal{A}^1(TM,\tau^*\tau)$. Hence $T^h(D)$ and $T^{\mathrm{v}}(D)$ belong to $\mathcal{A}^2(TM,\tau^*\tau)$. $T^h(D)$ does not depend on any horizontal map, while $T^{\mathrm{v}}(D)$ strongly depends on $\mathcal{H}$, so the naming of these forms seems to be illogical at first sight. The terms 'torsion' for $\mathcal{T}$ and 'Finsler torsion' for $\mathcal{S}$, as well as the notion of symmetry of D are borrowed from [27].

Remark 2. In view of 1.43, for any vector fields ξ, η on TM we have

$$\boxed{T^h(D)(\xi,\eta)=D_\xi\mathbf{j}\eta-D_\eta\mathbf{j}\xi-\mathbf{j}[\xi,\eta],\ T^{\mathrm{v}}(D)(\xi,\eta)=D_\xi\mathcal{V}\eta-D_\eta\mathcal{V}\xi-\mathcal{V}[\xi,\eta]}.$$

Lemma 1. *All of the partial torsions of a covariant derivative operator D in $\tau^*\tau$ are tensorial, namely $\mathcal{T}$, $\mathcal{S}$, $\mathbf{R}^1$, $\mathbf{P}^1$, $\mathbf{Q}^1$ are tensor fields of type $(1,2)$ along τ. For any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have*

$$\boxed{\begin{aligned}
&\mathcal{T}(\widetilde{X},\widetilde{Y}) = D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}],\\
&\mathcal{S}(\widetilde{X},\widetilde{Y}) = -D_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] = -D_{\mathbf{i}\widetilde{Y}}\widetilde{X} + \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X},\\
&\mathbf{R}^1(\widetilde{X},\widetilde{Y}) = -\mathcal{V}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}],\\
&\mathbf{P}^1(\widetilde{X},\widetilde{Y}) = D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \mathcal{V}[\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] = D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y},\\
&\mathbf{Q}^1(\widetilde{X},\widetilde{Y}) = -D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathcal{V}[\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}] = \mathrm{Tor}(D^{\mathrm{v}}),\\
&(\mathbf{R}^1)_0 = \Omega
\end{aligned}}$$

where ∇ is the Berwald derivative in $\tau^*\tau$ induced by $\mathcal{H}$, D^{v} is the v-covariant derivative arising from D, and Ω is the curvature of $\mathcal{H}$.

Proof. The verifications are all easy, and even trivial.

Corollary. *A covariant derivative operator in $\tau^*\tau$ is the Berwald derivative induced by a given horizontal map if, and only if, its Finsler torsion and v-mixed torsion vanish.*

Lemma 2. *The horizontal torsion $T^h(D)$ is completely determined by the torsion $\mathcal{T}$ and the Finsler torsion $\mathcal{S}$. Explicitly, for any vector fields ξ, η on TM we have*

$$T^h(D)(\xi,\eta) = \mathcal{T}(\mathbf{j}\xi,\mathbf{j}\eta) + \mathcal{S}(\mathbf{j}\xi,\mathcal{V}\eta) - \mathcal{S}(\mathbf{j}\eta,\mathcal{V}\xi).$$

Proof. This follows from Remark 1 and Lemma 1 by an immediate calculation.

Lemma 3. *Let D be a covariant derivative operator in $\tau^*\tau$. If D is associated to the horizontal map $\mathcal{H}$, then for every vector field $\widetilde{X}$ along τ we have*

$$\mathcal{S}(\delta,\widetilde{X}) = 0, \quad \mathbf{P}^1(\widetilde{X},\delta) = \widetilde{\mathbf{t}}(\widetilde{X}).$$

If, in addition, $\mathcal{H}$ is homogeneous then $\mathbf{P}^1(\,\cdot\,,\delta) = 0$.

Proof. Applying Lemma 1 and the condition $D\delta = \mathcal{V}$, we obtain:

$$\mathcal{S}(\delta, \widetilde{X}) = -D_{\mathbf{i}\widetilde{X}}\delta - \mathbf{j}[\mathcal{H}\delta, \mathbf{i}\widetilde{X}] = -\widetilde{X} + \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\delta].$$

In the second term of the right-hand side $J \circ \mathcal{H}(\delta) = C$, therefore 2.31, **5** implies that $\mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\delta] = \widetilde{X}$. This proves the first relation. Similarly,

$$\mathbf{P}^1(\widetilde{X}, \delta) \overset{\text{Lemma 1}}{=} D_{\mathcal{H}\widetilde{X}}\delta - \nabla_{\mathcal{H}\widetilde{X}}\delta = (D^h\delta)(\widetilde{X}) - (\nabla\delta)(\mathcal{H}\widetilde{X})$$
$$\overset{\text{2.44, Remark \& Ex. 2}}{=} -(\widetilde{\mathbf{t}} \circ \mathbf{j} + \mathcal{V})(\mathcal{H}\widetilde{X}) = -\widetilde{\mathbf{t}}(\widetilde{X}),$$

thus the second relation is also true. In the homogeneous case $\widetilde{\mathbf{t}}$ vanishes, and hence $\mathbf{P}^1(\,\cdot\,, \delta) = 0$. $\square$

Proposition 1. *Under our basic assumption that a horizontal map $\mathcal{H}$ is specified for τ, let two type $(1,2)$ tensor fields $\widetilde{S}$ and $\widetilde{P}$ be given along τ. Suppose that*

$$\widetilde{S}(\delta, \,\cdot\,) = 0 \quad \textit{and} \quad \widetilde{P}(\,\cdot\,, \delta) = -\widetilde{\mathbf{t}}.$$

Then there is a unique covariant derivative operator D in $\tau^*\tau$ such that

(1) *D is associated to $\mathcal{H}$,*

(2) *the Finsler torsion of D is $\widetilde{S}$,*

(3) *the v-mixed torsion of D is $\widetilde{P}$.*

Proof. Prescribing the Finsler torsion and the v-mixed torsion for a covariant derivative operator in $\tau^*\tau$, the rules for calculation are forced by Lemma 1 (see the second and the fourth relation there). So we have at most one possibility to define the desired operator, namely: let, for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ along τ,

$$D_{\mathbf{i}\widetilde{X}}\widetilde{Y} := \nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} - \widetilde{S}(\widetilde{X}, \widetilde{Y}),$$
$$D_{\mathcal{H}\widetilde{X}}\widetilde{Y} := \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} + \widetilde{P}(\widetilde{X}, \widetilde{Y}).$$

Then D is clearly a covariant derivative operator in $\tau^*\tau$ (cf. 1.41 (3)); our only task is to check that the requirements (1)–(3) are satisfied by D. But this is immediate:

(a) For any vector field $\widetilde{X}$ along τ we have

$$(D^{\mathrm{v}}\delta)(\widetilde{X}) = D^{\mathrm{v}}_{\widetilde{X}}\delta := D_{\mathbf{i}\widetilde{X}}\delta := \nabla_{\mathbf{i}\widetilde{X}}\delta - \widetilde{S}(\delta, \widetilde{X}) = \widetilde{X},$$

$$(D^h\delta)(\widetilde{X}) = D^h_{\widetilde{X}}\delta := D_{\mathcal{H}\widetilde{X}}\delta := \nabla_{\mathcal{H}\widetilde{X}}\delta + \widetilde{P}(\widetilde{X}, \delta) = -\mathbf{t}(\widetilde{X}) + \mathbf{t}(\widetilde{X}) = 0,$$

therefore $D\delta = \mathcal{V}$, i.e. D is associated to $\mathcal{H}$.

(b) Let, as above, $\mathcal{S}$ and $\mathbf{P}^1$ be the Finsler torsion and the v-mixed torsion of D, respectively. For any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have

$$\mathcal{S}(\widetilde{X},\widetilde{Y}) \overset{\text{Lemma 1}}{=} -D_{\mathbf{i}\widetilde{Y}}\widetilde{X} + \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} = -\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} + \widetilde{S}(\widetilde{X},\widetilde{Y}) + \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} = \widetilde{S}(\widetilde{X},\widetilde{Y}),$$

$$\mathbf{P}^1(\widetilde{X},\widetilde{Y}) \overset{\text{Lemma 1}}{=} D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} = \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} + \widetilde{P}(\widetilde{X},\widetilde{Y}) - \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} = \widetilde{P}(\widetilde{X},\widetilde{Y}),$$

which proves the Proposition.

Definition 2. Let D be a covariant derivative operator in $\tau^*\tau$. Then the maps $\mathbf{R}$, $\mathbf{P}$ and $\mathbf{Q}$ given by

$$\mathbf{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} := R^D(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y})\widetilde{Z},$$

$$\mathbf{P}(\widetilde{X},\widetilde{Y})\widetilde{Z} := R^D(\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y})\widetilde{Z},$$

$$\mathbf{Q}(\widetilde{X},\widetilde{Y})\widetilde{Z} := R^D(\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y})\widetilde{Z},$$

are said to be the *horizontal* or *Riemann curvature*, the *mixed* or *Berwald curvature* and the *vertical* or *Berwald-Cartan curvature* of D (with respect to $\mathcal{H}$), respectively.

Remark 3. It may be seen at once that $\mathbf{R}$, $\mathbf{P}$ and $\mathbf{Q}$ are type $(1,3)$ tensor fields along τ and R^D is completely determined by them.

Note. For a good, systematic study on the geometric interpretation of the 'partial curvatures' $\mathbf{R}$, $\mathbf{P}$ and $\mathbf{Q}$ in Finslerian case the reader is referred to the thesis [25] and the short communication [26] of J.-G. Diaz; see also [27]. The best recent account on the subject in a geometric flavour is probably Z. Shen's monograph [69].

Lemma 4. *Let D be a covariant derivative operator in $\tau^*\tau$. If D is associated to the horizontal map $\mathcal{H}$, then for any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have the following relations:*

$$\mathbf{R}(\widetilde{X},\widetilde{Y})\delta = \mathbf{R}^1(\widetilde{X},\widetilde{Y}), \quad \mathbf{P}(\widetilde{X},\widetilde{Y})\delta = \mathbf{P}^1(\widetilde{X},\widetilde{Y}), \quad \mathbf{Q}(\widetilde{X},\widetilde{Y})\delta = \mathbf{Q}^1(\widetilde{X},\widetilde{Y}),$$

If, in addition, the Finsler torsion is symmetric, then $\mathbf{Q}(\,\cdot\,,\,\cdot\,)\delta = \mathbf{Q}^1 = 0$.

Proof. Since $D\delta = \mathcal{V}$ and therefore

$$D_{\mathcal{H}\widetilde{X}}\delta = 0, \quad D_{\mathcal{V}\widetilde{X}}\delta = \widetilde{X}, \quad \text{for all } \widetilde{X} \in \mathfrak{X}(\tau),$$

the first three relations may be obtained by an easy calculation. To prove the remaining assertion, start from the relation

$$\mathbf{Q}(\widetilde{X},\widetilde{Y})\delta = \mathbf{Q}^1(\widetilde{X},\widetilde{Y}) = D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathcal{V}[\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}].$$

If the Finsler torsion $\mathcal{S}$ is symmetric, then

$$0 = \mathcal{S}(\widetilde{X},\widetilde{Y}) - \mathcal{S}(\widetilde{Y},\widetilde{X}) = D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] + \mathbf{j}[\mathcal{H}\widetilde{Y},\mathbf{i}\widetilde{X}].$$

Hence

$$\mathbf{Q}(\widetilde{X},\widetilde{Y})\delta = \mathbf{j}[\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] - \mathbf{j}[\mathcal{H}\widetilde{Y},\mathbf{i}\widetilde{X}] - \mathcal{V}[\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}].$$

Representing $\widetilde{X}$ and $\widetilde{Y}$ in the form $\widetilde{X} = \mathbf{j}\xi$, $\widetilde{Y} = \mathbf{j}\eta$ $(\xi,\eta \in \mathfrak{X}(TM))$, it follows that

$$\begin{aligned}\mathbf{i}\big(\mathbf{Q}(\mathbf{j}\xi,\mathbf{j}\eta)\delta\big) &= J[\mathbf{h}\xi, J\eta] - J[\mathbf{h}\eta, J\xi] - \mathbf{v}[J\xi, J\eta]\\ &= J[\xi, J\eta] + J[J\xi,\eta] - [J\xi, J\eta] = -N_J(\xi,\eta) = 0,\end{aligned}$$

which completes the proof of the lemma.

Proposition 2. *Let D be a covariant derivative operator $\tau^*\tau$. Assume that D is associated to the horizontal map $\mathcal{H}$ and that D is symmetric. Then for any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have the following relations:*

(1) $\qquad [\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}] = \mathbf{i}(D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X}),$

(2) $\qquad [\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] = \mathbf{i}(D_{\mathcal{H}\widetilde{X}} - \mathbf{P}^1(\widetilde{X},\widetilde{Y})) - \mathcal{H}(\mathcal{S}(\widetilde{X},\widetilde{Y}) + D_{\mathbf{i}\widetilde{Y}}\widetilde{X}),$

(3) $\qquad [\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}] = -\mathbf{i}\mathbf{R}^1(\widetilde{X},\widetilde{Y}) + \mathcal{H}(D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X}).$

Proof. (a) In virtue of Lemma 4, $\mathbf{Q}^1 = 0$. This yields immediately relation (1).

(b) The vanishing of $\mathcal{T}$ implies by Lemma 2 and Remark 2 that for any vector fields ξ, η on TM we have

$(*) \qquad D_\xi\mathbf{j}\eta - D_\eta\mathbf{j}\xi - \mathbf{j}[\xi,\eta] = \mathcal{S}(\mathbf{j}\xi,\mathcal{V}\eta) - \mathcal{S}(\mathbf{j}\eta,\mathcal{V}\xi).$

With the choice $\xi := \mathcal{H}\widetilde{X}$, $\eta := \mathbf{i}\widetilde{Y}$ from this it follows that

$$-D_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] = \mathcal{S}(\widetilde{X},\widetilde{Y}),$$

hence

$(**) \qquad \mathbf{h}[\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] = -\mathcal{H}(\mathcal{S}(\widetilde{X},\widetilde{Y}) + D_{\mathbf{i}\widetilde{Y}}\widetilde{X}).$

On the other hand, according to Lemma 1,

$$\mathbf{P}^1(\widetilde{X}, \widetilde{Y}) = D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}],$$

and so

$$(***) \qquad \mathbf{v}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] = \mathbf{i}(D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \mathbf{P}^1(\widetilde{X}, \widetilde{Y})).$$

Adding the relations $(**)$ and $(***)$, the desired formula (2) drops out.

Next, let in $(*)$ $\xi := \mathcal{H}\widetilde{X}$, $\eta := \mathcal{H}\widetilde{Y}$. Then we find:

$$D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] = 0;$$

hence

$$\mathbf{h}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] = \mathcal{H}\big(D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X}\big).$$

On the other hand, using Lemma 1 again, we get

$$\mathbf{v}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] = \mathbf{i} \circ \mathcal{V}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] = -\mathbf{i}\mathbf{R}^1(\widetilde{X}, \widetilde{Y}).$$

Adding the last two relations, we obtain the relation (3). This concludes the proof.

2.46. The Bianchi identities. Ricci formulae.

Notation. We shall follow the convention mentioned in 1.47, Remark: if we are given an expression $A(X_1, X_2, X_3)$, we shall denote by $\underset{1,2,3}{\mathfrak{S}} A(X_1, X_2, X_3)$ the cyclic sum

$$A(X_1, X_2, X_3) + A(X_2, X_3, X_1) + A(X_3, X_1, X_2).$$

Theorem. *Let D be a covariant derivative operator in $\tau^*\tau$. Then its curvature tensor field $R^D \in \mathcal{T}^1_3(\tau)$ satisfies*

$$\boxed{\begin{array}{lcl} d^D d^D \mathbf{j} & = & R^D[\mathbf{j}] \ - \text{ the algebraic Bianchi identity,} \\ d^D R^D & = & 0 \ - \text{ the differential Bianchi identity} \end{array}}.$$

Proof. These results are obvious consequences of 1.43, Lemma 2 and 1.43, Proposition, as special cases.

Proposition 1. *Let D be a covariant derivative operator in $\tau^*\tau$. Assume that D is associated to the horizontal map $\mathcal{H}$ and that D is symmetric. Then* *we have the following set of* Bianchi identities:

I	$\underset{1,2,3}{\mathfrak{S}}\mathbf{R}(\widetilde{X}_1,\widetilde{X}_2)\widetilde{X}_3 = -\underset{1,2,3}{\mathfrak{S}}\mathcal{S}(\mathbf{R}^1(\widetilde{X}_1,\widetilde{X}_2),\widetilde{X}_3)$
II	$\underset{1,2,3}{\mathfrak{S}}(D_{\mathcal{H}\widetilde{X}_1}\mathbf{R})(\widetilde{X}_2,\widetilde{X}_3,\widetilde{Y}) = \underset{1,2,3}{\mathfrak{S}}\mathbf{P}(\widetilde{X}_3,\mathbf{R}^1(\widetilde{X}_1,\widetilde{X}_2))\widetilde{Y}$
III	$\underset{1,2,3}{\mathfrak{S}}(D_{\mathbf{i}\widetilde{X}_1}\mathbf{Q})(\widetilde{X}_2,\widetilde{X}_3,\widetilde{Y}) = 0$
IV	$(D_{\mathbf{i}\widetilde{X}_1}\mathbf{R})(\widetilde{X}_2,\widetilde{X}_3,\widetilde{Y}) + (D_{\mathcal{H}\widetilde{X}_2}\mathbf{P})(\widetilde{X}_3,\widetilde{X}_1,\widetilde{Y}) - (D_{\mathcal{H}\widetilde{X}_3}\mathbf{P})(\widetilde{X}_2,\widetilde{X}_1,\widetilde{Y})$ $+\mathbf{Q}(\mathbf{R}^1(\widetilde{X}_2,\widetilde{X}_3),\widetilde{X}_1)\widetilde{Y} + \mathbf{R}(\mathcal{S}(\widetilde{X}_1,\widetilde{X}_3),\widetilde{X}_2)\widetilde{Y} - \mathbf{R}(\mathcal{S}(\widetilde{X}_1,\widetilde{X}_2),\widetilde{X}_3)\widetilde{Y}$ $+\mathbf{P}(\widetilde{X}_3,\mathcal{T}(\widetilde{X}_1,\widetilde{X}_2))\widetilde{Y} - \mathbf{P}(\widetilde{X}_2,\mathcal{T}(\widetilde{X}_1,\widetilde{X}_3))\widetilde{Y} = 0$
V	$(D_{\mathbf{i}\widetilde{X}_2}\mathbf{P})(\widetilde{X}_1,\widetilde{X}_3,\widetilde{Y}) - (D_{\mathbf{i}\widetilde{X}_3}\mathbf{P})(\widetilde{X}_1,\widetilde{X}_2,\widetilde{Y}) + (D_{\mathcal{H}\widetilde{X}_1}\mathbf{Q})(\widetilde{X}_3,\widetilde{X}_2,\widetilde{Y})$ $+\mathbf{Q}(\mathcal{T}(\widetilde{X}_1,\widetilde{X}_3),\widetilde{X}_2)\widetilde{Y} - \mathbf{Q}(\mathcal{T}(\widetilde{X}_1,\widetilde{X}_2),\widetilde{X}_3)\widetilde{Y} + \mathbf{P}(\mathcal{S}(\widetilde{X}_1,\widetilde{X}_3),\widetilde{X}_2)\widetilde{Y}$ $-\mathbf{P}(\mathcal{S}(\widetilde{X}_2,\widetilde{X}_1),\widetilde{X}_3)\widetilde{Y} = 0$

($\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{Y}$ are arbitrary vector fields along τ).

Proof. We check only the first three relations, in the remaining two cases the calculation is similar.

(1) We evaluate both sides of the algebraic Bianchi identity on a triplet $(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3)$. Applying 1.43 and the vanishing of $\mathcal{T}$, we get

$$[d^D(d^D\mathbf{j})](\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3) = D_{\mathcal{H}\widetilde{X}_1}(d^D\mathbf{j})(\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3) - D_{\mathcal{H}\widetilde{X}_2}(d^D\mathbf{j})(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_3)$$
$$+ D_{\mathcal{H}\widetilde{X}_3}(d^D\mathbf{j})(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2) - (d^D\mathbf{j})([\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3) + (d^D\mathbf{j})([\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_3],\mathcal{H}\widetilde{X}_2)$$
$$- (d^D\mathbf{j})([\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3],\mathcal{H}\widetilde{X}_1) = -\underset{1,2,3}{\mathfrak{S}}(d^D\mathbf{j})([\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3).$$

Since

$$(d^D\mathbf{j})([\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3) = (d^D\mathbf{j})(\mathcal{H}\circ\mathbf{j}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3)$$
$$+ (d^D\mathbf{j})(\mathbf{i}\circ\mathcal{V}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3) = -(d^D\mathbf{j})(\mathcal{H}\widetilde{X}_3,\mathbf{i}\circ\mathcal{V}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2])$$
$$= -\mathcal{S}(\widetilde{X}_3,\mathcal{V}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2]) = -\mathcal{S}(-\mathbf{R}^1(\widetilde{X}_1,\widetilde{X}_2),\widetilde{X}_3) = \mathcal{S}(\mathbf{R}^1(\widetilde{X}_1,\widetilde{X}_2),\widetilde{X}_3),$$

it follows that

$$d^D(d^D\mathbf{j})(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3) = -\underset{1,2,3}{\mathfrak{S}}\mathcal{S}(\mathbf{R}^1(\widetilde{X}_1,\widetilde{X}_2),\widetilde{X}_3).$$

On the other hand, according to the definition given in 1.37 (3),

$$R^D[\mathbf{j}](\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3)=\frac{1}{2}(R^D(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2)\mathbf{j}(\mathcal{H}\widetilde{X}_3)+R^D(\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3)\mathbf{j}(\mathcal{H}\widetilde{X}_1)$$
$$+R^D(\mathcal{H}\widetilde{X}_3,\mathcal{H}\widetilde{X}_1)\mathbf{j}(\mathcal{H}\widetilde{X}_2)-R^D(\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_1)\mathbf{j}(\mathcal{H}\widetilde{X}_3)-R^D(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_3)\mathbf{j}(\mathcal{H}\widetilde{X}_2)$$
$$-R^D(\mathcal{H}\widetilde{X}_3,\mathcal{H}\widetilde{X}_2)\mathbf{j}(\mathcal{H}\widetilde{X}_1))=\underset{1,2,3}{\mathfrak{S}}R^D(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2)\widetilde{X}_3=\underset{1,2,3}{\mathfrak{S}}\mathbf{R}(\widetilde{X}_1,\widetilde{X}_2)\widetilde{X}_3.$$

This concludes the proof of the first Bianchi identity.

(2) In view of the differential Bianchi identity,

$$0=d^DR^D(\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2,\mathcal{H}\widetilde{X}_3)=\underset{1,2,3}{\mathfrak{S}}(D_{\mathcal{H}\widetilde{X}_1}[\mathbf{R}(\widetilde{X}_2,\widetilde{X}_3)]-R^D([\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3)).$$

Using the product rule 1.32 (3), we get

$$\underset{1,2,3}{\mathfrak{S}}D_{\mathcal{H}\widetilde{X}_1}[\mathbf{R}(\widetilde{X}_2,\widetilde{X}_3)]=\underset{1,2,3}{\mathfrak{S}}(D_{\mathcal{H}\widetilde{X}_1}\mathbf{R})(\widetilde{X}_2,\widetilde{X}_3)+\underset{1,2,3}{\mathfrak{S}}\mathbf{R}(D_{\mathcal{H}\widetilde{X}_1}\widetilde{X}_2,\widetilde{X}_3)$$
$$+\underset{1,2,3}{\mathfrak{S}}\mathbf{R}(\widetilde{X}_2,D_{\mathcal{H}\widetilde{X}_1}\widetilde{X}_3).$$

Observe that from 2.45, Proposition 2 (3) it follows that

$$\mathbf{j}[\mathcal{H}\widetilde{X}_i,\mathcal{H}\widetilde{X}_j]=D_{\mathcal{H}\widetilde{X}_i}\widetilde{X}_j-D_{\mathcal{H}\widetilde{X}_j}\widetilde{X}_i\quad(1\leqq i\neq j\leqq 3).$$

Thus

$$\underset{1,2,3}{\mathfrak{S}}R^D([\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3)=\underset{1,2,3}{\mathfrak{S}}\mathbf{R}^D(\mathcal{H}\circ\mathbf{j}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3)$$
$$+\underset{1,2,3}{\mathfrak{S}}R^D(\mathbf{i}\circ\mathcal{V}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2],\mathcal{H}\widetilde{X}_3)=\underset{1,2,3}{\mathfrak{S}}\mathbf{R}(D_{\mathcal{H}\widetilde{X}_1}\widetilde{X}_2,\widetilde{X}_3)$$
$$-\underset{1,2,3}{\mathfrak{S}}\mathbf{R}(D_{\mathcal{H}\widetilde{X}_2}\widetilde{X}_1,\widetilde{X}_3)-\underset{1,2,3}{\mathfrak{S}}\mathbf{P}(\widetilde{X}_3,\mathcal{V}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2])$$
$$=\underset{1,2,3}{\mathfrak{S}}(\mathbf{R}(D_{\mathcal{H}\widetilde{X}_1}\widetilde{X}_2,\widetilde{X}_3)+\mathbf{R}(\widetilde{X}_2,D_{\mathcal{H}\widetilde{X}_1}\widetilde{X}_3))+\underset{1,2,3}{\mathfrak{S}}\mathbf{P}(\widetilde{X}_3,\mathbf{R}(\widetilde{X}_1,\widetilde{X}_2)),$$

and we obtain the desired relation.

(3) Again, our starting point is the differential Bianchi identity. Applying 2.45, Proposition 2 (1), we get

$$0=d^DR^D(\mathbf{i}\widetilde{X}_1,\mathbf{i}\widetilde{X}_2,\mathbf{i}\widetilde{X}_3)=\underset{1,2,3}{\mathfrak{S}}D_{\mathbf{i}\widetilde{X}_1}[\mathbf{Q}(\widetilde{X}_2,\widetilde{X}_3)]-\underset{1,2,3}{\mathfrak{S}}R^D([\mathbf{i}\widetilde{X}_1,\mathbf{i}\widetilde{X}_2],\mathbf{i}\widetilde{X}_3)$$
$$=\underset{1,2,3}{\mathfrak{S}}(D_{\mathbf{i}\widetilde{X}_1}\mathbf{Q})(\widetilde{X}_2,\widetilde{X}_3)+\underset{1,2,3}{\mathfrak{S}}(\mathbf{Q}(D_{\mathbf{i}\widetilde{X}_1}\widetilde{X}_2,\widetilde{X}_3)+\mathbf{Q}(\widetilde{X}_2,D_{\mathbf{i}\widetilde{X}_1}\widetilde{X}_3))$$
$$-\underset{1,2,3}{\mathfrak{S}}R^D(\mathbf{i}(D_{\mathbf{i}\widetilde{X}_1}\widetilde{X}_2-D_{\mathbf{i}\widetilde{X}_2}\widetilde{X}_1),\mathbf{i}\widetilde{X}_3)=\underset{1,2,3}{\mathfrak{S}}(D_{\mathbf{i}\widetilde{X}_1}\mathbf{Q})(\widetilde{X}_2,\widetilde{X}_3)$$
$$+\underset{1,2,3}{\mathfrak{S}}\mathbf{Q}(D_{\mathbf{i}\widetilde{X}_1}\widetilde{X}_2,\widetilde{X}_3)-\underset{1,2,3}{\mathfrak{S}}\mathbf{Q}(D_{\mathbf{i}\widetilde{X}_2}\widetilde{X}_1,\widetilde{X}_3)-\underset{1,2,3}{\mathfrak{S}}\mathbf{Q}(D_{\mathbf{i}\widetilde{X}_1}\widetilde{X}_2,\widetilde{X}_3)$$
$$+\underset{1,2,3}{\mathfrak{S}}\mathbf{Q}(D_{\mathbf{i}\widetilde{X}_2}\widetilde{X}_1,\widetilde{X}_3)=\underset{1,2,3}{\mathfrak{S}}(D_{\mathbf{i}\widetilde{X}_1}\mathbf{Q})(\widetilde{X}_2,\widetilde{X}_3),$$

thereby proving the third Bianchi identity.

Corollary. *Under the hypotheses of the Proposition,*

$$\underset{1,2,3}{\mathfrak{S}} \mathbf{Q}(\widetilde{X}_1, \widetilde{X}_2)\widetilde{X}_3 = 0.$$

Proof. Replacing $\widetilde{Y} := \delta$ in the third Bianchi identity, and taking into account 2.45, Lemma 4, the formula drops out. □

Proposition 2. *Hypothesis as above.*

(1) *The Berwald curvature* $\mathbf{P}$*, the Finsler torsion* $\mathcal{S}$ *and the v-mixed torsion* $\mathbf{P}^1$ *of* D *satisfy*

$$\begin{aligned}\mathbf{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z} - \mathbf{P}(\widetilde{Z}, \widetilde{Y})\widetilde{X} &= (D_{\mathcal{H}\widetilde{Z}}\mathcal{S})(\widetilde{X}, \widetilde{Y}) - (D_{\mathcal{H}\widetilde{X}}\mathcal{S})(\widetilde{Y}, \widetilde{Z}) \\ &+ \mathcal{S}(\mathbf{P}^1(\widetilde{Z}, \widetilde{Y}), \widetilde{X}) - \mathcal{S}(\mathbf{P}^1(\widetilde{X}, \widetilde{Y}), \widetilde{Z})\end{aligned}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\tau)$.

(2) *The Berwald-Cartan curvature* $\mathbf{Q}$ *is related with the Finsler torsion* $\mathcal{S}$ *by the following formula:*

$$\begin{aligned}\mathbf{Q}(\widetilde{X}, \widetilde{Y})\widetilde{Z} &= \mathcal{S}(\mathcal{S}(\widetilde{Z}, \widetilde{X}), \widetilde{Y}) - \mathcal{S}(\mathcal{S}(\widetilde{Y}, \widetilde{Z}), \widetilde{X}) \\ &+ (D_{\mathbf{i}\widetilde{Y}}\mathcal{S})(\widetilde{X}, \widetilde{Z}) - (D_{\mathbf{i}\widetilde{X}}\mathcal{S})(\widetilde{Y}, \widetilde{Z});\end{aligned}$$

$\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ *are arbitrary vector fields along* τ.

Proof. Applying Proposition 2 in 2.45, evaluate both sides of the algebraic Bianchi identity on a triplet of form $(\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}, \mathcal{H}\widetilde{Z})$ in the first case, and on a triplet $(\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}, \mathcal{H}\widetilde{Z}) \in (\mathfrak{X}(TM))^3$ in the second case. □

Proposition 3. *Let* D *be a symmetric covariant derivative operator in* $\tau^*\tau$*, associated to the horizontal map* $\mathcal{H}$*. If* A *is a type* $(1,2)$ *tensor field along* τ*, then we have the following* Ricci formulae:

(1) $$\begin{aligned}&\big(D_{\mathcal{H}\widetilde{X}}D_{\mathbf{i}\widetilde{Y}}A - D_{\mathbf{i}\widetilde{Y}}D_{\mathcal{H}\widetilde{X}}A - D_{[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]}A\big)(\widetilde{Z}_1, \widetilde{Z}_2) \\ &= \mathbf{P}(\widetilde{X}, \widetilde{Y})A(\widetilde{Z}_1, \widetilde{Z}_2) - A(\mathbf{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z}_1, \widetilde{Z}_2) - A(\widetilde{Z}_1, \mathbf{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z}_2),\end{aligned}$$

(2) $$\begin{aligned}&\big(D_{\mathbf{i}\widetilde{X}}D_{\mathbf{i}\widetilde{Y}}A - D_{\mathbf{i}\widetilde{Y}}D_{\mathbf{i}\widetilde{X}}A - D_{[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}]}A\big)(\widetilde{Z}_1, \widetilde{Z}_2) \\ &= \mathbf{Q}(\widetilde{X}, \widetilde{Y})A(\widetilde{Z}_1, \widetilde{Z}_2) - A(\mathbf{Q}(\widetilde{X}, \widetilde{Y})\widetilde{Z}_1, \widetilde{Z}_2) - A(\widetilde{Z}_1, \mathbf{Q}(\widetilde{X}, \widetilde{Y})\widetilde{Z}_2),\end{aligned}$$

(3) $$\begin{aligned}&\big(D_{\mathcal{H}\widetilde{X}}D_{\mathcal{H}\widetilde{Y}}A - D_{\mathcal{H}\widetilde{Y}}D_{\mathcal{H}\widetilde{X}}A - D_{[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]}A\big)(\widetilde{Z}_1, \widetilde{Z}_2) \\ &= \mathbf{R}(\widetilde{X}, \widetilde{Y})A(\widetilde{Z}_1, \widetilde{Z}_2) - A(\mathbf{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}_1, \widetilde{Z}_2) - A(\widetilde{Z}_1, \mathbf{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}_2)\end{aligned}$$

($\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}_1$, $\widetilde{Z}_2$ are arbitrary vector fields along τ).

We omit the straightforward but lengthy proof.

2.47. The Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$. Let ∇ be the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$ according to 2.44, Example 1. In that example we also described the intimate relation between the Berwald derivatives arising from $\mathcal{H}$ in τ_{TM} on the one hand, and in $\tau^*\tau$ on the other. In 2.22 we have found a canonical correspondence between the semibasic covariant (and vector-valued covariant) tensor fields on TM and the covariant (resp. vector-valued covariant) tensor fields along τ. These facts enable us to translate the results of 2.33 into the pull-back framework without any difficulty.

The torsions and the curvatures of the Berwald derivative ∇ in $\tau^*\tau$ will be denoted by $\overset{\circ}{\mathcal{T}}$, $\overset{\circ}{\mathcal{S}}$, $\overset{\circ}{\mathbf{R}}{}^1$, $\overset{\circ}{\mathbf{P}}{}^1$, $\overset{\circ}{\mathbf{Q}}{}^1$, and $\overset{\circ}{\mathbf{R}}$, $\overset{\circ}{\mathbf{P}}$, $\overset{\circ}{\mathbf{Q}}$, respectively.

1. $$(\overset{\circ}{\mathcal{T}})_0 = \mathbf{T} := \textit{ the torsion of } \mathcal{H}, \quad (\overset{\circ}{\mathbf{R}}{}^1)_0 = \Omega := \textit{ the curvature of } \mathcal{H}.$$

Indeed, for any vector fields X, Y on M we have

$$\begin{aligned}(\overset{\circ}{\mathcal{T}})_0(X^c, Y^c) &:= \mathbf{i}\overset{\circ}{\mathcal{T}}(\mathbf{j}X^c, \mathbf{j}Y^c) = \mathbf{i}\mathcal{T}^0(\widehat{X}, \widehat{Y}) = \mathbf{i}(\nabla_{X^h}\widehat{Y} - \nabla_{Y^h}\widehat{X} - \mathbf{j}[X^h, Y^h]) \\ &= \mathbf{i}(\mathcal{V}[X^h, Y^{\mathrm{v}}] - \mathcal{V}[Y^h, X^{\mathrm{v}}] - \mathbf{j}[X^h, Y^h]) = \mathbf{v}[X^h, Y^{\mathrm{v}}] - \mathbf{v}[Y^h, X^{\mathrm{v}}] \\ &\quad - J[X^h, Y^h] = [X^h, Y^{\mathrm{v}}] - [Y^h, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}} = \mathbf{T}(X^c, Y^c).\end{aligned}$$

Similarly, for any vector fields ξ, η on TM,

$$(\overset{\circ}{\mathbf{R}}{}^1)_0(\xi, \eta) = \mathbf{i}\overset{\circ}{\mathbf{R}}{}^1(\mathbf{j}\xi, \mathbf{j}\eta) \overset{\text{2.45, Lemma 1}}{=} \Omega(\xi, \eta).$$

Thus *the torsion and the h-horizontal torsion of the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$ may be regarded as the torsion and the curvature of $\mathcal{H}$, respectively.*

2. $$\boxed{\overset{\circ}{\mathcal{S}} = 0, \quad \overset{\circ}{\mathbf{P}}{}^1 = 0, \quad \overset{\circ}{\mathbf{Q}}{}^1 = 0.}$$

In view of our above remarks, these are immediate consequences of 2.33, **11**; see also 2.45, Corollary.

3. $$\boxed{\begin{aligned}&\overset{\circ}{\mathbf{P}}(\widehat{X}, \widehat{Y})\widehat{Z} = \mathcal{V}\big[[X^h, Y^{\mathrm{v}}], \quad \overset{\circ}{\mathbf{Q}} = 0, \\ &(\nabla^h\nabla^{\mathrm{v}}\widetilde{Z})(\widehat{X}, \widehat{Y}) - (\nabla^{\mathrm{v}}\nabla^h\widetilde{Z})(\widehat{Y}, \widehat{X}) = \overset{\circ}{\mathbf{P}}(\widehat{X}, \widehat{Y})\widetilde{Z}\end{aligned}}$$

$$(X, Y, Z \in \mathfrak{X}(M),\ \widetilde{Z} \in \mathfrak{X}(\tau)).$$

In fact, the first two of these relations correspond to 2.33, **8** and 2.33, **10**, while the third may be checked by an easy calculation.

4. *The tension* $\widetilde{\mathbf{t}}$ *of* $\mathcal{H}$ *and the Berwald curvature* $\overset{\circ}{\mathbf{P}}$ *of* ∇ *are related by*

$$\overset{\circ}{\mathbf{P}}(\widehat{X},\widehat{Y})\delta = -(\nabla^{\mathrm{v}}\widetilde{\mathbf{t}})(\widehat{Y},\widehat{X}) \quad \textit{for all } X,Y \in \mathfrak{X}(M).$$

Indeed, taking into account 2.44, Example 2 and 2.31, **5**, we obtain:

$$\overset{\circ}{\mathbf{P}}(\widehat{X},\widehat{Y})\delta := \nabla_{X^h}\nabla_{Y^{\mathrm{v}}}\delta - \nabla_{Y^{\mathrm{v}}}\nabla_{X^h}\delta - \nabla_{[X^h,Y^{\mathrm{v}}]}\delta = \nabla_{X^h}\widehat{Y} - \nabla_{Y^{\mathrm{v}}}\widetilde{\mathbf{t}}(\widehat{X})$$
$$- \mathbf{j}[[X^h,Y^{\mathrm{v}}],\mathcal{H}\circ\delta] = \mathcal{V}[X^h,Y^{\mathrm{v}}] - (\nabla^{\mathrm{v}}\widetilde{\mathbf{t}})(\widehat{Y},\widehat{X}) - \mathcal{V}[X^h,Y^{\mathrm{v}}]$$
$$= -(\nabla^{\mathrm{v}}\widetilde{\mathbf{t}})(\widehat{Y},\widehat{X}).$$

$$\boxed{\text{We assume now that } \nabla \text{ is associated to the horizontal map } \mathcal{H}.}$$

Notice that in view of 2.44, Example 2 our assumption is equivalent to the homogeneity of $\mathcal{H}$.

5. *For any vector fields* $\widetilde{X}$, $\widetilde{Y}$ *along* τ *we have*:

$$[\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}] = \mathbf{i}(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} - \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X}),$$
$$[\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}] = \mathbf{i}\nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \mathcal{H}\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X},$$
$$[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}] = -\mathbf{i}\overset{\circ}{\mathbf{R}}{}^1(\widetilde{X},\widetilde{Y}) + \mathcal{H}(\nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X}).$$

This is an immediate consequence of 2.45, Proposition 2.

6. The Bianchi identities for the Berwald derivative ∇, induced by and associated to the horizontal map $\mathcal{H}$, reduce to the following:

I	$\underset{1,2,3}{\mathfrak{S}}\overset{\circ}{\mathbf{R}}(\widetilde{X}_1,\widetilde{X}_2)\widetilde{X}_3 = 0$
II	$\underset{1,2,3}{\mathfrak{S}}(\nabla_{\mathcal{H}\widetilde{X}_1}\overset{\circ}{\mathbf{R}})(\widetilde{X}_2,\widetilde{X}_3,\widetilde{Y}) = \underset{1,2,3}{\mathfrak{S}}\overset{\circ}{\mathbf{P}}(\widetilde{X}_3,\overset{\circ}{\mathbf{R}}{}^1(\widetilde{X}_1,\widetilde{X}_2))\widetilde{Y}$
IV	$(\nabla_{\mathbf{i}\widetilde{X}_1}\overset{\circ}{\mathbf{R}})(\widetilde{X}_2,\widetilde{X}_3,\widetilde{Y}) + (\nabla_{\mathcal{H}\widetilde{X}_2}\overset{\circ}{\mathbf{P}})(\widetilde{X}_3,\widetilde{X}_1,\widetilde{Y})$ $-(\nabla_{\mathcal{H}\widetilde{X}_3}\overset{\circ}{\mathbf{P}})(\widetilde{X}_2,\widetilde{X}_1,\widetilde{Y}) = 0$
V	$(\nabla_{\mathbf{i}\widetilde{X}_2}\overset{\circ}{\mathbf{P}})(\widetilde{X}_1,\widetilde{X}_3,\widetilde{Y}) = (\nabla_{\mathbf{i}\widetilde{X}_3}\overset{\circ}{\mathbf{P}})(\widetilde{X}_1,\widetilde{X}_2,\widetilde{Y})$

.

7. *Under the homogeneity of $\mathcal{H}$, the Berwald curvature $\overset{\circ}{\mathbf{P}}$ of ∇ is totally symmetric.*

This property has already been proved in the tangent bundle framework under the condition that the torsion of $\mathcal{H}$ vanishes; see 2.33, Property **8**. In the present situation another efficient reasoning, due to J.-G. Diaz [25], is also possible. Observe first that, by the vanishing of $\overset{\circ}{\mathcal{S}}$, 2.46, Proposition 2(1) implies that

$$\overset{\circ}{\mathbf{P}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \overset{\circ}{\mathbf{P}}(\widetilde{Z},\widetilde{Y})\widetilde{X} \quad \text{for all } \widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathfrak{X}(\tau).$$

Since $\overset{\circ}{\mathbf{P}}{}^1$ also vanishes, 2.45, Lemma 4 leads to the relation $\overset{\circ}{\mathbf{P}}(\,\cdot\,,\,\cdot\,)\delta = 0$. Thus, replacing $\widetilde{Y} := \delta$ in the fifth Bianchi identity, we obtain that $\overset{\circ}{\mathbf{P}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \overset{\circ}{\mathbf{P}}(\widetilde{X},\widetilde{Z})\widetilde{Y}$. This concludes the proof.

8. *Let the $(1,1)$ tensor field $\widetilde{\overset{\circ}{\mathbf{R}}}$ along τ be defined by $\widetilde{\overset{\circ}{\mathbf{R}}}(\widetilde{X}) := \overset{\circ}{\mathbf{R}}(\delta,\widetilde{X})\delta$. Then for all $\widetilde{X},\widetilde{Y} \in \mathfrak{X}(\tau)$ we have*

$$\overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\delta = \frac{1}{3}\Big[\big(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{\overset{\circ}{\mathbf{R}}}\big)(\widetilde{Y}) - \big(\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{\overset{\circ}{\mathbf{R}}}\big)(\widetilde{X})\Big].$$

Proof. We follow the argument presented in [25]. Since $\overset{\circ}{\mathbf{P}}(\,\cdot\,,\,\cdot\,)\delta = 0$ and $\nabla^h\delta = 0$, the fourth Bianchi identity yields

$$(\nabla_{\mathbf{i}\widetilde{X}}\overset{\circ}{\mathbf{R}})(\widetilde{Z},\widetilde{Y},\delta) = 0 \quad \text{for all } \widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathfrak{X}(\tau).$$

In detail,

$$\nabla_{\mathbf{i}\widetilde{X}}(\overset{\circ}{\mathbf{R}}(\widetilde{Z},\widetilde{Y})\delta) - \overset{\circ}{\mathbf{R}}(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Z},\widetilde{Y})\delta - \overset{\circ}{\mathbf{R}}(\widetilde{Z},\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y})\delta - \overset{\circ}{\mathbf{R}}(\widetilde{Z},\widetilde{Y})\widetilde{X} = 0.$$

Hence

$$\nabla_{\mathbf{i}\widetilde{X}}\Big(\widetilde{\overset{\circ}{\mathbf{R}}}(Y)\Big) = \overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\delta + \widetilde{\overset{\circ}{\mathbf{R}}}(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y}) + \overset{\circ}{\mathbf{R}}(\delta,\widetilde{Y})\widetilde{X},$$

therefore

$$\Big(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{\overset{\circ}{\mathbf{R}}}\Big)(\widetilde{Y}) = \overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\delta + \overset{\circ}{\mathbf{R}}(\delta,\widetilde{Y})\widetilde{X},$$

and, consequently,

$$\Big(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{\overset{\circ}{\mathbf{R}}}\Big)(\widetilde{Y}) - \Big(\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{\overset{\circ}{\mathbf{R}}}\Big)(\widetilde{X}) = 2\overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\delta + \overset{\circ}{\mathbf{R}}(\delta,\widetilde{Y})\widetilde{X} - \overset{\circ}{\mathbf{R}}(\delta,\widetilde{X})\widetilde{Y}.$$

Here, by the first Bianchi identity, $\overset{\circ}{\mathbf{R}}(\delta,\widetilde{Y})\widetilde{X} - \overset{\circ}{\mathbf{R}}(\delta,\widetilde{X})\widetilde{Y} = \overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\delta$, which concludes the proof.

9. *In the homogeneous case the Riemann curvature of the Berwald derivative is determined by the v-horizontal torsion, namely*

$$\overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = (\nabla_{\mathbf{i}\widetilde{Z}}\overset{\circ}{\mathbf{R}}{}^1)(\widetilde{X},\widetilde{Y}) \quad \textit{for all } \widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathfrak{X}(\tau)$$

(cf. 2.33, **6**).

Proof. In view of the assumption $\nabla^h\delta = 0$ and Property **7** above, the fourth Bianchi identity yields

$$\big(\nabla_{\mathbf{i}\widetilde{Z}}\overset{\circ}{\mathbf{R}}\big)(\widetilde{X},\widetilde{Y},\delta) = 0 \quad \text{for all } \widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathfrak{X}(\tau).$$

In detail, taking into account 2.45, Lemma 4,

$$\begin{aligned} 0 &= \nabla_{\mathbf{i}\widetilde{Z}}(\overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\delta) - \overset{\circ}{\mathbf{R}}(\nabla_{\mathbf{i}\widetilde{Z}}\widetilde{X},\widetilde{Y})\delta - \overset{\circ}{\mathbf{R}}(\widetilde{X},\nabla_{\mathbf{i}\widetilde{Z}}\widetilde{Y})\delta - \overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} \\ &= \nabla_{\mathbf{i}\widetilde{Z}}(\overset{\circ}{\mathbf{R}}{}^1(\widetilde{X},\widetilde{Y})) - \overset{\circ}{\mathbf{R}}{}^1(\nabla_{\mathbf{i}\widetilde{Z}}\widetilde{X},\widetilde{Y}) - \overset{\circ}{\mathbf{R}}(\widetilde{X},\nabla_{\mathbf{i}\widetilde{Z}}\widetilde{Y}) - \overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} \\ &= (\nabla_{\mathbf{i}\widetilde{Z}}\overset{\circ}{\mathbf{R}}{}^1)(\widetilde{X},\widetilde{Y}) - \overset{\circ}{\mathbf{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z}, \end{aligned}$$

whence the statement.

10. *In the homogeneous case for the Riemann curvature of the Berwald derivative the following are equivalent:*

(1) $\overset{\circ}{\mathbf{R}} = 0$;

(2) $\overset{\circ}{\mathbf{R}}(\,\cdot\,,\,\cdot\,)\delta = \overset{\circ}{\mathbf{R}}{}^1 = 0$;

(3) $\overset{\circ}{\mathbf{R}}(\delta,\,\cdot\,)\delta = 0$.

This is an immediate consequence of the above properties **8** and **9**; see also 2.33, Property 7.

2.48. A theorem of M. Crampin on Berwald derivatives.

Theorem. *Let D be a covariant derivative operator in $\tau^*\tau$ such that $D^{\mathrm{v}} = \nabla^{\mathrm{v}}$, i.e., the v-covariant derivative induced by D is the canonical v-covariant derivative.*

(a) *If $\mathcal{H}$ is a horizontal map for τ then the following two conditions are equivalent:*

(i) $\quad (D_{\mathbf{i}\widetilde{Y}}\mathbf{P}^1)(\widetilde{X},\widetilde{Z}) = (D_{\mathbf{i}\widetilde{Z}}\mathbf{P}^1)(\widetilde{X},\widetilde{Y}) \quad \textit{for all } \widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathfrak{X}(\tau)$;

(ii) $\quad \mathbf{P}(\widetilde{X},\widetilde{Z})\widetilde{Y} = \mathbf{P}(\widetilde{X},\widetilde{Y})\widetilde{Z} \quad \textit{for all } \widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathfrak{X}(\tau)$.

($\mathbf{P}^1$ and $\mathbf{P}$ are the v-mixed torsion and the Berwald curvature of D with respect to $\mathcal{H}$, respectively).

(b) *If either of conditions (i) and (ii) holds for one horizontal map, then both of them hold for all horizontal maps.*

(c) *Condition (i), and hence (ii), is necessary and sufficient for there to be some horizontal map $\mathcal{H}$ on M which induces D as the Berwald derivative arising from $\mathcal{H}$.*

Proof. (1) Note first that $D^{\mathrm{v}} = \nabla^{\mathrm{v}}$ implies that

$$\mathcal{S} = 0, \quad \mathbf{Q}^1 = 0 \quad \text{and} \quad \mathbf{Q} = 0,$$

see e.g. 2.45, Lemma 1 and 2.42, Example 2.

(2) To prove (a), we start from the Jacobi indentity

$$(*) \qquad \big[[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}], \mathbf{i}\widetilde{Z}\big] + \big[[\mathbf{i}\widetilde{Y}, \mathbf{i}\widetilde{Z}], \mathcal{H}\widetilde{X}\big] + \big[[\mathbf{i}\widetilde{Z}, \mathcal{H}\widetilde{X}], \mathbf{i}\widetilde{Y}\big] = 0$$
$$(\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\tau)).$$

Next we express the brackets in terms of covariant derivatives. The calculation is somewhat lengthy, but quite straightforward. For example, using the expression for $\mathbf{P}^1$ in 2.45, Lemma 1 and the vanishing of $\mathcal{S}$ and $\mathbf{Q}^1$, the first term in $(*)$ may be formed as follows:

$$\begin{aligned}\big[[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}], \mathbf{i}\widetilde{Z}\big] &= \big[\mathbf{i}\mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}], \mathbf{i}Z\big] + \big[\mathcal{H}\circ\mathbf{j}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}], \mathbf{i}\widetilde{Z}\big] = [\mathbf{i}D_{\mathcal{H}\widetilde{X}}\widetilde{Y}, \mathbf{i}\widetilde{Z}] \\ &- [\mathbf{i}\mathbf{P}^1(\widetilde{X}, \widetilde{Y}), \mathbf{i}\widetilde{Z}] - [\mathcal{H}D_{\mathbf{i}\widetilde{Y}}\widetilde{X}, \mathbf{i}\widetilde{Z}] = \mathbf{i}D_{\mathbf{i}D_{\mathcal{H}\widetilde{X}}\widetilde{Y}}\widetilde{Z} - \mathbf{i}D_{\mathbf{i}\widetilde{Z}}D_{\mathcal{H}\widetilde{X}}\widetilde{Y} \\ &- \mathbf{i}D_{\mathbf{i}\mathbf{P}^1(\widetilde{X},\widetilde{Y})}\widetilde{Z} + \mathbf{i}D_{\mathbf{i}\widetilde{Z}}(\mathbf{P}^1(\widetilde{X}, \widetilde{Y})) - \mathbf{i}D_{\mathcal{H}D_{\mathbf{i}\widetilde{Y}}\widetilde{X}}\widetilde{Z} + \mathbf{i}\mathbf{P}^1(D_{\mathbf{i}\widetilde{Y}}\widetilde{X}, \widetilde{Z}) + \mathcal{H}D_{\mathbf{i}\widetilde{Z}}D_{\mathbf{i}\widetilde{Y}}\widetilde{X}.\end{aligned}$$

Analogously, we get similar expressions for the second and the third term of $(*)$. Adding all these, and operate by $\mathcal{V}$ on both sides of the equality obtained, after some easy steps we arrive at the relation

$$\mathbf{P}(\widetilde{X}, \widetilde{Z})\widetilde{Y} - \mathbf{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = (D_{\mathbf{i}\widetilde{Y}}\mathbf{P}^1)(\widetilde{X}, \widetilde{Z}) - (D_{\mathbf{i}\widetilde{Z}}\mathbf{P}^1)(\widetilde{X}, \widetilde{Y}).$$

This concludes the proof of assertion (a).

(3) Suppose that (ii) holds for a horizontal map $\mathcal{H}$. We show that then (ii) is valid for any other horizontal map $\bar{\mathcal{H}}$.

Observe that $\bar{\mathcal{H}} - \mathcal{H}$ is vertical-valued, since

$$J \circ (\bar{\mathcal{H}} - \mathcal{H}) = \mathbf{i} \circ \mathbf{j} \circ \bar{\mathcal{H}} - \mathbf{i} \circ \mathbf{j} \circ \mathcal{H} = \mathbf{i} - \mathbf{i} = 0.$$

From this it follows that $\bar{\mathcal{H}}$ and $\mathcal{H}$ are related as follows:

$$\bar{\mathcal{H}} = \mathcal{H} + \mathbf{i} \circ \widetilde{A}, \quad \widetilde{A} \in \mathcal{T}_1^1(\tau).$$

If $\bar{\mathbf{P}}$ is the Berwald curvature of D with respect to $\bar{\mathcal{H}}$, then for any vector fields $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ along τ we have

$$\bar{\mathbf{P}}(\widetilde{X}, \widetilde{Z})\widetilde{Y} - \bar{\mathbf{P}}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = R^D(\bar{\mathcal{H}}\widetilde{X}, \mathbf{i}\widetilde{Z})\widetilde{Y} - R^D(\bar{\mathcal{H}}\widetilde{X}, \mathbf{i}\widetilde{Y})\widetilde{Z}$$
$$= \mathbf{P}(\widetilde{X}, \widetilde{Z})\widetilde{Y} - \mathbf{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z} + \mathbf{Q}(\widetilde{A}(\widetilde{X}), \widetilde{Z}) - \mathbf{Q}(\widetilde{A}(\widetilde{X}), \widetilde{Y})\widetilde{Z} = 0,$$

thus proving assertion (b).

(4) *Both of conditions (i) and (ii) are necessary.* Indeed, if D is the Berwald derivative induced by a horizontal map, then its v-mixed torsion vanishes according to 2.47, **2**, therefore (i) holds automatically. Notice that (ii) is valid also in its own rights in virtue of 2.33, **8**.

(5) Finally, we turn to the proof of the *sufficiency* of (i). Suppose that (i) holds for a horizontal map $\mathcal{H}$. In view of 2.45, Corollary, we have to find a horizontal map $\bar{\mathcal{H}}$ which has vanishing v-mixed torsion $\bar{\mathbf{P}}^1$. As we have just seen, $\bar{\mathcal{H}}$ and $\mathcal{H}$ are related by $\bar{\mathcal{H}} = \mathcal{H} + \mathbf{i} \circ \widetilde{A}$, where $\widetilde{A}$ is a type $(1,1)$ tensor field along τ. Then the relationship between the corresponding vertical maps is $\bar{\mathcal{V}} = \mathcal{V} - \widetilde{A} \circ \mathbf{j}$. Thus for any vector fields $\widetilde{X}$, $\widetilde{Y}$ in $\mathfrak{X}(\tau)$ we have

$$\bar{\mathbf{P}}^1(\widetilde{X}, \widetilde{Y}) = D_{\bar{\mathcal{H}}\widetilde{X}}\widetilde{Y} - \bar{\mathcal{V}}[\bar{\mathcal{H}}\widetilde{X}, \mathbf{i}\widetilde{Y}] = D_{\mathcal{H}\widetilde{X}}\widetilde{Y} + D_{\mathbf{i}\widetilde{A}(\widetilde{X})}\widetilde{Y} - \bar{\mathcal{V}}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$$
$$- \bar{\mathcal{V}}[\mathbf{i}\widetilde{A}(\widetilde{X}), \mathbf{i}\widetilde{Y}] = \mathbf{P}^1(\widetilde{X}, \widetilde{Y}) + D_{\mathbf{i}\widetilde{A}(\widetilde{X})}\widetilde{Y} + \widetilde{A} \circ \mathbf{j}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] - D_{\mathbf{i}\widetilde{A}(\widetilde{X})}\widetilde{Y}$$
$$+ D_{\mathbf{i}\widetilde{Y}}\widetilde{A}(X) = \mathbf{P}^1(\widetilde{X}, \widetilde{Y}) + (D_{\mathbf{i}\widetilde{Y}}\widetilde{A})(\widetilde{X}),$$

therefore $\widetilde{A}$ has to satisfy the partial differential equation

$$(D_{\mathbf{i}\widetilde{Y}}\widetilde{A})(\widetilde{X}) = -\mathbf{P}^1(\widetilde{X}, \widetilde{Y}).$$

Relation (i) is just the condition of complete integrability of this equation, which concludes the proof of the theorem.

2.49. ***h*-basic covariant derivatives in $\tau^*\tau$.**

> *We confirm our assumption that a horizontal map $\mathcal{H}$ is specified for τ. ∇ means the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$.*

Lemma 1. *If D is a covariant derivative operator in $\tau^*\tau$, then the map $\widetilde{D}$ defined by*

$$\widetilde{D}_\xi \widetilde{Y} := D_{\mathbf{h}\xi}\widetilde{Y} + \nabla_{\mathbf{v}\xi}\widetilde{Y}; \quad \xi \in \mathfrak{X}(TM), \ \widetilde{Y} \in \mathfrak{X}(\tau),$$

is also a covariant derivative operator in $\tau^*\tau$. For the Berwald curvature $\widetilde{\mathbf{P}}$ of $\widetilde{D}$ we have

$$\widetilde{\mathbf{P}}(\widehat{X},\widehat{Y})\widehat{Z} = -\mathbf{j}[Y^{\mathrm{v}}, \mathcal{H}D_{X^h}\widehat{Y}] \quad \textit{for all } X, Y, Z \in \mathfrak{X}(M).$$

Proof. $\widetilde{D}$ is obviously a covariant derivative operator in $\tau^*\tau$ (cf. 2.44, Example 1). Taking into account that $[X^h, Y^{\mathrm{v}}]$ is a vertical vector field and using the rules of calculation for ∇, we obtain:

$$\begin{aligned}\widetilde{\mathbf{P}}(\widehat{X},\widehat{Y})\widehat{Z} &= \widetilde{D}_{X^h}\widetilde{D}_{Y^{\mathrm{v}}}\widehat{Z} - \widetilde{D}_{Y^{\mathrm{v}}}\widetilde{D}_{X^h}\widehat{Z} - \widetilde{D}_{[X^h,Y^{\mathrm{v}}]}\widehat{Z} \\ &= -\widetilde{D}_{Y^{\mathrm{v}}}D_{X^h}\widehat{Z} = -\mathbf{j}[Y^{\mathrm{v}}, \mathcal{H}D_{X^h}\widehat{Z}].\end{aligned}$$

□

Definition. A covariant derivative operator D in $\tau^*\tau$ is said to be *h-basic* if there is a covariant derivative $\overset{\circ}{D}$ on M such that

$$D_{X^h}\widehat{Y} = \widehat{\overset{\circ}{D}_X Y} \quad \text{for all } X, Y, \in \mathfrak{X}(M).$$

Then $\overset{\circ}{D}$ is called the *base covariant derivative* belonging to D.

Lemma 2. *A covariant derivative operator D in $\tau^*\tau$ is h-basic if, and only if, the covariant derivative $\widetilde{D}$ defined in Lemma 1 has vanishing Berwald curvature.*

Proof. (a) *Necessity.* If D is h-basic with the base covariant derivative operator $\overset{\circ}{D}$, then by Lemma 1 for any vector fields X, Y, Z on M we have

$$\widetilde{\mathbf{P}}(\widehat{X},\widehat{Y})\widehat{Z} = -\mathbf{j}[Y^{\mathrm{v}}, \mathcal{H}\widehat{\overset{\circ}{D}_X Y}] = -\mathbf{j}\big[Y^{\mathrm{v}}, (\overset{\circ}{D}_X Y)^h\big] = 0,$$

since the Lie bracket gives a vertical vector field.

(b) *Sufficiency.* If $\widetilde{\mathbf{P}} = 0$, then, according to Lemma 1, $\mathbf{j}[Y^{\mathrm{v}}, \mathcal{H}D_{X^h}\widetilde{Z}] = 0$, or equivalently,

$$J[Y^{\mathrm{v}}, \mathcal{H}D_{X^h}\widehat{Z}] = 0 \quad \text{for all } X, Y, Z \in \mathfrak{X}(M).$$

Applying 2.31, **3** we have

$$0 = [J, Y^{\mathrm{v}}]\mathcal{H}D_{X^h}\widehat{Z} = [\mathbf{i}D_{X^h}\widehat{Z}, Y^{\mathrm{v}}] - J[\mathcal{H}D_{X^h}\widehat{Z}, Y^{\mathrm{v}}],$$

therefore

$$[\mathbf{i}D_{X^h}\widehat{Z}, Y^{\mathrm{v}}] = 0 \quad \text{for all } Y \in \mathfrak{X}(M).$$

In virtue of 2.31, Lemma (ii), from this it follows that

$$\mathbf{i}D_{X^h}\widehat{Z} \quad \textit{is a vertical vector field for all } X, Z \in \mathfrak{X}(M).$$

Thus there is a possibility to define a covariant derivative operator $\overset{\circ}{D}$ on M by the rule

$$(X, Z) \mapsto \overset{\circ}{D}_X Z, \quad \widehat{\overset{\circ}{D}_X Z} := D_{X^h}\widehat{Z} \quad \text{for all } X, Z \in \mathfrak{X}(M).$$

This proves that D is an h-basic covariant derivative operator with the base covariant derivative $\overset{\circ}{D}$. □

Example. Consider the Berwald derivative ∇ induced by $\mathcal{H}$ in $\tau^*\tau$.

(**1**) ∇ *is h-basic if, and only if, there is covariant derivative operator* $\overset{\circ}{D}$ *on* M *such that*

$$\widehat{\overset{\circ}{D}_X Y} = \mathcal{V}[X^h, Y^{\mathrm{v}}], \ \textit{or equivalenty}, \ (\overset{\circ}{D}_X Y)^{\mathrm{v}} = [X^h, Y^{\mathrm{v}}] \ \textit{for all } X, Y \in \mathfrak{X}(M).$$

Indeed, this is obvious from the rule of calculation $\nabla_{X^h}\widehat{Y} = \mathcal{V}[X^h, Y^{\mathrm{v}}]$. Now Lemma 2 implies immediately that ∇ *is h-basic if, and only if, the Berwald curvature* $\overset{\circ}{\mathbf{P}}$ *of* ∇ *vanishes.*

(**2**) Suppose that ∇ is h-basic with the base covariant derivative $\overset{\circ}{D}$. We express the Riemann curvature $\overset{\circ}{\mathbf{R}}$ and the h-horizontal torsion $\overset{\circ}{\mathcal{T}}$ of ∇ in terms of the curvature tensor field $R^{\overset{\circ}{D}}$ and the torsion tensor field $T^{\overset{\circ}{D}}$ of $\overset{\circ}{D}$. Let X, Y, Z be vector fields on M. Then

$$\begin{aligned}
&\mathbf{i}\,\overset{\circ}{\mathbf{R}}(\widehat{X}, \widehat{Y})\widehat{Z} = \mathbf{i}\,R^{\nabla}(X^h, Y^h)\widehat{Z} = \mathbf{i}\big(\nabla_{X^h}\nabla_{Y^h}\widehat{Z} - \nabla_{Y^h}\nabla_{X^h}\widehat{Z} - \nabla_{[X^h, Y^h]}\widehat{Z}\big) \\
&= \mathbf{i}\big(\nabla_{X^h}\mathcal{V}[Y^h, Z^{\mathrm{v}}] - \nabla_{Y^h}\mathcal{V}[X^h, Z^{\mathrm{v}}] - \nabla_{\mathbf{h}[X^h, Y^h]}\widehat{Z} - \nabla_{\mathbf{v}[X^h, Y^h]}\widehat{Z}\big) \\
&= \mathbf{i}\Big(\nabla_{X^h}\widehat{\overset{\circ}{D}_Y Z} - \nabla_{Y^h}\widehat{\overset{\circ}{D}_X Z} - \nabla_{[X,Y]^h}\widehat{Z}\Big) = \mathbf{v}\big([X^h, (\overset{\circ}{D}_Y Z)^{\mathrm{v}}] - [Y^h, (\overset{\circ}{D}_X Z)^{\mathrm{v}}] \\
&- [[X, Y]^h, Z^{\mathrm{v}}]\big) = \big(\overset{\circ}{D}_X\overset{\circ}{D}_Y Z - \overset{\circ}{D}_Y\overset{\circ}{D}_X Z - \overset{\circ}{D}_{[X,Y]}Z\big)^{\mathrm{v}} = (R^{\overset{\circ}{D}}(X, Y)Z)^{\mathrm{v}},
\end{aligned}$$

hence

$$\overset{\circ}{\mathbf{R}}(\widehat{X}, \widehat{Y})\widehat{Z} = \widehat{R^{\overset{\circ}{D}}(X, Y)Z} \qquad \text{for all} \quad X, Y, Z \in \mathfrak{X}(M).$$

Thus *an h-basic Berwald derivative has vanishing curvature if, and only if, the curvature tensor field of its base covariant derivative operator vanishes.*

According to 2.47, **1** and the Lemma in 2.32, we have

$$\begin{aligned}\overset{\circ}{\mathcal{T}}(\widehat{X},\widehat{Y}) &= \mathcal{V}\mathbf{T}(X^c,Y^c) = \mathcal{V}([X^h,Y^{\mathrm{v}}] - [Y^h,X^{\mathrm{v}}] - [X,Y]^{\mathrm{v}}) \\ &= \mathcal{V}(\overset{\circ}{D}_X Y - \overset{\circ}{D}_Y X - [X,Y])^{\mathrm{v}} = \mathcal{V}\circ\mathbf{i}\,\widehat{T^{\overset{\circ}{D}}(X,Y)} = \widehat{T^{\overset{\circ}{D}}(X,Y)},\end{aligned}$$

i.e.,

$$\overset{\circ}{\mathcal{T}}(\widehat{X},\widehat{Y}) = \mathcal{V}\mathbf{T}(X^c,Y^c) = \widehat{T^{\overset{\circ}{D}}(X,Y)} \quad \text{for all } X,Y\in\mathfrak{X}(M).$$

From this it follows that *for an h-basic Berwald derivative ∇ the following are equivalent:*

(i) ∇ has vanishing h-horizontal torsion.

(ii) The torsion of the given horizontal map vanishes.

(iii) The base covariant derivative belonging to ∇ is torsion-free.

(**3**) Next we show that ∇ *is h-basic if, and only if, the tension $\widetilde{\mathbf{t}}$ of $\mathcal{H}$ is basic.*

Necessity. Suppose that ∇ is h-basic, i.e. there is a covariant derivative operator $\overset{\circ}{D}$ on M such that

$$[X^h,Y^{\mathrm{v}}] = (\overset{\circ}{D}_X Y)^{\mathrm{v}} \quad \text{for all } X,Y\in\mathfrak{X}(M).$$

Then the Lie brackets $[X^h,Y^{\mathrm{v}}]$ $(X,Y\in\mathfrak{X}(M))$ are vertical lifts, hence, according to the Proposition in 2.6, are homogeneous of degree 0:

$$[C,[X^h,Y^{\mathrm{v}}]] = -[X^h,Y^{\mathrm{v}}] \quad \text{for all } X,Y\in\mathfrak{X}(M).$$

Thus, using the Jacobi identity, we obtain:

$$\begin{aligned}0 &= [X^h,[Y^{\mathrm{v}},C]] + [Y^{\mathrm{v}},[C,X^h]] + [C,[X^h,Y^{\mathrm{v}}]] \\ &= [X^h,Y^{\mathrm{v}}] + [Y^{\mathrm{v}},[C,X^h]] - [X^h,Y^{\mathrm{v}}] = [Y^{\mathrm{v}},[X^h,C]].\end{aligned}$$

This implies by 2.31, Lemma (ii) that $[X^h,C] = \mathbf{i}\,\widetilde{\mathbf{t}}(\widetilde{X})$ is a vertical lift. Therefore there is a type $(1,1)$ tensor field A on M such that $\widetilde{\mathbf{t}}(\widehat{X}) = \widehat{A(X)}$ for all $X\in\mathfrak{X}(M)$, proving that $\widetilde{\mathbf{t}}$ is basic.

Sufficiency. Suppose that there is a $(1,1)$ tensor field A on M satisfying

$$\widehat{A(X)} = \widetilde{\mathbf{t}}(\widehat{X}) = \mathcal{V}[X^h,C] \quad \text{for all } X\in\mathfrak{X}(M).$$

Then, again by the Jacobi identity,

$$0 = [X^{\mathrm{v}}, [Y^h, C]] + [Y^h, [C, X^{\mathrm{v}}]] + [C, [X^{\mathrm{v}}, Y^h]] = [X^{\mathrm{v}}, (A(Y))^{\mathrm{v}}] \\ - [Y^h, X^{\mathrm{v}}] + [C, [X^{\mathrm{v}}, Y^h]] = [X^{\mathrm{v}}, Y^h] + [C, [X^{\mathrm{v}}, Y^h]],$$

hence

$$[C, [X^{\mathrm{v}}, Y^h] = -[X^{\mathrm{v}}, Y^h] \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

By the Proposition in 2.6 from this it follows that

$$[X^{\mathrm{v}}, Y^h] \quad \textit{is a vertical lift for all } X, Y \in \mathfrak{X}(M).$$

Now, as in the proof of Lemma 2, we conclude that there is a (unique) covariant derivative operator $\overset{\circ}{D}$ on M such that

$$\nabla_{X^h} \widehat{Y} = \mathcal{V}[X^h, Y^{\mathrm{v}}] = \widehat{\overset{\circ}{D}_X Y} \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

therefore ∇ is h-basic. This concludes the proof of our statement.

Lemma 3. *Suppose that D is and h-basic covariant derivative operator in $\tau^*\tau$ with the base covariant derivative $\overset{\circ}{D}$. Let h_0 be the horizontal projector arising from $\overset{\circ}{D}$ according to 2.15, Example 3. Then we have*

$$\mathbf{i} D_{X^h} \delta = X^h - X^{h_0} \quad \textit{for all } X \in \mathfrak{X}(M);$$

therefore h_0 coincides with h if, and only if, the h-deflection $D^h\delta$ of D vanishes.

Proof. Notice first that in view of 2.16, Example 3 and 2.33, Proposition 3 we have

$$[X^{h_0}, C] = 0, \ (\overset{\circ}{D}_X Y)^{\mathrm{v}} = [X^{h_0}, Y^{\mathrm{v}}] \quad \textit{for all } X, Y \in \mathfrak{X}(M).$$

Keeping these in mind, we apply a local argument. Let $(\mathcal{U}, (u^i)_{i=1}^n)$ be a chart on M and $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ the induced chart in TM. Then the coordinate expression of δ is $y^i \widehat{\frac{\partial}{\partial u^i}}$, and over $\tau^{-1}(\mathcal{U})$ we get:

$$\begin{aligned} \mathbf{i}(D_{X^h}\delta) &= \mathbf{i} D_{X^h}\left(y^i \widehat{\tfrac{\partial}{\partial u^i}}\right) = \mathbf{i}\Big[(X^h y^i)\widehat{\tfrac{\partial}{\partial u^i}} + y^i \widehat{\overset{\circ}{D}_X \tfrac{\partial}{\partial u^i}}\Big] \\ &= (X^h y^i)\tfrac{\partial}{\partial y^i} + y^i \big(\overset{\circ}{D}_X \tfrac{\partial}{\partial u^i}\big)^{\mathrm{v}} = (X^h y^i)\tfrac{\partial}{\partial y^i} + y^i \big[X^{h_0}, \tfrac{\partial}{\partial y^i}\big] \\ &= (X^h y^i)\tfrac{\partial}{\partial y^i} + [X^{h_0}, C] - (X^{h_0} y^i)\tfrac{\partial}{\partial y^i} = X^h - X^{h_0}. \end{aligned}$$

□

Remark. Suppose that D is a covariant derivative oprator in $\tau^*\tau$ with vanishing h-deflection, i.e. $D^h\delta = 0$. In view of Lemma 3 in order that D be h-basic it is *necessary* that $\mathcal{H}$ be homogeneous and smooth on its *whole* domain of definition $TM \times_M TM$. In particular, if $D = \nabla$ is the Berwald derivative induced by $\mathcal{H}$, then the only condition is the smoothness of $\mathcal{H}$ on $TM \times_M TM$, since $\nabla^h\delta = \widetilde{\mathbf{t}} =$ *the tension of* $\mathcal{H}$ (see 2.42, Example 4).

Proposition 1. *Suppose that $\mathcal{H}$ is a homogeneous horizontal map and let D be an h-basic covariant derivative operator in $\tau^*\tau$. Then*

$$D^h\delta = 0 \quad \textit{if, and only if,} \; \mathbf{P}^1 = 0.$$

Proof. Let $\overset{\circ}{D}$ be the base covariant derivative belonging to D, and let us denote by h_0 the horizontal projector arising from $\overset{\circ}{D}$. According to 2.45, Lemma 1, for any vector fields X, Y on M we have

$$\begin{aligned}\mathbf{i}\,\mathbf{P}^1(X^h, Y^h) &= \mathbf{i}D_{X^h}\widehat{Y} - [X^h, Y^{\mathrm{v}}] = (\overset{\circ}{D}_X Y)^{\mathrm{v}} - [X^h, Y^{\mathrm{v}}] \\ &= [X^{h_0}, Y^{\mathrm{v}}] - [X^h, Y^{\mathrm{v}}],\end{aligned}$$

therefore

$$\mathbf{P}^1 = 0 \iff [X^{h_0}, Y^{\mathrm{v}}] = [X^h, Y^{\mathrm{v}}] \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

Next we show that the last property is equivalent to the coincidence of h_0 and h; this, in virtue of Lemma 3, yields the Proposition.

Since $X^h - X^{h_0}$ is obviously a vertical vector field, the property

$$[X^h - X^{h_0}, Y^{\mathrm{v}}] = 0 \quad \textit{for all } Y \in \mathfrak{X}(M)$$

implies by 2.31, Lemma (ii) that $X^h - X^{h_0}$ is a vertical lift. Hence, in view of 2.31, Corollary 2, $[J, X^h - X^{h_0}] = 0$. Choose a vector field ξ on TM such that $J\xi = C$. Then it follows that

$$\begin{aligned}0 &= [J, X^h - X^{h_0}]\xi = [C, X^h - X^{h_0}] - J[\xi, X^h - X^{h_0}] \\ &= -J[\xi, X^h - X^{h_0}] \overset{2.31,\,\mathbf{5}}{=} X^{h_0} - X^h\end{aligned}$$

(since h and h_0 are homogeneous). Thus $X^h = X^{h_0}$ for all $X \in \mathfrak{X}(M)$, hence $h = h_0$. This concludes the proof.

Proposition 2. *Suppose that the horizontal map $\mathcal{H}$ is everywhere continuous on its domain. Let D be an h-basic covariant derivative operator with the base covariant derivative $\overset{\circ}{D}$. Then*

$$D^h\delta = \widetilde{\mathbf{t}} \quad \textit{if, and only if, } \ \mathbf{P}^1 = 0.$$

Proof. (a) *Necessity.* Let $X \in \mathfrak{X}(M)$. By the condition, $\mathbf{i}D_{X^h}\delta = [X^h, C]$ (see also 2.42, Example 4). On the other hand, according to Lemma 3, $\mathbf{i}D_{X^h}\delta = X^h - X^{h_0}$. Thus, by the homogeneity of $\mathbf{h}_0$, we obtain

$$[C, X^h - X^{h_0}] = [C, X^h] = -(X^h - X^{h_0}),$$

i.e., $X^h - X^{h_0}$ is homogeneous of degree 0. Locally this means that the vertical vector field $X^h - X^{h_0}$ has positive-homogeneous component functions of degree zero. Since these functions are continuous on their domain of definition and smooth outside the zero section, it follows from 2.6 Lemma 2(1) that they are vertical lifts. (More directly, we may also refer to 2.6, Proposition (1).) Hence for any vector field Y on M we have

$$0 = [X^h - X^{h_0}, Y^{\mathrm{v}}] = [X^h, Y^{\mathrm{v}}] - [X^{h_0}, Y^{\mathrm{v}}],$$

therefore

$$\mathbf{i}\,\mathbf{P}^1(X^h, Y^h) = (\overset{\circ}{D}_X Y)^{\mathrm{v}} - [X^h, Y^{\mathrm{v}}] = [X^{h_0}, Y^{\mathrm{v}}] - [X^h, Y^{\mathrm{v}}] = 0.$$

Thus $\mathbf{P}^1 = 0$, as we claimed.

(b) *Sufficiency.* If $\mathbf{P}^1 = 0$, then, according to 2.45, Lemma 1,

$$D_{X^h}\widetilde{Y} = \mathcal{V}[X^h, \mathbf{i}\widetilde{Y}] \quad \text{for all } X \in \mathfrak{X}(M),\ \widetilde{Y} \in \mathfrak{X}(\tau).$$

In particular,

$$D_{X^h}\delta = \mathcal{V}[X^h, C] = \widetilde{\mathbf{t}}(\widehat{X}) \quad \text{for all } X \in \mathfrak{X}(M),$$

whence $D^h\delta = \widetilde{\mathbf{t}}$. This concludes the proof of the Proposition.

Comment. The concept of h-basic covariant derivatives (under the name 'linear Finsler connections') was introduced by M. Hashiguchi [39]. The importance of these covariant derivatives lies in the fact that a large class of special Finsler manifolds may be efficiently investigated with the help of an appropriate h-basic covariant derivative operator; see Y. Ichijyō's important papers [40], [41] and the papers of Sz. Szakál and J. Szilasi [73], [74]. In the next chapter we shall also present some application.

2.50. Hessian, gradients, Cartan tensors.

Definition 1. Let D be a covariant derivative operator in $\tau^*\tau$. The *Hessian* of a smooth function F on TM is the second v-covariant differential

$$H^D F := D^{\mathrm{v}} D^{\mathrm{v}} F,$$

where D^{v} is the v-covariant derivative induced by D.

Remark 1. For any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have

$$\begin{aligned} H^D F(\widetilde{X}, \widetilde{Y}) &= [D^{\mathrm{v}}(D^{\mathrm{v}} F)](\widetilde{X}, \widetilde{Y}) = D_{\mathbf{i}\widetilde{X}}(D^{\mathrm{v}} F(\widetilde{Y})) - D^{\mathrm{v}} F(D_{\mathbf{i}\widetilde{X}}\widetilde{Y}) \\ &= (\mathbf{i}\widetilde{X})(\mathbf{i}\widetilde{Y}(F)) - (\mathbf{i}D_{\mathbf{i}\widetilde{X}}\widetilde{Y})F. \end{aligned}$$

Let, in particular, ∇ be the Berwald derivative induced by a horizontal map in $\tau^*\tau$. Then, for any vector fields X, Y on M,

$$H^\nabla F(\widehat{X}, \widehat{Y}) = X^{\mathrm{v}}(Y^{\mathrm{v}} F) - (\mathbf{i}\nabla_{X^{\mathrm{v}}}\widehat{Y})F = X^{\mathrm{v}}(Y^{\mathrm{v}} F).$$

If $(\mathcal{U}, (u^i)_{i=1}^n)$ is a chart on M, and $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ is the induced chart on TM, then we have the following coordinate expression for $H^\nabla F$:

$$H^\nabla F\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \frac{\partial^2 F}{\partial y^i \partial y^j}, \quad 1 \leqq i,\ j \leqq n.$$

Lemma 1. *Let D be a covariant derivative operator in $\tau^*\tau$.*

(1) *If the v-vertical torsion $\mathbf{Q}^1$ of D vanishes, then the Hessian $H^D F$ of any smooth function F on TM is a symmetric type $(0,2)$ tensor field along τ.*

(2) *If D has vanishing v-vertical torsion $\mathbf{Q}^1$ and vertical curvature $\mathbf{Q}$, then $D^{\mathrm{v}} H^D F$ is a totally symmetric type $(0,3)$ tensor field along τ for all $F \in C^\infty(TM)$.*

Proof. Let $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ be vector fields along τ.

(1) $H^D F(\widetilde{X}, \widetilde{Y}) - H^D F(\widetilde{Y}, \widetilde{X}) = ([\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}] - \mathbf{i}(D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X}))F$ $= -\mathbf{i}\mathbf{Q}^1(\widetilde{X}, \widetilde{Y})F = 0$, so $H^D F$ is indeed symmetric.

(2) $D^{\mathrm{v}}(H^D F)(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = (D_{\mathbf{i}\widetilde{X}} H^D F)(\widetilde{Y}, \widetilde{Z}) = \mathbf{i}\widetilde{X}(H^D F(\widetilde{Y}, \widetilde{Z}))$ $-H^D F(D_{\mathbf{i}\widetilde{X}}\widetilde{Y}, \widetilde{Z}) - H^D F(\widetilde{Y}, D_{\mathbf{i}\widetilde{X}}\widetilde{Z}) = \mathbf{i}\widetilde{X}(\mathbf{i}\widetilde{Y}(\mathbf{i}\widetilde{Z})F) - \mathbf{i}\widetilde{X}(\mathbf{i}D_{\mathbf{i}\widetilde{Y}}\widetilde{Z})F$ $-(\mathbf{i}D_{\mathbf{i}\widetilde{X}}\widetilde{Y})(\mathbf{i}\widetilde{Z})F + (\mathbf{i}D_{\mathbf{i}D_{\mathbf{i}\widetilde{X}}\widetilde{Y}}\widetilde{Z})F - (\mathbf{i}\widetilde{Y})(\mathbf{i}D_{\mathbf{i}\widetilde{X}}\widetilde{Z})F + (\mathbf{i}D_{\mathbf{i}\widetilde{Y}}D_{\mathbf{i}\widetilde{X}}\widetilde{Z})F.$

Interchanging the variables $\widetilde{X}$, $\widetilde{Y}$ and subtract, we obtain:

$$D^{\mathrm{v}}(H^D F)(\widetilde{X},\widetilde{Y},\widetilde{Z}) - D^{\mathrm{v}}(H^D F)(\widetilde{Y},\widetilde{X},\widetilde{Z}) = [\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}](\mathbf{i}\widetilde{Z})F$$
$$- \mathbf{i}(D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X})(\mathbf{i}\widetilde{Z})F + \mathbf{i}(D_{\mathbf{i}(D_{\mathbf{i}\widetilde{X}}\widetilde{Y}-D_{\mathbf{i}\widetilde{Y}}\widetilde{X})}\widetilde{Z})F$$
$$- (\mathbf{i}(D_{\mathbf{i}\widetilde{X}}D_{\mathbf{i}\widetilde{Y}} - D_{\mathbf{i}\widetilde{Y}}D_{\mathbf{i}\widetilde{X}})\widetilde{Z})F = -\mathbf{i}\mathbf{Q}^1(\widetilde{X},\widetilde{Y})(\mathbf{i}\widetilde{Z})F - (\mathbf{i}\mathbf{Q}(\widetilde{X},\widetilde{Y})\widetilde{Z})F = 0.$$

Thus $D^{\mathrm{v}}(H^D F)$ is symmetric in its first two variables. Since $H^D F$ is symmetric, the symmetry of $D^{\mathrm{v}}(H^D F)$ in the second two variables is obvious. This concludes the proof.

In the rest of this subsection g will be a pseudo-Riemannian metric in $\tau^*\tau$, i.e., a Finsler metric on M (cf. 2.23). We continue to assume that a horizontal map $\mathcal{H}$ is specified for τ.

Lemma 2. *Let F be a smooth function on TM.*

(1) *Consider the Sasaki lift $g^S \in \mathcal{T}^0_2(TM)$ of g. There is a unique vector field* $\operatorname{grad} F$ *on TM such that*

$$g^S(\operatorname{grad} F, \xi) = (dF)(\xi) \quad \textit{for all } \xi \in \mathfrak{X}(TM).$$

(2) *There is a unique vector field* $\operatorname{grad}_{\mathrm{v}} F$ *along τ such that*

$$g(\operatorname{grad}_{\mathrm{v}} F, \widetilde{X}) = (d^{\mathrm{v}} F)(\widetilde{X}) \quad \textit{for all } \widetilde{X} \in \mathfrak{X}(\tau).$$

(3) *There is a unique vector field* $\operatorname{grad}_h F$ *along τ such that*

$$g(\operatorname{grad}_h F, \widetilde{X}) = (d^h F)(\widetilde{X}) \quad \textit{for all } \widetilde{X} \in \mathfrak{X}(\tau).$$

Proof. All assertions are obivious consequences of the non-degeneracy of g^S and g (cf. 1.30(5)). □

The vector field $\operatorname{grad} F$ on TM is called the *gradient* of F, while the vector fields $\operatorname{grad}_{\mathrm{v}} F$ and $\operatorname{grad}_h F$ along τ are said to be the *v-gradient* and the *h-gradient* of F, respectively.

Lemma 3. *Let F be a smooth function on TM. Then*

$$\operatorname{grad} F = \mathbf{i}\operatorname{grad}_{\mathrm{v}} F + \mathcal{H}\operatorname{grad}_h F.$$

Proof. For any vector field $\widetilde{X}$ along τ we have

$$g^S(\mathbf{v}\,\mathrm{grad}\,F, \mathbf{i}\widetilde{X}) \overset{2.23}{:=} g(\mathcal{V}\,\mathbf{v}\,\mathrm{grad}\,F, \mathcal{V}\,\mathbf{i}\widetilde{X}) + g(\mathbf{j}\,\mathbf{v}\,\mathrm{grad}\,F, \mathbf{j}\,\mathbf{i}\widetilde{X})$$
$$= g(\mathcal{V}\,\mathrm{grad}\,F, \widetilde{X}).$$

On the other hand, taking into account 2.23, Lemma,

$$g^S(\mathbf{v}\,\mathrm{grad}\,F, \mathbf{i}\widetilde{X}) = g^S(\mathrm{grad}\,F, \mathbf{i}\widetilde{X}) := dF(\mathbf{i}\widetilde{X}) = (d^{\mathrm{v}}F)(\widetilde{X});$$

therefore $g(\mathcal{V}\,\mathrm{grad}\,F, \widetilde{X}) = (d^{\mathrm{v}}F)(\widetilde{X})$. From these it follows that

$$\mathcal{V}\,\mathrm{grad}\,F = \mathrm{grad}_{\mathrm{v}}\,F, \quad \text{or equivalently,} \quad \mathbf{v}\,\mathrm{grad}\,F = \mathbf{i}\,\mathrm{grad}_{\mathrm{v}}\,F.$$

We obtain by a similar argument that $\mathbf{h}\,\mathrm{grad}\,F = \mathcal{H}\,\mathrm{grad}_h\,F$. This concludes the proof.

Remark 2. Let $f \in C^\infty(M)$. Then $\mathrm{grad}_{\mathrm{v}}\,f^{\mathrm{v}}$ is obviously zero, so $\mathrm{grad}\,f^{\mathrm{v}} = \mathcal{H}\,\mathrm{grad}_h\,f^{\mathrm{v}}$, i.e. *the gradient of a vertically lifted function is a horizontal vector field on TM.*

Lemma 3. *Let D be a covariant derivative operator in $\tau^*\tau$. Suppose that D has vanishing v-vertical torsion and that D is v-metrical, i.e., $D^{\mathrm{v}}g = 0$. Then*

$$H^D F(\widetilde{X}, \widetilde{Y}) = g(D_{\mathbf{i}\widetilde{X}}\,\mathrm{grad}_{\mathrm{v}}\,F, \widetilde{Y}) \quad \textit{for all } \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\tau).$$

Proof. As we have learnt,

$$H^D F(\widetilde{X}, \widetilde{Y}) = \mathbf{i}\widetilde{X}((\mathbf{i}\widetilde{Y})F) - (\mathbf{i}D_{\mathbf{i}\widetilde{X}}\widetilde{Y})F$$
$$= \mathbf{i}\widetilde{X}g(\mathrm{grad}_{\mathrm{v}}\,F, \widetilde{Y}) - g(\mathrm{grad}_{\mathrm{v}}\,F, D_{\mathbf{i}\widetilde{X}}Y).$$

Since D is v-metrical,

$$\mathbf{i}\widetilde{X}g(\mathrm{grad}_{\mathrm{v}}\,F, \widetilde{Y}) = g(D_{\mathbf{i}\widetilde{X}}\,\mathrm{grad}_{\mathrm{v}}\,F, \widetilde{Y}) + g(\mathrm{grad}_{\mathrm{v}}\,F, D_{\mathbf{i}\widetilde{X}}\widetilde{Y}),$$

whence the desired formula. □

Note. This result is a strict analogue of Lemma 49 in [61], Chapter 3. We shall see in the next subsection that v-metrical covariant derivatives with vanishing v-vertical torsion do exist in $\tau^*\tau$.

Definition 2. The *first Cartan tensor* of the Finsler metric g is the type $(1,2)$ tensor field $\mathcal{C}$ along τ given by

$$g(\mathcal{C}(\widetilde{X},\widetilde{Y}),\widetilde{Z}) = (\nabla^{\mathrm{v}}g)(\widetilde{X},\widetilde{Y},\widetilde{Z}) \quad \text{for all } \widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathfrak{X}(\tau),$$

where ∇^{v} is the canonical v-covariant derivative in $\tau^*\tau$. The lowered tensor $\mathcal{C}_\flat = \nabla^{\mathrm{v}}g$ is also called the first Cartan tensor of g.

Remark 3. (a) Since ∇^{v} is an 'intrinsic' operator in the calculus along τ, the first Cartan tensor introduced here depends only on g, contrary to the Cartan tensor defined in 1.46(b).

(b) *The lowered first Cartan tensor is symmetric in its last two variables.* Indeed, for any vector fields $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ along τ we have

$$\begin{aligned}\mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z}) &= (\nabla^{\mathrm{v}}g)(\widetilde{X},\widetilde{Y},\widetilde{Z}) = (\nabla_{\mathbf{i}\widetilde{X}}g)(\widetilde{Y},\widetilde{Z})\\ &= \mathbf{i}\widetilde{X}(g(\widetilde{Y},\widetilde{Z})) - g(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y},\widetilde{Z}) - g(\widetilde{Y},\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Z}) = \mathcal{C}_\flat(\widetilde{X},\widetilde{Z},\widetilde{Y}),\end{aligned}$$

since g is symmetric.

(c) For any vector fields X, Y, Z on M, we have

$$\mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) = X^{\mathrm{v}}g(\widehat{Y},\widehat{Z}).$$

If $(\mathcal{U},(u^i)_{i=n}^n)$ is a chart on M, and $(\tau^{-1}(\mathcal{U}),((x^i)_{i=1}^n,(y^i)_{i=1}^n))$ is the induced chart on TM, then we obtain the following coordinate expressions for $\mathcal{C}_\flat$ and $\mathcal{C}$:

$$\mathcal{C}_\flat\left(\frac{\widehat{\partial}}{\partial u^i},\frac{\widehat{\partial}}{\partial u^j},\frac{\widehat{\partial}}{\partial u^k}\right) = \frac{\partial g_{jk}}{\partial y^i}, \quad g_{jk} := g\left(\frac{\widehat{\partial}}{\partial u^j},\frac{\widehat{\partial}}{\partial u^k}\right);$$

$$\text{if} \quad \mathcal{C}\left(\frac{\widehat{\partial}}{\partial u^i},\frac{\widehat{\partial}}{\partial u^j}\right) = \mathcal{C}^\ell_{ij}\,\frac{\widehat{\partial}}{\partial u^\ell}, \quad \text{then} \quad \mathcal{C}^\ell_{ij} = g^{\ell k}\,\frac{\partial g_{jk}}{\partial y^i};\ 1 \leqq i,j,k \leqq n$$

((g^{ij}) is the inverse of (g_{ij}); summation convention is used).

Lemma 4. *The first Cartan tensor of a Finsler metric g vanishes if, and only if, g reduces to a pseudo-Riemannian metric on M (in the sense of 2.23, Example).*

Proof. (a) Suppose that $g = \widehat{g}_M$, where g_M is a pseudo-Riemannian metric on M. Then for any vector fields X, Y, Z on M we have

$$(\nabla^{\mathrm{v}}\widehat{g}_M)(\widehat{X},\widehat{Y},\widehat{Z}) = X^{\mathrm{v}}g_M(\widehat{Y},\widehat{Z}) = X^{\mathrm{v}}(g_M(Y,Z))^{\mathrm{v}} = 0,$$

thus $\mathcal{C}_\flat := \nabla^{\mathrm{v}}g = \nabla^{\mathrm{v}}\widehat{g}_M = 0$.

(b) Conversely, the vanishing of $\mathcal{C}_\flat$ implies immediately that g is basic, since the components of g are vertical lifts. $\square$

Definition 3. A Finsler metric g is said to be *variational* if there is a smooth function F on TM such that $g = H^{\nabla}F$.

Corollary. *The first Cartan tensor of a variational Finsler metric is symmetric, the lowered first Cartan tensor is totally symmetric.*

This is an immediate consequence of Lemma 1.

Remark 4. If $g = H^D F$ ($F \in C^\infty(TM)$), then for any vector fields X, Y, Z on M we have

$$\mathcal{C}_\flat(\widehat{X}, \widehat{Y}, \widehat{Z}) := (\nabla^{\mathrm{v}}(H^{\nabla}F))(\widehat{X}, \widehat{Y}, \widehat{Z}) = (\nabla_{X^{\mathrm{v}}} H^{\nabla}F)(\widehat{Y}, \widehat{Z})$$
$$= X^{\mathrm{v}}((H^{\nabla}F)(\widehat{Y}, \widehat{Z})) = X^{\mathrm{v}}(Y^{\mathrm{v}}(Z^{\mathrm{v}}F)).$$

In local coordinates (cf. Remark 3, (c)),

$$\mathcal{C}_\flat\left(\frac{\widehat{\partial}}{\partial u^i}, \frac{\widehat{\partial}}{\partial u^j}, \frac{\widehat{\partial}}{\partial u^k}\right) = \frac{\partial^3 F}{\partial y^i \partial y^j \partial y^k}, \quad 1 \leqq i, j, k \leqq n.$$

Definition 4. The *second Cartan tensor* of the Finsler metric g with respect to the horizontal map $\mathcal{H}$ is the type $(1, 2)$ tensor field $\mathcal{C}^h$ along τ given by

$$g(\mathcal{C}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) = (\nabla^h g)(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) \quad \text{for all } \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\tau),$$

where ∇ is the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$. The lowered tensor $\mathcal{C}^h_\flat = \nabla^h g$ will also be mentioned as the second Cartan tensor of g.

Remark 5. It may be checked immediately that *the lowered second Cartan tensor of a Finsler metric is symmetric in its last two variables.*

Lemma 5. *Let g be a variational Finsler metric, namely $g = H^{\nabla}F$, $F \in C^\infty(TM)$. Then the second Cartan tensor of g operates as follows:*

$$\mathcal{C}^h_\flat(\widehat{X}, \widehat{Y}, \widehat{Z}) = -(\mathbf{i}\overset{\circ}{\mathbf{P}}(\widehat{X}, \widehat{Y})\widehat{Z})F + Z^{\mathrm{v}}(Y^{\mathrm{v}}(X^h F)) \quad \textit{for all } X, Y, Z \in \mathfrak{X}(M).$$

Proof. Taking into account 2.44, Example 1, we get:

$$\mathcal{C}^h_\flat(\widehat{X}, \widehat{Y}, \widehat{Z}) := (\nabla^h g)(\widehat{X}, \widehat{Y}, \widehat{Z}) = (\nabla_{X^h} g)(\widehat{Y}, \widehat{Z}) = X^h(g(\widehat{Y}, \widehat{Z})) - g(\nabla_{X^h}\widehat{Y}, \widehat{Z})$$
$$- g(\widehat{Y}, \nabla_{X^h}\widehat{Z}) = X^h(Y^{\mathrm{v}}(Z^{\mathrm{v}}F)) - g(\mathcal{V}[X^h, Y^{\mathrm{v}}], \widehat{Z}) - g(\widehat{Y}, \mathcal{V}[X^h, Z^{\mathrm{v}}])$$
$$= X^h(Y^{\mathrm{v}}(Z^{\mathrm{v}}F)) - [X^h, Y^{\mathrm{v}}](Z^{\mathrm{v}}F) + (\mathbf{i}\nabla_{[X^h, Y^{\mathrm{v}}]}\widehat{Z})F - Y^{\mathrm{v}}([X^h, Z^{\mathrm{v}}]F)$$
$$+ (\mathbf{i}D_{Y^{\mathrm{v}}}\mathcal{V}[X^h, Z^{\mathrm{v}}])F = Y^{\mathrm{v}}(Z^{\mathrm{v}}(X^h F)) + J[Y^{\mathrm{v}}, \mathcal{H} \circ \mathcal{V}[X^h, Z^{\mathrm{v}}]]F.$$

Since $[Y^{\mathrm{v}}, Z^{\mathrm{v}}] = 0$, the first term in the right hand side equals to $Z^{\mathrm{v}}(Y^{\mathrm{v}}(X^h F))$. In the second term

$$\mathcal{H} \circ \mathcal{V}[X^h, Z^{\mathrm{v}}] \overset{2.19}{=} (\mathbf{F} + J)[X^h, Z^{\mathrm{v}}] = \mathbf{F}[X^h, Z^{\mathrm{v}}],$$

therefore, applying 2.28, Corollary 2; 2.31, **3** and 2.47, **3**,

$$J[Y^{\mathrm{v}}, \mathcal{H} \circ \mathcal{V}[X^h, Z^{\mathrm{v}}]] = -J[\mathbf{F}[X^h, Z^{\mathrm{v}}], Y^{\mathrm{v}}] = [J, Y^{\mathrm{v}}]\mathbf{F}[X^h, Z^{\mathrm{v}}]$$
$$- [J \circ \mathbf{F}[X^h, Z^{\mathrm{v}}], Y^{\mathrm{v}}] = -[[X^h, Z^{\mathrm{v}}], Y^{\mathrm{v}}] = [[Z^{\mathrm{v}}, Y^{\mathrm{v}}], X^h] + [[Y^{\mathrm{v}}, X^h], Z^{\mathrm{v}}]$$
$$= -\mathbf{i}\mathcal{V}[[X^h, Y^{\mathrm{v}}], Z^{\mathrm{v}}] = -\mathbf{i}\,\overset{\circ}{\mathbf{P}}(\widehat{X}, \widehat{Y})\widehat{Z}.$$

This concludes the proof.

A note on terminology. In a classical Finslerian context the first and the second Cartan tensor were called the *Cartan torsion* and the *Landsberg curvature*, respectively, by Z. Shen in [69].

2.51. A Miron-type metric derivative.

In this subsection (M, g) is a generalized Finsler manifold in the sense of 2.23. We assume that a horizontal map $\mathcal{H}$ is also given on M.

Lemma. *Let $\mathcal{C}$ and $\mathcal{C}^h$ be the first and the second Cartan tensor of g, respectively. If $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are defined by*

$$g(\overset{\circ}{\mathcal{C}}(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) := \mathcal{C}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) + \mathcal{C}_\flat(\widetilde{Y}, \widetilde{Z}, \widetilde{X}) - \mathcal{C}_\flat(\widetilde{Z}, \widetilde{X}, \widetilde{Y})$$

and

$$g(\overset{\circ}{\mathcal{C}}{}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) := \mathcal{C}^h_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) + \mathcal{C}^h_\flat(\widetilde{Y}, \widetilde{Z}, \widetilde{X}) - \mathcal{C}^h_\flat(\widetilde{Z}, \widetilde{X}, \widetilde{Y})$$

$(\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\tau))$, then $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are well-defined symmetric type $(1, 2)$ tensor fields along τ.

Proof. Well-definedness is assured by the non-degeneracy of g, the symmetries are consequences of Remarks 3 and 5 in 2.50. □

Proposition. *Suppose that the horizontal map $\mathcal{H}$ has vanishing torsion and let ∇ be the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$. If $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are the tensors given by the Lemma, then the rules*

$$\boxed{\begin{aligned} D_{\mathbf{i}\widetilde{X}}\widetilde{Y} &:= \nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}(\widetilde{X},\widetilde{Y}) \\ D_{\mathcal{H}\widetilde{X}}\widetilde{Y} &:= \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\widetilde{X},\widetilde{Y}) \end{aligned}}$$

$(\widetilde{X},\widetilde{Y} \in \mathfrak{X}(\tau))$ define a symmetric, metric derivative D in $\tau^*\tau$. More explicitly, for any vector fields ξ on TM and $\widetilde{Y}$ in $\mathfrak{X}(\tau)$ we have

$$\boxed{D_\xi\widetilde{Y} = \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] + \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}] + \frac{1}{2}\big(\overset{\circ}{\mathcal{C}}(\mathcal{V}\xi,\widetilde{Y}) + \overset{\circ}{\mathcal{C}}{}^h(\mathbf{j}\xi,\widetilde{Y})\big)}.$$

The partial torsions of D are

$$\boxed{\begin{gathered} \mathcal{T} = 0, \quad \mathcal{S} = -\frac{1}{2}\overset{\circ}{\mathcal{C}}, \quad (\mathbf{R}^1)_0 = \Omega = -\frac{1}{2}[\mathbf{h},\mathbf{h}], \\ \mathbf{P}^1 = \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h, \quad \mathbf{Q}^1 = 0 \end{gathered}}.$$

Proof. (1) First we chek that D is a *v-metric* covariant derivative operator, i.e. $D^{\mathrm{v}}g = 0$. This is an immediate calculation: for any vector fields X, Y, Z on M we have

$$\begin{aligned} &(D^{\mathrm{v}}g)(\widehat{X},\widehat{Y},\widehat{Z}) = (D_{X^{\mathrm{v}}}g)(\widehat{Y},\widehat{Z}) = X^{\mathrm{v}}(g(\widehat{Y},\widehat{Z})) - g(D_{X^{\mathrm{v}}}\widehat{Y},\widehat{Z}) - g(\widehat{Y},D_{X^{\mathrm{v}}}\widehat{Z}) \\ &= \mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) - \frac{1}{2}g(\overset{\circ}{\mathcal{C}}(\widehat{X},\widehat{Y}),\widehat{Z}) - \frac{1}{2}g(\widehat{Y},\overset{\circ}{\mathcal{C}}(\widehat{X},\widehat{Z})) = \mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) \\ &- \frac{1}{2}g(\overset{\circ}{\mathcal{C}}(\widehat{X},\widehat{Y}),\widehat{Z}) - \frac{1}{2}g(\overset{\circ}{\mathcal{C}}(\widehat{Z},\widehat{X}),\widehat{Y}) = \mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) - \mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) = 0. \end{aligned}$$

(2) We obtain similarly that D *is h-metric*, i.e. $D^hg = 0$. Indeed,

$$\begin{aligned} &(D^hg)(\widehat{X},\widehat{Y},\widehat{Z}) = (\nabla^hg)(\widehat{X},\widehat{Y},\widehat{Z}) - \frac{1}{2}g(\overset{\circ}{\mathcal{C}}{}^h(\widehat{X},\widehat{Y}),\widehat{Z}) - \frac{1}{2}g(\widehat{Y},\overset{\circ}{\mathcal{C}}{}^h(\widehat{X},\widehat{Z})) \\ &= \mathcal{C}^h_\flat(\widehat{X},\widehat{Y},\widehat{Z}) - \frac{1}{2}(\mathcal{C}^h_\flat(\widehat{X},\widehat{Y},\widehat{Z}) + \mathcal{C}^h_\flat(\widehat{Y},\widehat{Z},\widehat{X}) - \mathcal{C}^h_\flat(\widehat{Z},\widehat{X},\widehat{Y})) \\ &- \frac{1}{2}(\mathcal{C}^h_\flat(\widehat{Z},\widehat{X},\widehat{Y}) + \mathcal{C}^h_\flat(\widehat{X},\widehat{Y},\widehat{Z}) - \mathcal{C}^h_\flat(\widehat{Y},\widehat{Z},\widehat{X})) = 0. \end{aligned}$$

(3) Now we calculate the (h-horizontal) torsion $\mathcal{T}$ of D. According to 2.45, Lemma 1, for any vector fields $\widetilde{X}$, $\widetilde{Y}$ along τ we have

$$\mathcal{T}(\widetilde{X},\widetilde{Y}) = D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}] = \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}]$$
$$+\frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\widetilde{X},\widetilde{Y}) - \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\widetilde{Y},\widetilde{X}) = \overset{\circ}{\mathcal{T}}(\widetilde{X},\widetilde{Y}).$$

In view of 2.47, **1** and the condition of the Proposition

$$(\overset{\circ}{\mathcal{T}})_0 = \mathbf{T} = \text{ torsion of } \mathcal{H} = 0,$$

therefore $\mathcal{T} = 0$, as we claimed.

(4) For the Finsler torsion of D we get

$$\mathcal{S}(\widetilde{X},\widetilde{Y}) = -D_{\mathbf{i}\widetilde{Y}}\widetilde{X} + \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} = -\frac{1}{2}\overset{\circ}{\mathcal{C}}(\widetilde{Y},\widetilde{X}) = -\frac{1}{2}\overset{\circ}{\mathcal{C}}(\widetilde{X},\widetilde{Y}) = \mathcal{S}(\widetilde{Y},\widetilde{X})$$

($\widetilde{X},\widetilde{Y} \in \mathfrak{X}(\tau)$), thus $\mathcal{S}$ is symmetric.

(5) 2.45, Lemma 1 and the definition of D imply immediately that $\mathbf{P}^1 = 0$. Finally, for any vector fields X, Y on M we have

$$\mathbf{Q}^1(\widehat{X},\widehat{Y}) = D_{X^{\mathrm{v}}}\widehat{Y} - D_{Y^{\mathrm{v}}}\widehat{X} - \mathcal{V}[X^{\mathrm{v}},Y^{\mathrm{v}}] = \frac{1}{2}\overset{\circ}{\mathcal{C}}(\widehat{X},\widehat{Y}) - \frac{1}{2}\overset{\circ}{\mathcal{C}}(\widehat{Y},\widehat{X}) = 0,$$

so D has vanishing v-vertical torsion.

This concludes the proof.

Comment. The idea of the above construction of the metric derivative D is from R. Miron's paper [57], for a coordinate version see also [59], pp. 184–185. As a matter of fact, under some further regularity conditions on g, which will be discussed in 3.9, and by means of the classical tensor calculus, Miron was also able to derive a horizontal map that depends only on the metric. His metric derivative is built upon this canonical horizontal map, whose intrinsic construction will be indicated in 3.15, Remark.

Chapter 3

Applications to Second-order Vector Fields and Finsler Metrics

A. Horizontal maps generated by second-order vector fields

3.1. Second order vector fields and geodesics.

(1) In this subsection I will denote an open interval of $\mathbb{R}$. The velocity field $\dot{c} : t \in I \mapsto \dot{c}(t) \in T_{c(t)}M$ of a curve $c : I \longrightarrow M$ will also be mentioned as the *canonical lift* of c into TM. The velocity field $\ddot{c}$ of the canonical lift $\dot{c}$ is said to be the *acceleration vector field* of c.

(2) As in the foregoing, the Einstein summation convention will be used in coordinate calculations.

Lemma 1. *Consider a curve $c : I \longrightarrow M$. Choose a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M such that $c^{-1}(\mathcal{U}) \neq \emptyset$. Let $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ be the induced chart on TM. If $c^i := u^i \circ c$ $(1 \leqq i \leqq n)$, then*

$$\ddot{c}(t) = (c^i)'(t) \left(\frac{\partial}{\partial x^i}\right)_{\dot{c}(t)} + (c^i)''(t) \left(\frac{\partial}{\partial y^i}\right)_{\dot{c}(t)} \quad \textit{for all } t \in c^{-1}(\mathcal{U}). \qquad \square$$

Lemma 2. *A curve $\gamma : I \longrightarrow TM$ is the canonical lift of a curve $c : I \longrightarrow M$ if, and only if, $\gamma = \tau_* \circ \dot{\gamma}$.*

Proof. If $\gamma = \dot{c}$, then $\tau \circ \gamma = \tau \circ \dot{c} = c$, and we have

$$\gamma = \dot{c} = c_* \circ \frac{d}{dr} = (\tau \circ \gamma)_* \circ \frac{d}{dr} = \tau_* \circ \left(\gamma_* \circ \frac{d}{dr}\right) = \tau_* \circ \dot{\gamma}.$$

Conversely, suppose that $\gamma = \tau_* \circ \dot{\gamma}$. Let $c := \tau \circ \gamma$. Then

$$\dot{c} = (\tau \circ \gamma)_* \circ \frac{d}{dr} = \tau_* \circ \left(\gamma_* \circ \frac{d}{dr}\right) = \tau_* \circ \dot{\gamma} = \gamma,$$

so γ is the canonical lift of c. (As for the representation of $\dot{c}$, see 1.8, Example and 1.26, Remark.) $\square$

Definition 1. A vector field ξ on TM is said to be a *second-order vector field over* M if it satisfies the condition $J\xi = C$ (or, equivalently, $\mathbf{j}\xi = \delta$).

Note. Second-order vector fields are frequently mentioned as *semisprays* or *second-order differential equations* (abbreviated as SODE). We follow here the usage of Lang's book [45], but we shall also use the term 'semispray' with a seemingly slight, but essential difference; see 3.5.

Remark. Any (finite) *convex combination of second-order vector fields is a second-order vector field:* if $(\xi_i)_{i=1}^k$ is a family of second-order vector fields over M and $(f_i)_{i=1}^k$ is a family of non-negative smooth functions on TM such that $\sum_{i=1}^k f_i = \mathbf{1} :=$ *the 1-valued constant function on* TM, then $\sum_{i=1}^k f_i\xi_i$ is also a second-order vector field over M.

Lemma 3 and definition. *A vector field $\xi : TM \longrightarrow TTM$ is a second order vector field, if and only if, the coordinate expression of ξ with respect to any induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ is of the following form:*

$$\xi \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial y^i}; \quad \xi^i \in C^\infty(\tau^{-1}(\mathcal{U})), \ 1 \leqq i \leqq n.$$

The functions $\xi^i = \xi(y^i)$ (or $G^i := -\frac{1}{2}\xi(y^i)$, $1 \leqq i \leqq n$) are called the forces *defined by ξ with respect to the given chart.*

The proof is an easy calculation.

Proposition. *For a vector field ξ on TM, the following properties are equivalent:*

(1) *ξ is a second-order vector field.*

(2) $\tau_* \circ \xi = 1_{TM}$.

(3) *Each integral curve γ of ξ is equal to the canonical lift of $\tau \circ \gamma$, in other words, $(\tau \circ \gamma)^{\cdot} = \gamma$.*

Proof. (1) $\Longrightarrow$ (2) Using the coordinate expression of ξ, according to Lemma 3 for any vector $w \in TM$ we have

$$\tau_*(\xi(w)) = y^i(w)\left(\frac{\partial}{\partial u^i}\right)_{\tau(w)} = w,$$

so $\tau_* \circ \xi = 1_{TM}$.

(2) $\Longrightarrow$ (1) Let $\xi \restriction \tau^{-1}(\mathcal{U}) = X^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial y^i}$. Then for all $w \in \tau^{-1}(\mathcal{U})$ we have

$$y^i(w)\left(\frac{\partial}{\partial u^i}\right)_{\tau(w)} = w = \tau_*\left(X^i(w)\left(\frac{\partial}{\partial x^i}\right)_w + \xi^i(w)\left(\frac{\partial}{\partial y^i}\right)_w\right) = X^i(w)\left(\frac{\partial}{\partial u^i}\right)_{\tau(w)},$$

hence $X^i = y^i$ $(1 \leqq i \leqq n)$. In view of Lemma 3 this means that ξ is a second-order vector field.

(2) $\Longrightarrow$ (3) Let $\gamma : I \longrightarrow TM$ be an integral curve of ξ, i.e. $\dot{\gamma} = \xi \circ \gamma$. Then $\tau_* \circ \dot{\gamma} = \tau_* \circ \xi \circ \gamma = \gamma$, therefore γ is the canonical lift of $\tau \circ \gamma$ by Lemma 2.

(3) $\Longrightarrow$ (2) We again consider an integral curve $\gamma : I \longrightarrow TM$ of ξ, and assume that γ is a canonical lift. Then, according to Lemma 2,

$$\gamma = \tau_* \circ \dot{\gamma} = (\tau_* \circ \xi) \circ \gamma.$$

Now we use the fact that given a vector $w \in TM$, there is an integral curve $\gamma := \gamma_w$ with $\gamma_w(0) = w$ (see 1.28). This implies immediately that $\tau_* \circ \xi = 1_{TM}$, and concludes the proof.

Definition 2. Let a second-order vector field ξ over M be given. A curve $c : I \longrightarrow M$ is said to be a *geodesic* with respect to ξ if its canonical lift is an integral curve of ξ, in other words $\ddot{c} = \xi \circ \dot{c}$. This relation is also called the *second-order differential equation* for the curve c, determined by ξ.

Coordinate expression. Let $\xi \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial y^i}$. If $c : I \longrightarrow M$ is a geodesic with respect to ξ, then applying Lemma 1 and Lemma 3 we get

$$c^{i\prime\prime} = \xi^i \circ \dot{c}; \quad c^i := u^i \circ c, \quad 1 \leqq i \leqq n.$$

If we introduce the functions $G^i := -\frac{1}{2}\xi^i$, the above relation takes the form

G. $$\boxed{c^{i\prime\prime} + 2G^i \circ \dot{c} = 0 \quad (1 \leqq i \leqq n)}.$$

By a slight abuse of language, **G** is also called the *differential equation of the geodesics* with respect to ξ.

3.2. Some technical results.

Lemma 1. *Let ξ be a second-order vector field over M. Then*

$$(1)\qquad J[J\eta,\xi]=J\eta \qquad \textit{for all}\quad \eta\in\mathfrak{X}(TM).$$

$$(2)\qquad J[X^c,\xi]=0 \qquad \textit{for all}\quad X\in\mathfrak{X}(M).$$

Proof. The first relation is merely a restatement of 2.31,**5**. According to 2.31, **3** and 2.20, Lemma 2, we have

$$0=[J,X^c]\xi=[J\xi,X^c]-J[\xi,X^c]=[C,X^c]-J[\xi,X^c]=J[X^c,\xi],$$

thus proving the second relation.

Corollary. *If $\xi: TM\longrightarrow TTM$ is a second-order vector field, then for any vector field X on M*

$[X^c,\xi]$ is a vertical vector field;

$[X^{\mathrm{v}},\xi]$ is a projectable vector field, namely $[X^{\mathrm{v}},\xi]\underset{\tau}{\sim} X$.

Proof. The first assertion is obvious from the above relation (2). According to the relation (1), $J[X^{\mathrm{v}},\xi]=J[JX^c,\xi]=JX^c$, so $[X^{\mathrm{v}},\xi]$ and X^c differ in a vertical vector field:

$$[X^{\mathrm{v}},\xi]=X^c+\eta,\quad \eta\in\mathfrak{X}^{\mathrm{v}}(TM).$$

Since $X^c+\eta\underset{\tau}{\sim} X^{\mathrm{v}}$, the second assertion is also true.

Lemma 2. *Let $\mathcal{H}$ be a horizontal map for τ. If ξ is a second-order vector field over M, then*

$$(1)\qquad X^h=\mathbf{h}[X^{\mathrm{v}},\xi] \qquad \textit{for all}\quad X\in\mathfrak{X}(M);$$

$$(2)\qquad \mathbf{h}[\xi,\mathbf{h}]=\mathcal{H}\circ\mathcal{V},\quad \textit{therefore}\quad \mathbf{F}=\mathbf{h}[\xi,\mathbf{h}]-J,$$

where $\mathbf{h}$, $\mathcal{V}$ and $\mathbf{F}$ are the horizontal projector, the vertical map and the almost complex structure belonging to $\mathcal{H}$, respectively.

Proof. (1) As we have just seen, $[X^{\mathrm{v}},\xi]=X^c+\eta$, where $\eta\in\mathfrak{X}^{\mathrm{v}}(TM)$. Hence $\mathbf{h}[X^{\mathrm{v}},\xi]=\mathbf{h}\,X^c=X^h$.

(2) For any vector field X on M we have

$$\mathbf{h}[\xi,\mathbf{h}]X^{\mathrm{v}}=\mathbf{h}[X^{\mathrm{v}},\xi]\overset{(1)}{=}X^h,\ \mathcal{H}\circ\mathcal{V}(X^{\mathrm{v}})=\mathcal{H}\circ\mathcal{V}\circ\mathbf{i}(\widehat{X})=\mathcal{H}(\widehat{X})=X^h;$$

$$\mathbf{h}[\xi,\mathbf{h}]X^h=-\mathbf{h}[X^h,\xi]+\mathbf{h}[X^h,\xi]=0,\ \mathcal{H}\circ\mathcal{V}(X^h)=\mathcal{H}\circ\mathcal{V}\circ\mathcal{H}(\widehat{X})=0,$$

therefore $\mathbf{h}[\xi,\mathbf{h}]=\mathcal{H}\circ\mathcal{V}$. $\square$

Definition. Let $\mathcal{H}$ be a horizontal map for τ and let ξ be a second-order vector field over M. The horizontal Lie-derivative $\mathcal{L}_\xi^h$ with respect to ξ (see 2.39 (2)) is said to be the *dynamical derivative* with respect to ξ.

Note. The operator $\mathcal{L}_\xi^h$ is usually called the *dynamical covariant derivative*, see e.g. [49], [50], [51]. It was introduced by J. F. Cariñena and E. Martínez in [14] under the name 'generalized covariant derivative'.

Lemma 3. *With the above notation we have*

$$\mathcal{L}_\xi^h \delta = \mathcal{V}\xi.$$

Proof. According to 2.39(2), $\mathcal{L}_\xi^h \delta = \mathbf{j}[\xi, \mathcal{H} \circ \delta]$, or equivalently, $\mathbf{i}\mathcal{L}_\xi^h \delta = J[\xi, \mathcal{H} \circ \delta]$. The vector field $\xi - \mathcal{H} \circ \delta$ is vertical, since

$$J(\xi - \mathcal{H} \circ \delta) = C - \mathbf{i} \circ \mathbf{j} \circ \mathcal{H} \circ \delta = C - C = 0.$$

Hence, by Lemma 1(1), $\xi - \mathcal{H} \circ \delta = J[\xi - \mathcal{H} \circ \delta, \xi] = J[\xi, \mathcal{H} \circ \delta]$, therefore

$$\mathcal{L}_\xi^h \delta = \mathcal{V}\big(\mathbf{i}\,\mathcal{L}_\xi^h \delta\big) = \mathcal{V}\xi - \mathcal{V} \circ \mathcal{H} \circ \delta = \mathcal{V}\xi.$$

□

Proposition. *If ξ is a second-order vector field over M, then*

(1) $[i_\xi, i_J] = i_C$,

(2) $[i_\xi, d_J] = d_C + i_{[J,\xi]}$,

(3) $[i_J, d_\xi] = i_{[J,\xi]}$.

Proof. Relations (1), (2) and (3) are immediate consequences of 2.30 Corollary (i), (ii) and the second fundamental formula in 2.29, respectively. □

3.3. Second-order vector fields and horizontal maps.

Remark 1. If ξ is a second-order vector field, then the vector field $\xi^* = [C, \xi] - \xi$ is called the *deviation* of ξ. Clearly, ξ^* 'measures' the non-homogeneity of ξ. ξ^* is a vertical vector field, since, applying 3.2, Lemma 1(1), $J\xi^* = J[C, \xi] - J\xi = C - C = 0$.

Lemma 1 and definition. *Let $\mathcal{H} : TM \times_M TM \longrightarrow TTM$ be a horizontal map for τ. Then $\xi_{\mathcal{H}} := \mathcal{H} \circ \delta$ is a second-order vector field, called* associated to $\mathcal{H}$. *$\xi_{\mathcal{H}}$ satisfies the relation*

$$\mathbf{h}[C, \xi_{\mathcal{H}}] = \xi_{\mathcal{H}}.$$

Proof. The first observation is obvious: $\mathbf{j} \circ \xi_{\mathcal{H}} = \mathbf{j} \circ \mathcal{H} \circ \delta = \delta$. To prove the second, consider the deviation $\xi^*_{\mathcal{H}} := [C, \xi_{\mathcal{H}}] - \xi_{\mathcal{H}}$ of $\xi_{\mathcal{H}}$. Since it is vertical,

$$0 = \mathbf{h}\,\xi^*_{\mathcal{H}} = \mathbf{h}[C, \xi_{\mathcal{H}}] - \mathbf{h}\,\xi_{\mathcal{H}} = \mathbf{h}[C, \xi_{\mathcal{H}}] - \mathcal{H} \circ \mathbf{j} \circ \xi_{\mathcal{H}} = \mathbf{h}[C, \xi_{\mathcal{H}}] - \xi_{\mathcal{H}},$$

whence the statement.

Coordinate description. Let a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M be given. Let us consider the induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$. If (Γ_i^j) is the matrix of the Christoffel symbols of $\mathcal{H}$ with respect to $(\mathcal{U}, (u^i)_{i=1}^n)$, then the second-order vector field ξ_h associated to $\mathcal{H}$ may be represented as follows:

$$\xi_{\mathcal{H}} \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} - y^i \Gamma_i^j \frac{\partial}{\partial y^j}.$$

In other words, the forces defined by $\xi_{\mathcal{H}}$ with respect to the chart $(\tau^{-1}(\mathcal{U}), (x, y))$ are the functions $-\sum_{i=1}^{n} y^i \Gamma_i^j$, $1 \leqq j \leqq n$.

Definition. Let $\mathcal{H}$ be a horizontal map for τ, and let $\mathcal{V}$ be the vertical map belonging to $\mathcal{H}$. A curve $c : I \longrightarrow M$ is said to be a *geodesic of the nonlinear connection* given by $\mathcal{H}$ (or briefly *a geodesics of* $\mathcal{H}$) if $\mathcal{V} \circ \ddot{c} = 0$, i.e. if *the acceleration vector field of c is 'horizontal with respect to* $\mathcal{H}$'.

Proposition 1. *The geodesics of a horizontal map $\mathcal{H}$ are exactly the geodesics of the second-order vector field associated to $\mathcal{H}$.*

Proof. Let $\xi_{\mathcal{H}}$ be the second-order vector field associated to $\mathcal{H}$. Consider a curve $c : I \longrightarrow M$ on M.

(a) If c is a geodesic of $\xi_{\mathcal{H}}$, i.e. $\ddot{c} = \xi_{\mathcal{H}} \circ \dot{c}$, then $\mathcal{V} \circ \ddot{c} = \mathcal{V} \circ \mathcal{H} \circ \delta \circ \dot{c} = 0$, so c is also geodesic with respect to ξ.

(b) Conversely, let c be a geodesic of $\mathcal{H}$, i.e. let $\mathcal{V} \circ \ddot{c} = 0$. For any $t \in \mathbb{R}$, the following statements are true:

(1) $\ddot{c}(t) = \mathbf{i} \circ \mathcal{V}\ddot{c}(t) + \mathbf{h}\,\ddot{c}(t) = \mathbf{h}\,\ddot{c}(t)$.

(2) $\xi_{\mathcal{H}}\dot{c}(t) = \mathcal{H} \circ \delta(\dot{c}(t)) = (\mathcal{H} \circ \mathbf{j}) \circ (\mathcal{H} \circ \delta)\dot{c}(t) = \mathbf{h}\,\xi_{\mathcal{H}}(\dot{c}(t))$.

(3) $$J(\xi_{\mathcal{H}}\dot{c}(t) - \ddot{c}(t)) = C(\dot{c}(t)) - \mathbf{i} \circ \mathbf{j}(\ddot{c}(t)) = C(\dot{c}(t)) - \mathbf{i}(\dot{c}(t), \tau_*(\ddot{c}(t))$$
$$= C(\dot{c}(t)) - \mathbf{i}(\dot{c}(t), \dot{c}(t)) \overset{2.4(3)}{=} C(\dot{c}(t)) - C(\dot{c}(t)) = 0.$$

Thus

$$0 \overset{(3)}{=} \mathbf{h}(\xi_{\mathcal{H}}\dot{c}(t) - \ddot{c}(t)) \overset{(2),(1)}{=} \xi_{\mathcal{H}}(\dot{c}(t)) - \ddot{c}(t),$$

therefore $\xi_{\mathcal{H}} \circ \dot{c} = \ddot{c}$, proving that c is a geodesic with respect to $\xi_{\mathcal{H}}$. $\square$

Theorem 1. (a) *Any second-order vector field ξ over M determines a horizontal map $\mathcal{H}$ for τ with the horizontal projector*

$$\mathbf{h} = \frac{1}{2}\big(1_{TTM} + [J, \xi]\big).$$

The horizontal lift by $\mathcal{H}$ of a vector field X on M is

$$X^h := \mathcal{H}\widehat{X} = \mathbf{h}X^c = \frac{1}{2}\big(X^c + [X^{\mathrm{v}}, \xi]\big).$$

(b) *The second-order vector field associated to $\mathcal{H}$ is*

$$\xi_{\mathcal{H}} = \xi + \frac{1}{2}\xi^* = \frac{1}{2}\big(\xi + [C, \xi]\big).$$

If ξ is homogeneous of degree 2, then $\xi_{\mathcal{H}} = \xi$ and $\mathcal{H}$ is a homogeneous horizontal map.

(c) *The torsion of $\mathcal{H}$ vanishes.*

Proof. (1) Applying 3.2, Lemma 1(2), we get

$$[J, \xi]X^c = [X^{\mathrm{v}}, \xi] - J[X^c, \xi] = [X^{\mathrm{v}}, \xi].$$

So if $\mathbf{h}$ is a horizontal projector in the sense of 2.12 and $\mathcal{H}$ is the horizontal map induced by $\mathbf{h}$ (see 2.12, Lemma 1), then the horizontal lift of a vector field X on M with respect to $\mathcal{H}$ is indeed $\frac{1}{2}(X^c + [X^{\mathrm{v}}, \xi])$.

(2) Next we show that $\mathbf{h}$ is indeed a horizontal projector, i.e. $\mathbf{h}^2 = \mathbf{h}$ and $\operatorname{Ker}\mathbf{h} = \mathfrak{X}^{\mathrm{v}}(TM)$.

For any vector field X on M we have

$$\mathbf{h}(X^{\mathrm{v}}) = \frac{1}{2}\big(X^{\mathrm{v}} - J[X^{\mathrm{v}}, \xi]\big) = \frac{1}{2}(X^{\mathrm{v}} - X^{\mathrm{v}}) = 0,$$

therefore $\mathfrak{X}^{\mathrm{v}}(TM) \subset \operatorname{Ker}\mathbf{h}$, and $\mathbf{h}^2 \restriction \mathfrak{X}^{\mathrm{v}}(TM) = \mathbf{h} \restriction \mathfrak{X}^{\mathrm{v}}(TM)$ holds automatically. Now we prove the converse relation $\operatorname{Ker}\mathbf{h} \subset \mathfrak{X}^{\mathrm{v}}(TM)$.

If $\eta \in \operatorname{Ker}\mathbf{h}$, then

$$0 = 2\mathbf{h}(\eta) := \eta + [J\eta, \xi] - J[\eta, \xi],$$

hence

$$J\eta = J^2[\eta, \xi] - J[J\eta, \xi] = -J\eta.$$

From this we obtain that $J\eta = 0$, and so $\eta \in \mathfrak{X}^{\mathrm{v}}(TM)$. Thus $\operatorname{Ker}\mathbf{h} \subset \mathfrak{X}^{\mathrm{v}}(TM)$ is also true, therefore $\operatorname{Ker}\mathbf{h} = \mathfrak{X}^{\mathrm{v}}(TM)$.

Now we turn to the proof of $\mathbf{h}^2(X^c) = \mathbf{h}(X^c)$. Starting with the definition of $\mathbf{h}$,

$$\mathbf{h}^2(X^c) = \frac{1}{2}\mathbf{h}(X^c + [J,\xi]X^c) = \frac{1}{2}\mathbf{h}(X^c + [X^{\mathrm{v}},\xi] - J[X^c,\xi]) \overset{3.2,\,\text{Lemma 1}}{=}$$
$$= \frac{1}{2}(\mathbf{h}\,X^c + \mathbf{h}[X^{\mathrm{v}},\xi]).$$

Since by 3.2, Corollary, $[X^{\mathrm{v}},\xi] \underset{\tau}{\sim} X$, furthermore $X^c \underset{\tau}{\sim} X$, it follows that $[X^{\mathrm{v}},\xi] - X^c \underset{\tau}{\sim} 0$. Hence $[X^{\mathrm{v}},\xi] - X^c \in \mathfrak{X}^{\mathrm{v}}(TM) = \operatorname{Ker}\mathbf{h}$, therefore $\mathbf{h}[X^{\mathrm{v}},\xi] = \mathbf{h}\,X^c$, and we obtain the desired relation $\mathbf{h}^2(X^c) = \mathbf{h}\,X^c$.

This concludes the proof of part (a).

(3) $\xi_{\mathcal{H}} := \mathcal{H} \circ \delta = \mathcal{H} \circ \mathbf{j} \circ \widetilde{\xi} = \mathbf{h}\,\widetilde{\xi}$, where $\widetilde{\xi}$ is an arbitrary second-order vector field over M. According to the definition of $\mathbf{h}$, and arguing as in the proof of 3.2, Lemma 3, we obtain:

$$\mathbf{h}\widetilde{\xi} = \frac{1}{2}\big(\widetilde{\xi} + [C,\xi] - J[\widetilde{\xi},\xi]\big) = \frac{1}{2}(\widetilde{\xi} + [C,\xi] - \widetilde{\xi} + \xi)$$
$$= \frac{1}{2}\big(\xi + [C,\xi]\big) = \xi + \frac{1}{2}\,\xi^*.$$

If ξ is homogeneous of degree 2, i.e. $[C,\xi] = \xi$, then $\xi_{\mathcal{H}} = \xi$. In this case $\mathcal{H}$ is homogeneous as well. Indeed, for any vector field X on M we have

$$[X^h, C] = \frac{1}{2}[X^c, C] + \frac{1}{2}[[X^{\mathrm{v}},\xi], C] = \frac{1}{2}[[X^{\mathrm{v}},\xi], C]$$
$$= -\frac{1}{2}\big([[\xi, C], X^{\mathrm{v}}] + [[C, X^{\mathrm{v}}],\xi]\big) = \frac{1}{2}\big([\xi, X^{\mathrm{v}}] + [X^{\mathrm{v}},\xi]\big) = 0.$$

This proves part (b).

(4) We check that $\mathcal{H}$ has vanishing torsion.

$$\mathbf{T} \overset{2.32}{:=} [J,\mathbf{h}] = \frac{1}{2}\big([J, 1_{TTM}] + [J,[J,\xi]]\big) = \frac{1}{2}[J,[J,\xi]].$$

In view of the graded Jacobi identity,

$$0 = (-1)^{1\cdot 0}[J,[J,\xi]] + (-1)^{1\cdot 1}[J,[\xi,J]] + (-1)^{0\cdot 1}[\xi,[J,J]] = 2[J,[J,\xi]],$$

therefore $\mathbf{T} = 0$.

This concludes the proof.

Remark 2. The horizontal map (or the horizontal projector) described by Theorem 1 will usually be mentioned as the horizontal map (or horizontal projector) *generated by* the given second-order vector field. Notice that *two second-order vector fields generate the same horizontal map if, and only if, their difference is a vertical lift.*

Indeed, let the horizontal maps $\mathcal{H}_1$ and $\mathcal{H}_2$ be generated by the second-order vector fields ξ_1 and ξ_2, respectively. Obviously,

$$\mathcal{H}_1 = \mathcal{H}_2 \iff [J, \xi_1 - \xi_2] = 0.$$

Since $\xi_1 - \xi_2$ is a vertical vector field, the last relation is equivalent to the fact that $\xi_1 - \xi_2$ is the vertical lift of a vector field on M.

Coordinate description. As usual, choose a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M, and consider the induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ on TM. Let $\xi \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$. Then for the horizontal map $\mathcal{H}$ generated by ξ we have

$$\begin{aligned}\mathcal{H}\left(\widehat{\tfrac{\partial}{\partial u^i}}\right) &= \left(\tfrac{\partial}{\partial u^i}\right)^h = \tfrac{1}{2}\left(\left(\tfrac{\partial}{\partial u^i}\right)^c + \left[\left(\tfrac{\partial}{\partial u^i}\right)^{\mathrm{v}}, \xi\right]\right) \\ &= \tfrac{1}{2}\left(\tfrac{\partial}{\partial x^i} + \left[\tfrac{\partial}{\partial y^i}, y^j \tfrac{\partial}{\partial x^j} - 2G^j \tfrac{\partial}{\partial y^j}\right]\right) = \tfrac{\partial}{\partial x^i} - \tfrac{\partial G^j}{\partial y^i}\tfrac{\partial}{\partial y^j} \qquad (1 \leqq i \leqq n),\end{aligned}$$

therefore the Christoffel symbols of $\mathcal{H}$ with respect to $(\mathcal{U}, (u^i)_{i=1}^n)$ are the functions

$$\boxed{G_i^j := \frac{\partial G^j}{\partial y^i}, \quad 1 \leqq i, j \leqq n}.$$

The second-order vector field $\xi_{\mathcal{H}}$ asssociated to $\mathcal{H}$ has the coordinate expression

$$\xi_{\mathcal{H}} \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} - y^j \frac{\partial G^i}{\partial y^j}\frac{\partial}{\partial y^i}.$$

The components of the torsion of $\mathcal{H}$ are

$$\mathbf{T}\left(\left(\frac{\partial}{\partial u^i}\right)^c, \left(\frac{\partial}{\partial u^j}\right)^c\right) = \left(\frac{\partial G_i^k}{\partial y^j} - \frac{\partial G_j^k}{\partial y^i}\right)\frac{\partial}{\partial y^k} = \left(\frac{\partial^2 G^k}{\partial y^j \partial y^i} - \frac{\partial^2 G^k}{\partial y^i \partial y^j}\right)\frac{\partial}{\partial y^k} = 0$$

$$(1 \leqq i, j \leqq k),$$

as we expected.

Corollary 1. *A second-order vector field ξ over M is horizontal with respect to the horizontal map $\mathcal{H}$ generated by ξ according to Theorem 1 if, and only if, ξ is homogeneous of degree two.*

Proof. $\mathbf{h}\xi = \mathcal{H} \circ \delta = \xi_{\mathcal{H}} = \xi + \frac{1}{2}\xi^*$, so $\mathbf{h}\xi = \xi$ if, and only if, $\xi^* = [C, \xi] - \xi = 0$, i.e. ξ is homogeneous of degree two. $\square$

Corollary 2. *Let ξ be a second-order vector field on TM, and let $\mathbf{h}$ be the horizontal projector arising from ξ according to Theorem 1. Then for any smooth function F on TM we have*

$$\boxed{2d_{\mathbf{h}}F = d(F - CF) - i_\xi dd_J F + d_J i_\xi dF}\,.$$

Proof. $2d_{\mathbf{h}}F = d_{1_{TTM}}F + d_{[J,\xi]}F \overset{2.26,\ \text{Remark}}{=} dF + d_{[J,\xi]}F = dF + d_J d_\xi F - d_\xi d_J F \overset{1.40\,(3)}{=} dF + d_J i_\xi dF + d_J d i_\xi F - i_\xi dd_J F - di_\xi d_J F = dF + d_J i_\xi dF - i_\xi dd_J F - di_\xi d_J F$.

Here $i_\xi d_J F = d_J F(\xi) = (dF \circ J)(\xi) = dF(C) = CF$, whence the desired formula. $\square$

Corollary 3. *Suppose that the horizontal map $\mathcal{H}$ is generated by the second-order vector field ξ. Then for any vector field X on M we have*

$$J[X^h, \xi] = X^h - X^c.$$

Proof. In view of Theorem 1(a), $X^h = (2\mathbf{h} - 1_{TTM})X^h = [J, \xi]X^h = [X^{\mathrm{v}}, \xi] - J[X^h, \xi] = 2X^h - X^c - J[X^h, \xi]$, hence $J[X^h, \xi] = X^h - X^c$. $\square$

Corollary 4. *If ξ is a second-order vector field on TM and $\mathcal{H}$ is the horizontal map generated by ξ, then $\mathcal{L}^h_\xi = \mathcal{L}^{\mathrm{v}}_\xi$.*

Proof. We have only to show that $\mathcal{L}^h_\xi$ and $\mathcal{L}^{\mathrm{v}}_\xi$ operate in the same way on $\mathfrak{X}(\tau)$ (see 2.39). For any vector field Y on M we have

$$\mathbf{i}\mathcal{L}^h_\xi \widehat{Y} = J[\xi, Y^h] \overset{\text{Cor. 3}}{=} Y^c - Y^h.$$

On the other hand,

$$\mathbf{i}\mathcal{L}^{\mathrm{v}}_\xi \widehat{Y} = \mathbf{v}[\xi, Y^{\mathrm{v}}] \overset{\text{Th. 1}}{=} \mathbf{v}(Y^c - 2Y^h) = \mathbf{v}\, Y^c = Y^c - \mathbf{h}\, Y^c = Y^c - Y^h,$$

therefore

$$\mathcal{L}^h_\xi \widehat{Y} = \mathcal{L}^{\mathrm{v}}_\xi \widehat{Y} \qquad \text{for all} \quad Y \in \mathfrak{X}(M).$$

To complete the proof we check that

$$\mathcal{L}^h_\xi f\widehat{Y} = \mathcal{L}^{\mathrm{v}}_\xi f\widehat{Y} \qquad \text{for all} \quad f \in C^\infty(TM).$$

Indeed,

$$\mathcal{L}^h_\xi f\widehat{Y} := \mathbf{j}[\xi, \mathcal{H}(f\widehat{Y})] = f\mathcal{L}^h_\xi \widehat{Y} + \mathbf{j}(\xi f)Y^h = f\mathcal{L}^h_\xi \widehat{Y} + (\xi f)\widehat{Y},$$

$$\mathcal{L}^{\mathrm{v}}_\xi f\widehat{Y} := \mathcal{V}[\xi, \mathbf{i}(f\widehat{Y})] = f\mathcal{L}^{\mathrm{v}}_\xi \widehat{Y} + \mathcal{V}(\xi f)Y^{\mathrm{v}} = f\mathcal{L}^{\mathrm{v}}_\xi \widehat{Y} + (\xi f)\widehat{Y},$$

and this ends the proof.

Corollary 5. *With the same assumption as above, we have*

$$[\xi, \mathbf{i}\widetilde{Y}] = -\mathcal{H}\widetilde{Y} + \mathbf{i}\mathcal{L}^h_\xi \widetilde{Y} \quad \textit{for all } \widetilde{Y} \in \mathfrak{X}(\tau).$$

Proof. $[\xi, \mathbf{i}\,\widetilde{Y}] = \mathcal{H} \circ \mathbf{j}[\xi, \mathbf{i}\,\widetilde{Y}] + \mathbf{i} \circ \mathcal{V}[\xi, \mathbf{i}\,\widetilde{Y}] = \mathcal{H} \circ \mathbf{j}[\xi, \mathbf{i}\,\widetilde{Y}] + \mathbf{i}\,\mathcal{L}^{\mathrm{v}}_\xi \widetilde{Y}$. Since $\mathbf{i} \circ \mathbf{j}[\xi, \mathbf{i}\,\widetilde{Y}] = -J[\mathbf{i}\,\widetilde{Y}, \xi] = -\mathbf{i}\,\widetilde{Y}$ by 3.2, Lemma 1(1), and $\mathcal{L}^{\mathrm{v}}_\xi = \mathcal{L}^h_\xi$ according to the previous Corollary, we obtain the desired formula. □

Theorem 2. *Any horizontal map with vanishing torsion is generated by a second-order vector field according to Theorem 1(a).*

Proof (after E. Ayassou [6]). Let $\mathcal{H}$ be the given horizontal map and $\mathbf{h}$ the horizontal projector belongig to $\mathcal{H}$. Consider the vector-valued one-form $L = 2\mathbf{h} - 1_{TTM}$. Since

$$\mathbf{T} = [J, \mathbf{h}] =: \theta_J \mathbf{h} = 0,$$

it follows that

$$\theta_J L = 2\theta_J \mathbf{h} - \theta_J 1_{TTM} = 0,$$

thus L is θ_J-closed. On the other hand, using the first box in 2.19,

$$J^* L + J \circ L = L \circ J + J \circ L = 2\mathbf{h} \circ J - J + 2J \circ \mathbf{h} - J = -2J + 2J = 0,$$

so L satisfies the condition of Ayassou's theorem in 2.31. Therefore L *is locally θ_J-exact*: there is a vector-valued 0-form, i.e. a vector field $\widetilde{\xi}$ defined on an open set $\mathcal{O} \subset TM$ such that

$$L \restriction \mathcal{O} = \theta_J \widetilde{\xi} = [J, \widetilde{\xi}].$$

We may suppose that $\mathcal{O}$ is the domain of an induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$. Next we show that among these solutions there is a second-order vector field. Let $\widetilde{\xi} \restriction \tau^{-1}(\mathcal{U}) = \widetilde{X}^i \frac{\partial}{\partial x^i} + \widetilde{\xi}^i \frac{\partial}{\partial y^i}$. Then

$$[J, \widetilde{\xi}] \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial \widetilde{X}^j}{\partial y^i} \frac{\partial}{\partial x^j} + \left(\frac{\partial \widetilde{\xi}^j}{\partial y^i} - \frac{\partial \widetilde{X}^j}{\partial x^i} \right) \frac{\partial}{\partial y^j}, \quad [J, \widetilde{\xi}] \frac{\partial}{\partial y^i} = -\frac{\partial \widetilde{X}^j}{\partial y^i} \frac{\partial}{\partial y^j},$$

so L may be represented over $\tau^{-1}(\mathcal{U})$ by the $2n \times 2n$ matrix

$$\begin{pmatrix} \left(\frac{\partial \widetilde{X}^j}{\partial y^i}\right) & 0 \\ \left(\frac{\partial \widetilde{\xi}^j}{\partial y^i} - \frac{\partial \widetilde{X}^j}{\partial x^i}\right) & \left(-\frac{\partial \widetilde{X}^j}{\partial y^i}\right) \end{pmatrix}.$$

If the Christoffel symbols of $\mathcal{H}$ with respect to $\mathcal{U}$ constitute the $n \times n$ matrix (Γ_i^j), then, according to the local description of $\mathbf{h}$ given in 2.32, L can be represented by the matrix

$$\begin{pmatrix} (\delta_i^j) & 0 \\ -2(\Gamma_i^j) & -(\delta_i^j) \end{pmatrix}.$$

The equality of these matrices yields the relations

$$\text{(i)} \qquad \frac{\partial \widetilde{X}^j}{\partial y^i} = \delta_i^j,$$

$$\text{(ii)} \qquad \frac{\partial \widetilde{\xi}^j}{\partial y^i} - \frac{\partial \widetilde{X}^j}{\partial x^i} = -2\Gamma_i^j \quad (1 \leqq i, j \leqq n).$$

The first of them implies

$$\widetilde{X}^j = y^j + (f^j)^{\mathrm{v}}, \qquad 1 \leqq j \leqq n,$$

with some smooth functions f^j over $\mathcal{U}$.

Now we introduce the following new functions over $\mathcal{U}$:

$$X^j := y^j, \quad \xi^j := \widetilde{\xi}^j - y^k \left(\frac{\partial f^j}{\partial u^k}\right)^{\mathrm{v}} \qquad (1 \leqq j \leqq n).$$

Then

$$\xi_{\mathcal{U}} := X^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial y^i} = y^i \frac{\partial}{\partial x^i} + \xi^i \frac{\partial}{\partial y^i}$$

is a second-order vector field over $\mathcal{U}$ (see 3.1, Lemma 3), and solves the problem, since

$$\begin{aligned} \frac{\partial \xi^j}{\partial y^i} - \frac{\partial X^j}{\partial x^i} &= \frac{\partial}{\partial y^i}\left(\widetilde{\xi}^j - y^k \left(\frac{\partial f^j}{\partial u^k}\right)^{\mathrm{v}}\right) - \frac{\partial}{\partial x^i}(\widetilde{X}^j - (f^j)^{\mathrm{v}}) \\ &= \frac{\partial \widetilde{\xi}^j}{\partial y^i} - \frac{\partial \widetilde{X}^j}{\partial x^i} - \left(\frac{\partial f^j}{\partial u^i}\right)^{\mathrm{v}} + \frac{\partial (f^j)^{\mathrm{v}}}{\partial x^i} = -2\Gamma_i^j, \end{aligned}$$

i.e. the functions X^j, ξ^j satisfy (ii), while (i) holds automatically.

From this local result we conclude that there is a locally finite open cover $(\mathcal{U}_i)_{i\in I}$ of M and a family $(\xi_i)_{i\in I}$ of second-order vector fields such that $\tau^{-1}(\mathcal{U}_i)$ is the domain of ξ_i and $L \upharpoonright \tau^{-1}(\mathcal{U}_i) = [J, \xi_i]$ $(i \in I)$. Now let $(f_i)_{i\in I}$ be a partition of unity subordinate to $(\mathcal{U}_i)_{i\in I}$ (see 1.5). Then $\xi := \sum_{i\in I}(f_i)^{\mathrm{v}}\xi_i$ is a well-defined second-order vector field over M (cf. 3.1, Remark) satisfying $L = [J, \xi]$. Thus $\mathbf{h} = \frac{1}{2}\big(1_{TTM} + [J, \xi]\big)$, therefore $\mathcal{H}$ is generated by ξ. □

Note. Theorem 1, at least in this intrinsic presentation, was discovered independently by M. Crampin and J. Grifone; see [16], [17] and [36]. Theorem 2 is due to M. Crampin [18], but it was also deduced by E. Ayassou, independently, in his thesis [6].

Proposition 2. *The horizontal map generated by a second-order vector field ξ is homogeneous if, and only if, there exists a second order vector field $\bar{\xi}$ on TM and a vector field X on M such that $\bar{\xi}$ is homogeneous of degree two, i.e. $[C, \bar{\xi}] = \bar{\xi}$ and $\xi = \bar{\xi} + X^{\mathrm{v}}$.*

Proof. Let $\mathcal{H}$ be the horizontal map generated by ξ; $\mathbf{h}$ and $\mathbf{v}$ the horizontal and the vertical projector belonging to $\mathcal{H}$; $\mathbf{t}$ the tension of $\mathcal{H}$. Then
$2\mathbf{h} = 1_{TTM} + [J, \xi]$.

(a) *Sufficiency.* Suppose that $\xi = \bar{\xi} + X^{\mathrm{v}}$, where $J\bar{\xi} = C$, $[C, \bar{\xi}] = \bar{\xi}$ and $X \in \mathfrak{X}(M)$. Then for any vector field Y on M we have

$$\begin{aligned}
2\mathbf{t}(Y^c) = 2[Y^h, C] &\overset{\text{Theorem 1(a)}}{=} [Y^c, C] + [[Y^{\mathrm{v}}, \bar{\xi} + X^{\mathrm{v}}], C] \overset{\text{2.20, Lemma 2}}{=} \\
&= [[Y^{\mathrm{v}}, \bar{\xi}\,], C] \overset{\text{Jacobi identity}}{=} -[[\bar{\xi}, C], Y^{\mathrm{v}}] - [[C, Y^{\mathrm{v}}], \bar{\xi}\,] \overset{\text{2.20, Lemma 2}}{=} \\
&= [\bar{\xi}, Y^{\mathrm{v}}] + [Y^{\mathrm{v}}, \bar{\xi}] = 0,
\end{aligned}$$

hence $\mathcal{H}$ is homogeneous.

(b) *Necessity.* Suppose that $\mathcal{H}$ is homogeneous, and so $\mathbf{t} = [\mathbf{h}, C] = 0$. Let $\bar{\xi} := \mathbf{h}\,\xi$. Then $\bar{\xi}$ is a second-order vector field again, namely $\bar{\xi} = \xi_{\mathcal{H}}$. We show that $\bar{\xi}$ is homogeneous of degree two.

Indeed, $2\mathbf{h}\,\xi = \xi + [C, \xi]$, hence $\bar{\xi} = \mathbf{h}\xi = \mathbf{h}[C, \xi]$. On the other hand, $0 = [\mathbf{h}, C]\xi = [\mathbf{h}\xi, C] - \mathbf{h}[\xi, C]$, therefore $\mathbf{h}[C, \xi] = [C, \mathbf{h}\xi] = [C, \bar{\xi}]$. Thus we conclude that $[C, \bar{\xi}] = \bar{\xi}$.

Next we prove that $\mathbf{v}\xi$ is a vertical lift. Let η be an arbitrary vector field on TM. Then we obtain

$$[J,\mathbf{v}\xi]\eta = [J\eta,\mathbf{v}\xi] - J[\eta,\mathbf{v}\xi] \overset{(1)}{=} [J\eta,\mathbf{v}\xi] - J[\eta,\mathbf{v}\xi] - [J,\mathbf{v}](\eta,\xi) = -[\mathbf{v}\,\eta, C]$$
$$- J[\eta,\xi] + J[\mathbf{v}\,\eta,\xi] + \mathbf{v}[J\eta,\xi] + \mathbf{v}[\eta,C] = -[\mathbf{v},C]\eta - J[\mathbf{h}\,\eta,\xi] + \mathbf{v}[J\eta,\xi]$$
$$\overset{(2)}{=} \mathbf{v}[J\eta,\xi] - J[\mathbf{h}\,\eta,\xi] = \frac{1}{2}\big([J\eta,\xi] - [J,\xi]([J\eta,\xi]) - J[\eta,\xi] - J[[J,\xi]\eta,\xi]\big)$$
$$= \frac{1}{2}\big([J\eta,\xi] + J[[J\eta,\xi],\xi] - [J[J\eta,\xi],\xi] - J[\eta,\xi] - J[[J\eta,\xi] - J[\eta,\xi],\xi]\big)$$
$$\overset{(3)}{=} \frac{1}{2}\big([J\eta,\xi] - [J\eta,\xi] - J[\eta,\xi] + J[\eta,\xi]\big) = 0,$$

taking into account that $\mathbf{T} = [J,\mathbf{h}] = 0$ implies $[J,\mathbf{v}] = 0$ (step (1)); $[\mathbf{v},C] = 0$ according to the homogeneity of $\mathcal{H}$ (step (2)), and applying 3.2, Lemma 1(1) (step (3)). Thus $[J,\mathbf{v}\,\xi] = 0$, therefore by 2.31, Corollary 2, $\mathbf{v}\,\xi$ is indeed a vertical lift: there is a vector field X on M, such that $\mathbf{v}\,\xi = X^{\mathrm{v}}$. So we get

$$\xi = \mathbf{h}\,\xi + \mathbf{v}\,\xi = \bar{\xi} + X^{\mathrm{v}},$$

where the second-order vector field $\bar{\xi}$ is homogeneous of degree two. This concludes the proof.

Corollary 6. *If $\mathcal{H}$ is a homogeneous horizontal map with vanishing torsion, then there exists a second-order vector field which is homogeneous of degree two and generates $\mathcal{H}$ according to Theorem 1.*

Indeed, this is an immediate consequence of Theorem 2 and Proposition 2.

3.4. The Berwald derivative induced by a second-order vector field. Let ξ be a second-order vector field over M and $\mathcal{H}$ the horizontal map generated by ξ. Then the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$ or on TM is said to be the *Bervald derivative induced by* ξ in $\tau^*\tau$ or on TM, respectively.

Coordinate description. Let $(\mathcal{U},(u^i)_{i=1}^n)$ be a chart on M and $(\tau^{-1}(\mathcal{U}),((x^i)_{i=1}^n,(y^i)_{i=1}^n))$ be the induced chart on TM. Suppose that

$$\xi \restriction \tau^{-1}(\mathcal{U}) = y^k\,\frac{\partial}{\partial x^k} - 2G^k\,\frac{\partial}{\partial y^k}.$$

Then, as we have learnt in 3.3, the Christoffel symbols of the horizontal map $\mathcal{H}$ generated by ξ are the functions

$$G_i^k := \frac{\partial G^k}{\partial y^i}, \qquad 1 \leqq i,k \leqq n.$$

Let ∇ be the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$. Then for all $i, j \in \{1, \ldots, n\}$ we have

$$\nabla_{\left(\frac{\partial}{\partial u^i}\right)^h} \widehat{\frac{\partial}{\partial u^j}} = \mathcal{V}\left[\frac{\partial}{\partial x^i} - G_i^k \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right] = \frac{\partial G_i^k}{\partial y^j} \mathcal{V} \circ \mathbf{i}\, \widehat{\frac{\partial}{\partial u^k}} = G_{ij}^k \widehat{\frac{\partial}{\partial u^k}},$$

where $G_{ij}^k := \frac{\partial G_i^k}{\partial y^j} = \frac{\partial^2 G^k}{\partial y^j \partial y^i}$. These functions are called the *Christoffel symbols* of the Berwald derivative ∇ with respect to the chart $(\mathcal{U}, (u^i)_{i=1}^n)$. Thus

$$\boxed{\nabla_{\left(\frac{\partial}{\partial u^i}\right)^h} \widehat{\frac{\partial}{\partial u^j}} = G_{ij}^k \widehat{\frac{\partial}{\partial u^k}}, \quad 1 \leqq i, j \leqq n}\,.$$

Remark. In terms of the Berwald derivative, Theorems 1, 2 in 3.3 may be partly restated as follows:

The necessary and sufficient condition for a horizontal map $\mathcal{H}$ to be derived from a second-order vector field is that the (h-horizontal) torsion of the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$ vanishes.

Indeed, this is obvious from the relation $(\overset{\circ}{\mathcal{T}})_0 = \mathbf{T}$ (see 2.47, **1**).

Corollary 1. *If ∇ is the Berwald derivative in $\tau^*\tau$ induced by a second-order vector field, then its Berwald curvature is totally symmetric.*

This is an immediate consequence of Property **8** in 2.33.

Corollary 2. *Suppose that ∇ is an h-basic Berwald derivative induced by a second-order vector field. Then the base covariant derivative $\overset{\circ}{D}$ belonging to ∇ has vanishing torsion.*

Indeed, according to 2.49, Example, for any vector fields X, Y on M we have $(T^{\overset{\circ}{D}}(X, Y))^{\mathrm{v}} = \mathbf{T}(X^c, Y^c) = 0$.

Lemma 1. *Let ∇ be the Berwald derivative induced by the second-order vector field ξ in $\tau^*\tau$. Then*

$$\nabla_\xi \widetilde{X} = \mathcal{L}_\xi^h \widetilde{X} + \widetilde{\mathbf{t}}(\widetilde{X}) \qquad \textit{for all} \quad \widetilde{X} \in \mathfrak{X}(\tau).$$

Proof. It is enough to check the formula for an arbitrary basic vector field $\widehat{X}$. Starting with the definitions, we have

$$\begin{aligned}\mathbf{i}(\nabla_\xi \widehat{X} - \mathcal{L}_\xi^h \widehat{X}) = \mathbf{i}(\nabla_{\mathbf{h}\xi} \widehat{X} + \nabla_{\mathbf{v}\xi} \widehat{X}) - J[\xi, X^h] = \mathbf{v}[\mathbf{h}\,\xi, X^{\mathrm{v}}] + J[\mathbf{v}\,\xi, X^h] \\ - J[\xi, X^h] = \mathbf{v}[\mathbf{h}\,\xi, X^{\mathrm{v}}] - J[\mathbf{h}\,\xi, X^h].\end{aligned}$$

Since the torsion of the horizontal map generated by ξ vanishes, we get

$$0 = [J, \mathbf{h}](\xi, X^h) = [C, X^h] + [\mathbf{h}\,\xi, X^{\mathrm{v}}] + J[\xi, X^h] - J[\mathbf{h}\,\xi, X^h] - J[\xi, X^h]$$
$$- \mathbf{h}[C, X^h] - \mathbf{h}[\xi, X^{\mathrm{v}}] = [C, X^h] + \mathbf{v}[\mathbf{h}\,\xi, X^{\mathrm{v}}] - J[\mathbf{h}\,\xi, X^h].$$

Hence $\mathbf{v}[\mathbf{h}\,\xi, X^{\mathrm{v}}] - J[\mathbf{h}\,\xi, X^h] = [X^h, C] = \mathbf{t}(X^c) = \mathbf{i}\,\widetilde{\mathbf{t}}(\widehat{X})$, and we obtain the desired relation. □

Corollary 3. *Under the hypothesis of Lemma 1 we have*

$$\mathbf{i}\nabla_\xi \widehat{X} = [X^h, C] + X^c - X^h, \text{ or equivalently, } \nabla_\xi \widehat{X} = \mathcal{V}\mathbf{t}(X^c) + \mathcal{V}(X^c).$$

Indeed, we have seen in the proof of 3.3, Corollary 4 that $\mathbf{i}\,\mathcal{L}^h_\xi \widehat{X} = X^c - X^h$.

Lemma 2. *Let $\mathcal{H}$ and ∇ be the horizontal map and the Berwald derivative generated by the second-order vector field ξ. Then the tension of $\mathcal{H}$ may be expressed as follows:*

$$\widetilde{\mathbf{t}} = \nabla^{\mathrm{v}} \mathcal{L}^h_\xi \delta.$$

Proof. For any vector field X on M we have

$$\mathbf{i}(\nabla^{\mathrm{v}} \mathcal{L}^h_\xi \delta)(\widehat{X}) = \mathbf{i}(\nabla_{X^{\mathrm{v}}} \mathcal{L}^h_\xi \delta) \overset{3.2,\text{ Lemma 3}}{=} \mathbf{i}(\nabla_{X^{\mathrm{v}}} \mathcal{V}\xi) = J[X^{\mathrm{v}}, \mathcal{H} \circ \mathcal{V}\xi].$$

Since

$$0 = [J, X^{\mathrm{v}}]\mathcal{H} \circ \mathcal{V}\xi = [J \circ \mathcal{H} \circ \mathcal{V}\xi, X^{\mathrm{v}}] - J[\mathcal{H} \circ \mathcal{V}\xi, X^{\mathrm{v}}] = [\mathbf{v}\,\xi, X^{\mathrm{v}}] + J[X^{\mathrm{v}}, \mathcal{H} \circ \mathcal{V}\xi],$$

it follows that

$$(\nabla^{\mathrm{v}} \mathcal{L}^h_\xi \delta)(\widehat{X}) = \mathcal{V}[X^{\mathrm{v}}, \mathbf{v}\,\xi] \quad \text{for all } X \in \mathfrak{X}(M).$$

We have learnt that $\widetilde{\mathbf{t}}(\widehat{X}) = \mathcal{V}[X^h, C]$, so our only task is to check the equality $[X^{\mathrm{v}}, \mathbf{v}\,\xi] = [X^h, C]$. We apply the vanishing of the torsion of $\mathcal{H}$. This gives

$$0 = [J, \mathbf{h}](X^h, \xi) = [X^{\mathrm{v}}, \mathbf{h}\,\xi] + [X^h, C] + J[X^h, \xi] - J[X^h, \xi] - J[X^h, \mathbf{h}\,\xi]$$
$$- \mathbf{h}[X^{\mathrm{v}}, \xi] - \mathbf{h}[X^h, C].$$

On the right-hand side

$$\mathbf{h}[X^h, C] = 0,\ J[X^h, \mathbf{h}\,\xi] = J[X^h, \xi] \overset{3.3,\text{ Cor. }3}{=} X^h - X^c,\ \mathbf{h}[X^{\mathrm{v}}, \xi] \overset{3.2,\text{ Lemma }2}{=} X^h,$$

therefore

$$[X^h, C] = 2X^h - X^c - [X^{\mathrm{v}}, \mathbf{h}\,\xi] \overset{3.3,\text{ Th.}1}{=} [X^{\mathrm{v}}, \xi] - [X^{\mathrm{v}}, \xi] + [X^{\mathrm{v}}, \mathbf{v}\,\xi] = [X^{\mathrm{v}}, \mathbf{v}\,\xi].$$

This completes the proof.

3.5. Semisprays and sprays. By a *semispray* on the manifold M we mean a map $\xi : TM \longrightarrow TTM$, $v \in TM \mapsto \xi(v) \in T_vTM$, satisfying the following conditions:

SPR1. $J\xi = C$ (*or, equivalently*, $\mathbf{j}\xi = \delta$).

SPR2. ξ *is smooth on* $\overset{\circ}{T}M$.

A *spray* on M is a semispray ξ having the following properties:

SPR3. ξ *is of class* C^1 *over* TM.

SPR4. ξ *is homogeneous of degree 2, i.e.* $[C, \xi] = \xi$.

A spray is called *quadratic* if it is of class C^2 over TM.

Remarks. (1) According to 2.6, Lemma 2(3), the quadratic sprays are those second-order vector fields which are homogeneous of degree 2.

(2) The concepts and results treated in 3.1–3.4 remain valid either without any change or with immediate modifications in the formulation if the second-order vector fields are replaced by semisprays, and the second-order vector fields with homogeneity of degree 2 are replaced by sprays. For their importance, we reformulate here the main results of 3.3.

Theorem of M. Crampin and J. Grifone. (i) *Any semispray* $\xi : TM \longrightarrow TTM$ *generates a horizontal map on* M *with the horizontal projector* $\frac{1}{2}(1_{TTM} + [J, \xi])$. *The associated semispray of this horizontal map is* $\frac{1}{2}(\xi + [C, \xi])$. *If, in particular,* ξ *is a spray, then the associated semispray is* ξ *itself and the horizontal map generated by* ξ *is homogeneous.*

(ii) *A horizontal map is generated by a semispray in the above sense if, and only if, its torsion vanishes.* □

Proposition. *Suppose* ξ *is a spray over* M, *and* ∇ *is the Berwald derivative induced by* ξ *in* $\tau^*\tau$. *Then for any vector fields* $\widetilde{X}$, $\widetilde{Y}$ *along* τ *we have*

$$R^{\nabla}(\xi, \mathbf{i}\,\widetilde{X})\widetilde{Y} = 0.$$

Proof. It is enough to check that for any vector fields X, Y on M we have $R^{\nabla}(\xi, X^{\mathrm{v}})\widehat{Y} = 0$. To see this, we start with the definition:

$$R^{\nabla}(\xi, X^{\mathrm{v}})\widehat{Y} = \nabla_\xi \nabla_{X^{\mathrm{v}}} \widehat{Y} - \nabla_{X^{\mathrm{v}}} \nabla_\xi \widehat{Y} - \nabla_{[\xi, X^{\mathrm{v}}]} \widehat{Y} = -\nabla_{X^{\mathrm{v}}} \nabla_\xi \widehat{Y} - \nabla_{[\xi, X^{\mathrm{v}}]} \widehat{Y}.$$

Since ξ generates a homogeneous horizontal map, $\widetilde{\mathbf{t}} = 0$. Thus, applying 3.4, Corollary 3, we get

$$\begin{aligned}\mathbf{i}\,\nabla_{X^{\mathrm{v}}}\nabla_{\xi}\widehat{Y} &= J[X^{\mathrm{v}}, \mathcal{H}\nabla_{\xi}\widehat{Y}] = J[X^{\mathrm{v}}, \mathcal{H}\mathcal{V}Y^{c}] = [X^{\mathrm{v}}, J \circ \mathcal{H} \circ \mathcal{V}Y^{c}] \\ &= [X^{\mathrm{v}}, \mathbf{v}\,Y^{c}] = [X^{\mathrm{v}}, Y^{c}] - [X^{\mathrm{v}}, \mathbf{h}\,Y^{c}] = [X, Y]^{\mathrm{v}} - [X^{\mathrm{v}}, Y^{h}].\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{i}\,\nabla_{[\xi, X^{\mathrm{v}}]}\widehat{Y} &= \mathbf{i}\,\nabla_{\mathbf{h}[\xi, X^{\mathrm{v}}]}\widehat{Y} + \mathbf{i}\,\nabla_{\mathbf{v}[\xi, X^{\mathrm{v}}]}\widehat{Y} \overset{3.2,\,\mathrm{Lemma}\,2(1)}{=} -\mathbf{i}\,\nabla_{X^{h}}\widehat{Y} \\ &= -\mathbf{i} \circ \mathcal{V}[X^{h}, Y^{\mathrm{v}}] = -[X^{h}, Y^{\mathrm{v}}],\end{aligned}$$

taking into account that $\mathbf{v}[\xi, X^{\mathrm{v}}]$ can be combined from vertical lifts. Hence

$$\mathbf{i}\,R^{\nabla}(\xi, X^{\mathrm{v}})\widehat{Y} = -[Y^{h}, X^{\mathrm{v}}] + [X^{h}, Y^{\mathrm{v}}] + [Y, X]^{\mathrm{v}} = -\mathbf{T}(Y^{c}, X^{c}) = 0.$$

This gives the desired conclusion. $\square$

B. Linearization of second-order vector fields

In this section ξ will be a second-order vector field on the manifold M, $\mathcal{H}$ and ∇ denote the horizontal map and the Berwald derivative generated by ξ. We define the notion of v-linearizability and linearizability of a second-order vector field, and establish necessary and sufficient conditions for the characterizations of these properties. Except for a minor modification, our treatment follows the line of the paper [49] of E. Martínez and J. Cariñena.

3.6. The Jacobi endomorphism.

Definition. The type $(1,1)$ tensor field $\widetilde{\Phi}$ along τ given by

$$\widetilde{\Phi}(\widetilde{X}) := \mathcal{V}[\xi, \mathcal{H}\widetilde{X}] \quad \text{for all } \widetilde{X} \in \mathfrak{X}(\tau)$$

is said to be the *Jacobi endomorphism* determined by ξ. The semibasic tensor field $\Phi := (\widetilde{\Phi})_0 \in \mathcal{B}^1_0(TM)$ corresponding to $\widetilde{\Phi}$ according to 2.22, Lemma 1 is called the Jacobi endomorphism determined by ξ *on* TM.

Coordinate description. With the hypothesis and notations of 3.3, let

$$\xi \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i};\ G^i_j := \frac{\partial G^i}{\partial y^j},\ G^i_{jk} := \frac{\partial^2 G^i}{\partial y^j \partial y^k} \quad (1 \leqq i,j,k \leqq n).$$

As we have learnt, $\mathcal{H}\widehat{\frac{\partial}{\partial u^i}} = \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j}$, and so $\mathcal{V}\frac{\partial}{\partial x^i} = G^j_i \widehat{\frac{\partial}{\partial u^j}}$, $\mathcal{V}\frac{\partial}{\partial y^i} = \widehat{\frac{\partial}{\partial u^i}}$ $(1 \leqq i \leqq n)$. If

$$\widetilde{\Phi}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = \Phi^j_i \widehat{\frac{\partial}{\partial u^j}} \qquad (1 \leqq i \leqq n),$$

then we get by an easy calculation that

$$\Phi^j_i = 2\frac{\partial G^j}{\partial x^i} - G^j_k G^k_i + 2G^k G^j_{ik} - y^k \frac{\partial G^j_i}{\partial x^k} = 2\frac{\partial G^j}{\partial x^i} - G^j_k G^k_i - \xi(G^j_i),$$

therefore, locally,

$$\boxed{\widetilde{\Phi} = \left(2\frac{\partial G^j}{\partial x^i} - G^j_k G^k_i - \xi(G^j_i)\right) \widehat{du^i} \otimes \widehat{\frac{\partial}{\partial u^j}}}\,.$$

(Φ^j_i) is the *matrix of* $\widetilde{\Phi}$ with respect to the chart $(\mathcal{U}, (u^i)^n_{i=1})$ on M; then the matrix of the semibasic tensor Φ is the $2n \times 2n$ matrix $\begin{pmatrix} 0 & 0 \\ (\Phi^i_j) & 0 \end{pmatrix}$.

Remark. For any vector field $\widetilde{Y}$ along τ we have

$$[\xi, \mathcal{H}\widetilde{Y}] = \mathcal{H} \circ \mathbf{j}[\xi, \mathcal{H}\widetilde{Y}] + \mathbf{i} \circ \mathcal{V}[\xi, \mathcal{H}\widetilde{Y}] = \mathcal{H}\,\mathcal{L}^h_\xi \widetilde{Y} + \mathbf{i}\,\widetilde{\Phi}(\widetilde{Y}).$$

Due to this observation and 3.3, Corollary 5, we obtain the following useful relations:

$$\boxed{\begin{array}{rcl} [\xi, \mathbf{i}\,\widetilde{Y}] & = & -\mathcal{H}\widetilde{Y} + \mathbf{i}\,\mathcal{L}^h_\xi \widetilde{Y}, \\ [\xi, \mathcal{H}\widetilde{Y}] & = & \mathcal{H}(\mathcal{L}^h_\xi \widetilde{Y}) + \mathbf{i}\,\widetilde{\Phi}(\widetilde{Y}) \end{array}}\,.$$

Lemma 1. $\Phi = -\mathbf{v}[\mathbf{h}, \xi] = -([\mathbf{h}, \xi] + \mathbf{F} + J)$.

Proof. Let X be a vector field on M. Since Φ is semibasic, $\Phi(X^{\mathrm{v}}) = 0$. On the other hand, $\Phi(X^h) := \mathbf{i}\,\widetilde{\Phi}(\mathbf{j}\,X^h) = \mathbf{i}\,\widetilde{\Phi}(\widehat{X}) = \mathbf{v}[\xi, X^h] = -\mathbf{v}[\mathbf{h}, \xi](X^h)$, $\mathbf{v}[\mathbf{h}, \xi](X^{\mathrm{v}}) = 0$; whence the first equality. Furthermore,

$$([\mathbf{h}, \xi] + \mathbf{F} + J)(X^{\mathrm{v}}) = -\mathbf{h}[X^{\mathrm{v}}, \xi] + X^h \overset{3.2,\text{ Lemma } 2}{=} -X^h + X^h = 0,$$

$$([\mathbf{h}, \xi] + \mathbf{F} + J)(X^h) = [X^h, \xi] - \mathbf{h}[X^h, \xi] - X^{\mathrm{v}} + X^{\mathrm{v}} = \mathbf{v}[X^h, \xi] = -\Phi(X^h),$$

so the second equality is also valid.

Lemma 2. *For any vector field $\widetilde{X}$ along τ we have*

$$\widetilde{\Phi}(\widetilde{X}) = \overset{\circ}{\mathbf{R}}{}^1(\widetilde{X}, \delta) - \nabla^h_{\widetilde{X}} \mathcal{L}^h_\xi \delta.$$

Proof. Taking into account 3.2, Lemma 3, $\nabla^h_{\widetilde{X}} \mathcal{L}^h_\xi \delta + \widetilde{\Phi}(\widetilde{X}) = \nabla^h_{\widetilde{X}} \mathcal{V}\xi +$ $\mathcal{V}[\xi, \mathcal{H}\widetilde{X}] = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{v}\,\xi] + \mathcal{V}[\xi, \mathcal{H}\widetilde{X}] = \mathcal{V}[\mathcal{H}\widetilde{X}, \xi] - \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{h}\,\xi] +$ $\mathcal{V}[\xi, \mathcal{H}\widetilde{X}] = -\mathcal{V}[\mathcal{H}\widetilde{X}, \mathcal{H} \circ \delta] = \overset{\circ}{\mathbf{R}}{}^1(\widetilde{X}, \delta)$. □

Lemma 3. *The curvature of the horizontal map generated by ξ and the Jacobi endomorphism determined by ξ on TM are related as follows:*

$$3\Omega = -[J, \Phi].$$

Proof. Since the torsion $\mathbf{T} := [J, \mathbf{h}]$ of $\mathcal{H}$ vanishes, Corollary 2(2) in 2.32 implies that $[J, \mathbf{F}] = -\Omega$. Thus Lemma 1 leads to the relation

$$[J, \Phi] = -[J, [\mathbf{h}, \xi]] - [J, \mathbf{F}] - [J, J] = -[J, [\mathbf{h}, \xi]] + \Omega.$$

Applying the graded Jacobi identity and 3.3, Theorem 1(a), we get

$$\begin{aligned} 0 = (-1)^{1\cdot 0}[J, [\mathbf{h}, \xi]] + (-1)^{1\cdot 1}[\mathbf{h}, [\xi, J]] + (-1)^{0\cdot 1}[\xi, [J, \mathbf{h}]] = [J, [\mathbf{h}, \xi]] \\ + [\mathbf{h}, [J, \xi]] = [J, [\mathbf{h}, \xi]] + [\mathbf{h}, 2\mathbf{h} - 1_{TTM}] = [J, [\mathbf{h}, \xi]] - 4\Omega; \end{aligned}$$

therefore $[J, \Phi] = -3\Omega$, as was to be proved.

Corollary 1. *The h-horizontal torsion of the Berwald derivative induced by ξ in $\tau^*\tau$ and the Jacobi endomorphism $\widetilde{\Phi}$ determined by ξ are related as follows:*

$$-3\overset{\circ}{\mathbf{R}}{}^1(\widehat{X},\widehat{Y}) = (\nabla^{\mathrm{v}}_{\widehat{X}}\widetilde{\Phi})(\widehat{Y}) - (\nabla^{\mathrm{v}}_{\widehat{Y}}\widetilde{\Phi})(\widehat{X}) \quad \textit{for all } X, Y \in \mathfrak{X}(M).$$

Proof. In view of the preceding lemma, for any vector fields X, Y on M we have

$$\begin{aligned}
&-3\Omega(X^h,Y^h) = [J,\Phi](X^h,Y^h) = [X^{\mathrm{v}},\Phi(Y^h)] + [\Phi(X^h),Y^{\mathrm{v}}]\\
&+(J\circ\Phi+\Phi\circ J)[X^h,Y^h] - J[\Phi(X^h),Y^h] - J[X^h,\Phi(Y^h)] - \Phi([X^{\mathrm{v}},Y^h])\\
&-\Phi([X^h,Y^{\mathrm{v}}]) = [X^{\mathrm{v}},\Phi(Y^h)] - [Y^{\mathrm{v}},\Phi(X^h)]
\end{aligned}$$

(repeatedly using the fact that Φ is semibasic). Since

$$\Omega(X^h,Y^h) \overset{2.45,\text{ Lemma 1}}{=} (\overset{\circ}{\mathbf{R}}{}^1)_0(X^h,Y^h) = \mathbf{i}\,\overset{\circ}{\mathbf{R}}{}^1(\mathbf{j}\circ\mathcal{H}\widehat{X},\mathbf{j}\circ\mathcal{H}\widehat{Y}) = \mathbf{i}\,\overset{\circ}{\mathbf{R}}{}^1(\widehat{X},\widehat{Y}),$$

$$\Phi(X^h) := (\widetilde{\Phi}_0)(X^h) = \mathbf{i}\,\widetilde{\Phi}(\mathbf{j}\circ\mathcal{H}\widehat{X}) = \mathbf{i}\,\widetilde{\Phi}(\widehat{X}),$$

it follows that

$$-3\overset{\circ}{\mathbf{R}}{}^1(\widehat{X},\widehat{Y}) = \mathcal{V}[X^{\mathrm{v}},\mathbf{i}\,\widetilde{\Phi}(\widehat{Y})] - \mathcal{V}[Y^{\mathrm{v}},\mathbf{i}\,\widetilde{\Phi}(\widehat{X})].$$

Now observe that

$$0 = [J,X^{\mathrm{v}}]\mathcal{H}\widetilde{\Phi}(\widehat{Y}) = [\mathbf{i}\,\widetilde{\Phi}(\widehat{Y}),X^{\mathrm{v}}] - J[\mathcal{H}\widetilde{\Phi}(\widehat{Y}),X^{\mathrm{v}}].$$

Hence

$$\begin{aligned}
\mathcal{V}[X^{\mathrm{v}},\mathbf{i}\,\widetilde{\Phi}(\widehat{Y})] &= \mathcal{V}\circ J[X^{\mathrm{v}},\mathcal{H}\widetilde{\Phi}(\widehat{Y})] = \mathbf{j}[X^{\mathrm{v}},\mathcal{H}\widetilde{\Phi}(\widehat{Y})]\\
&= \nabla_{X^{\mathrm{v}}}(\widetilde{\Phi}(\widehat{Y})) = \big(\nabla^{\mathrm{v}}_{\widehat{X}}\widetilde{\Phi}\big)(\widehat{Y}).
\end{aligned}$$

Similarly, $\mathcal{V}[Y^{\mathrm{v}},\mathbf{i}\,\widetilde{\Phi}(\widehat{X})] = (\nabla^{\mathrm{v}}_{\widehat{Y}}\widetilde{\Phi})(\widehat{X})$. This concludes the proof.

Corollary 2. *If the Jacobi endomorphism determined by ξ is v-parallel, i.e. $\nabla^{\mathrm{v}}\widetilde{\Phi} = 0$, then the Riemann curvature of ∇, as well as the curvature of the horizontal map generated by ξ vanish. We have the same conclusion if $\widetilde{\Phi}$ is basic, i.e. $\widetilde{\Phi} = \widehat{\varphi}$, $\varphi \in \mathcal{T}^1_1(M)$.* □

Proposition. *Suppose that the Berwald derivative arising from the second-order vector field ξ has vanishing Berwald curvature and the Jacobi endomorphism determined by ξ is v-parallel. Then the tension of the horizontal map generated by ξ is parallel.*

Proof. In view of 2.47, Property 4, the vanishing of $\overset{\circ}{\mathbf{P}}$ implies that $\nabla^{\mathrm{v}}\widetilde{\mathbf{t}} = 0$. Also due to the vanishing of $\overset{\circ}{\mathbf{P}}$, from 2.47, Property 3 we obtain

$$0 = (\nabla^h\nabla^{\mathrm{v}}\widetilde{Z})(\widehat{Y},\widehat{X}) - (\nabla^{\mathrm{v}}\nabla^h\widetilde{Z})(\widehat{X},\widehat{Y}) = \nabla_{Y^h}\nabla_{X^{\mathrm{v}}}\widetilde{Z} - (\nabla^{\mathrm{v}}\widetilde{Z})(\nabla_{Y^h}\widehat{X}) \\ - \nabla_{X^{\mathrm{v}}}\nabla_{Y^h}\widetilde{Z},$$

hence for all $X, Y \in \mathfrak{X}(M)$, $\widetilde{Z} \in \mathfrak{X}(\tau)$ we have

$$(*) \qquad \nabla_{X^{\mathrm{v}}}\nabla_{Y^h}\widetilde{Z} = \nabla_{Y^h}\nabla_{X^{\mathrm{v}}}\widetilde{Z} - (\nabla^{\mathrm{v}}\widetilde{Z})(\nabla_{Y^h}\widehat{X}).$$

Since $\widetilde{\Phi}$ is v-parallel, Corollary 1 yields $\overset{\circ}{\mathbf{R}}{}^1 = 0$. Thus, by Lemma 2, $\widetilde{\Phi} = -\nabla^h\mathcal{L}^h_\xi\delta$. Now, for any vector fields X, Y on M we get

$$0 = (\nabla^{\mathrm{v}}\widetilde{\Phi})(\widehat{X},\widehat{Y}) = \big(\nabla_{X^{\mathrm{v}}}\widetilde{\Phi}\big)(\widehat{Y}) = \nabla_{X^{\mathrm{v}}}(\widetilde{\Phi}(\widehat{Y})) = -\nabla_{X^{\mathrm{v}}}\nabla_{Y^h}\mathcal{L}^h_\xi\delta \\ \overset{(*)}{=} (\nabla^{\mathrm{v}}\mathcal{L}^h_\xi\delta)(\nabla_{Y^h}\widehat{X}) - \nabla_{Y^h}\nabla_{X^{\mathrm{v}}}\mathcal{L}^h_\xi\delta \overset{3.4,\text{ Lemma 2}}{=} \widetilde{\mathbf{t}}(\nabla_{Y^h}\widehat{X}) - \nabla_{Y^h}(\widetilde{\mathbf{t}}(\widehat{X})) \\ = \widetilde{\mathbf{t}}(\nabla_{Y^h}\widehat{X}) - (\nabla_{Y^h}\widetilde{\mathbf{t}})\widehat{X} - \widetilde{\mathbf{t}}(\nabla_{Y^h}\widehat{X}) = -(\nabla^h\widetilde{\mathbf{t}})(\widehat{Y},\widehat{X}),$$

therefore $\nabla^h\widetilde{\mathbf{t}} = 0$. This completes the proof.

3.7. The v-linearizability of a second-order vector field.

Definition. A second-order vector field ξ over M is said to be *linearizable in velocities*, or briefly *v-linearizable*, if around any point of M there is a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ such that the forces $G^i := -\frac{1}{2}\xi(u^i)^c$ $(1 \leqq i \leqq n)$ defined by ξ are of the form

$$\boxed{G^i = (A^i_j \circ \tau)y^j + b^i \circ \tau; \quad A^i_j, b^i \in C^\infty(\mathcal{U}) \quad (1 \leqq i, j \leqq n)}.$$

Theorem (E. Martínez and J. Cariñena)**.** *Let ξ be a second-order vector field on the manifold M. The following assertions are equivalent:*

(1) *ξ is v-linearisable.*

(2) *The Berwald derivative induced by ξ has vanishing curvature, i.e. its Riemann curvature and Berwald curvature vanish.*

(3) *The tension of the horizontal map generated by ξ is basic and the Berwald derivative induced by ξ has vanishing Riemann curvature.*

(4) *The Berwald derivative induced by ξ is h-basic and its base covariant derivative has vanishing curvature.*

Proof. Our reasoning follows the scheme

$$(1) \implies (2) \iff (3) \iff (4) \implies (1).$$

(1) $\implies$ *(2)* If ξ is v-linearizable, then in a suitable induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ the coordinate expression of ξ is

$$\xi \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}; \quad G^i = (A^i_j \circ \tau) y^j + b^i \circ \tau \quad (1 \leqq i \leqq n)$$

(see the above box-formula). According to the coordinate descriptions presented in 2.32 and 3.3, the Christoffel symbols of the horizontal map and the Berwald derivative generated by ξ are

$$G^i_j = A^i_j \circ \tau \quad \text{and} \quad G^i_{jk} = 0 \quad (1 \leqq i, j, k \leqq n),$$

respectively. Thus for all $i, j, k \in \{1, \dots, n\}$ we have

$$\overset{\circ}{\mathbf{P}} \left(\widehat{\tfrac{\partial}{\partial u^i}}, \widehat{\tfrac{\partial}{\partial u^j}} \right) \widehat{\tfrac{\partial}{\partial u^k}} \overset{2.47,\ \mathbf{3}}{=} \mathcal{V} \left[\left[\left(\tfrac{\partial}{\partial u^i} \right)^h, \tfrac{\partial}{\partial y^j} \right], \tfrac{\partial}{\partial y^k} \right]$$
$$= \mathcal{V} \left[\left[\tfrac{\partial}{\partial x^i} - G^\ell_i \tfrac{\partial}{\partial y^\ell}, \tfrac{\partial}{\partial y^j} \right], \tfrac{\partial}{\partial y^k} \right] = \mathcal{V} \left[G^\ell_{ji} \tfrac{\partial}{\partial y^\ell}, \tfrac{\partial}{\partial y^k} \right] = 0,$$

therefore the Berwald curvature of ∇ vanishes. Similarly, an easy calculation shows that $\overset{\circ}{\mathbf{R}} = 0$.

(2) $\implies$ *(3)* Indeed, according to 2.49, Example, the tension $\widetilde{\mathbf{t}}$ is basic if, and only if, ∇ is h-basic, i.e. $\overset{\circ}{\mathbf{P}} = 0$.

(3) $\implies$ *(4)* This is also an immediate consequence of the results in 2.49, Example.

(4) $\implies$ *(1)* Let $\overset{\circ}{D}$ be the base covariant derivative belonging to ∇. Since the horizontal map generated by ξ has vanishing torsion, $\overset{\circ}{D}$ is also torsion-free (see 2.49, Example). As a torsion-free covariant derivative operator on M with zero curvature, according to 1.48, Proposition, $\overset{\circ}{D}$ is arising from a locally affine structure $(\mathcal{U}_\alpha, (u^i_\alpha)_{i=1}^n)_{\alpha \in A}$ on M such that for any vector fields X and $Y = Y^i \frac{\partial}{\partial u^i_\alpha}$ on $\mathcal{U}_\alpha$ we have $\overset{\circ}{D}_X Y = X(Y^i) \frac{\partial}{\partial u^i_\alpha}$. Then, in particular, $\overset{\circ}{D}_{\frac{\partial}{\partial u^i_\alpha}} \frac{\partial}{\partial u^j_\alpha} = 0$ $(1 \leqq i, j \leqq n,\ \alpha \in A)$. Now choose an arbitrary member $(\mathcal{U}, (u^i)_{i=1}^n)$ of the locally affine structure (the family index is omitted for simplicity). Let, as usual, $\xi \restriction \tau^{-1}(\mathcal{U}) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$. Since ∇ is h-basic, for any indices $i, j \in \{1, \dots, n\}$ we have

$$0 = \left(\overset{\circ}{D}_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} \right)^{\mathrm{v}} = \left[\left(\frac{\partial}{\partial u^i} \right)^h, \left(\frac{\partial}{\partial u^j} \right)^{\mathrm{v}} \right] = \left[\frac{\partial}{\partial x^i} - G^k_i \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j} \right] = G^k_{ji} \frac{\partial}{\partial y^k},$$

therefore $G^k_{ij} = \frac{\partial G^k_j}{\partial y^i} = 0$ $(1 \leqq i, j, k \leqq 0)$. This shows that the Christoffel symbols of $\mathcal{H}$ are vertical lifts. In other words, there exist smooth functions $A^i_j \in C^\infty(\mathcal{U})$ such that

$$G^i_j = A^i_j \circ \tau, \qquad 1 \leqq i, j \leqq n.$$

Since $G^i_j := \frac{\partial G^i}{\partial y^j}$, we conclude immediately that

$$G^i = (A^i_j \circ \tau) y^j + b^i \circ \tau; \quad b^i \in C^\infty(\mathcal{U}), \qquad 1 \leqq i \leqq n.$$

This completes the proof of the Theorem.

3.8. The linearizability of second-order vector fields.

Lemma. *Let $\widetilde{\Phi}$ be the Jacobi endomorphism determined by a second-order vector field ξ. Over any induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)^n_{i=1}, (y^i)^n_{i=1}))$ we have*

$$(\nabla^{\mathrm{v}} \widetilde{\Phi}) \left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}} \right) = \frac{\partial \Phi^k_j}{\partial y^i} \widehat{\frac{\partial}{\partial u^k}},$$

$$(\nabla^h \widetilde{\Phi}) \left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}} \right) = \left(\frac{\partial \Phi^k_j}{\partial x^i} - G^\ell_i \frac{\partial \Phi^k_j}{\partial y^\ell} + G^k_{i\ell} \Phi^\ell_j - G^\ell_{ij} \Phi^k_\ell \right) \widehat{\frac{\partial}{\partial u^k}} \quad (1 \leqq i, j \leqq n),$$

where (Φ^j_i) is the matrix of $\widetilde{\Phi}$ with respect to the chart $(\mathcal{U}, (u^i)^n_{i=1})$.

Proof. Immediate calculation.

Definition. A second-order vector field ξ on M is said to be *linearizable* if around any point of M there is a chart such that the forces defined by ξ are of the form

$$\boxed{G^i = A^i_j y^j + B^i_j x^j + C^i; \quad A^i_j, B^i_j, C^i \in \mathbb{R}, \quad 1 \leqq i, j \leqq n}.$$

Theorem (E. Martínez and J. Cariñena)**.** *A second-order vector field ξ is linearizable if, and only if, it induces an h-basic Berwald derivative and the Jacobi endomorphism determined by ξ is parallel with respect to the Berwald derivative.*

Proof. (a) *Necessity.* Suppose that ξ is linearizable. Then, obviously, ξ is v-linearizable as well. Thus the preceding theorem guarantees that ξ induces an h-basic Berwald derivative in $\tau^*\tau$. It remains only to check that $\nabla \widetilde{\Phi} = 0$. This can be done by an easy calculation.

Let $(\mathcal{U}, (u^i)_{i=1}^n)$ be a chart on M such that the forces defined by ξ with respect to this chart have the form given by the box-formula. Then

$$G_i^j := \frac{\partial G^j}{\partial y^i} = A_i^j,\ G_{ki}^j := \frac{\partial G_i^j}{\partial y^k} = 0,\ \frac{\partial G^j}{\partial x^i} = B_i^j,\ \frac{\partial G_i^j}{\partial x^k} = 0 \quad (1 \leqq i, j, k \leqq n),$$

therefore the components of the Jacobi endomorphism $\widetilde{\Phi}$ (see 3.6) are the constant functions

$$\Phi_i^j = 2B_i^j - A_k^j A_i^k \qquad (1 \leqq i, j \leqq n).$$

Taking into account the preliminary Lemma, it follows at once that $\nabla\widetilde{\Phi} = 0$.

(b) *Sufficiency.* Suppose that the Berwald derivative ∇ induced by ξ is h-basic with the base covariant derivative $\overset{\circ}{D}$, and let the Jacobi endomorphism $\widetilde{\Phi}$ determined by ξ be parallel. Then by 3.7, Corollary 2, $\overset{\circ}{\mathbf{R}} = 0$. Hence, according to 2.49, Example, $R^{\overset{\circ}{D}} = 0$. Since the torsion of the horizontal map generated by ξ vanishes, we also have $T^{\overset{\circ}{D}} = 0$. Thus, as in the proof of the preceding theorem, $\overset{\circ}{D}$ is arising from a locally affine structure on M. It follows that around any point of M there is a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ such that $\overset{\circ}{D}_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0$, and consequently $G_{jk}^i = \frac{\partial^2 G^i}{\partial y^j \partial y^k} = 0$ $(1 \leqq i, j, k \leqq n;$ $G^i := -\frac{1}{2}\xi(u^i)^c$, as above). Then, over $\tau^{-1}(\mathcal{U})$, the coordinate expression of the tension $\widetilde{\mathbf{t}}$ reduces to

$$\widetilde{\mathbf{t}} = -G_i^j \widehat{du^i} \otimes \widehat{\frac{\partial}{\partial u^j}}$$

(cf. 2.32, **4**). In view of the Proposition in 3.6, $\widetilde{\mathbf{t}}$ is parallel, i.e. $\nabla\widetilde{\mathbf{t}} = 0$. Hence for all $i, k \in \{1, \dots, n\}$ we have

$$0 = (\nabla^h \widetilde{\mathbf{t}})\left(\widehat{\frac{\partial}{\partial u^k}}, \widehat{\frac{\partial}{\partial u^i}}\right) = \left(\nabla_{\left(\frac{\partial}{\partial u^k}\right)^h} \widetilde{\mathbf{t}}\right)\widehat{\frac{\partial}{\partial u^i}} = -\nabla_{\left(\frac{\partial}{\partial u^k}\right)^h} G_i^j \widehat{\frac{\partial}{\partial u^j}} - \widetilde{\mathbf{t}}\left(G_{ki}^j \widehat{\frac{\partial}{\partial u^j}}\right)$$

$$= -\frac{\partial G_i^j}{\partial x^k} \frac{\partial}{\partial y^j},$$

therefore the functions $G_i^j = \frac{\partial G^j}{\partial y^i}$ $(1 \leqq i, j \leqq n)$ are constant over $\tau^{-1}(\mathcal{U})$.

Since $\frac{\partial G_j^i}{\partial x^k} = \frac{\partial G_j^i}{\partial y^k} = 0$ $(1 \leqq i, j \leqq n)$, the components of $\widetilde{\Phi}$ (see 3.6) reduce to

$$\Phi_i^j = 2\frac{\partial G^j}{\partial x^i} - G_k^j G_i^k \quad (1 \leqq i, j \leqq n).$$

Applying the Lemma, by the condition $\nabla\widetilde{\Phi} = 0$ we obtain that

$$\frac{\partial \Phi_i^j}{\partial x^k} = \frac{\partial \Phi_i^j}{\partial y^k} = 0, \qquad 1 \leqq i, j, k \leqq n;$$

hence the functions Φ_i^j are constant. This implies that $\frac{\partial G^j}{\partial x^i}$ $(1 \leqq i, j \leqq n)$ are also constant functions.

From these we infer immediately that the forces defined by ξ have the desired form

$$G^i = A^i_j y^j + B^i_j x^j + C^i; \quad A^i_j, B^i_j, C^i \in \mathbb{R}, \qquad 1 \leqq i, j \leqq n. \qquad \square$$

C. Second-order vector fields generated by Finsler metrics

In this section by a *Finsler metric* we mean a pseudo-Riemannian metric in the deleted bundle $\overset{\circ}{\tau}{}^*\tau$. Thus a Finsler metric $g \in \Gamma(\mathbf{S}_2(\overset{\circ}{\tau}{}^*\tau))$ is a smooth map $v \in \overset{\circ}{T}M \mapsto g_v \in L^2_{\mathrm{sym}}(T_{\overset{\circ}{\tau}(v)}M)$ such that g_v is non-degenerate. As we have learnt, g may also be interpreted as a non-degenerate, symmetric $C^\infty(TM)$-bilinear map $\mathfrak{X}(\overset{\circ}{\tau}) \times \mathfrak{X}(\overset{\circ}{\tau}) \longrightarrow C^\infty(TM)$. According to 2.23, a pair (M, g) is called a *generalized Finsler manifold*, if g is a Finsler metric in $\overset{\circ}{\tau}{}^*\tau$. Then, by a slight abuse of language, we also say that M is a *generalized Finsler manifold* with the Finsler metric g. We shall frequently need the smooth function

$$E := \frac{1}{2} g(\delta, \delta) : \overset{\circ}{T}M \longrightarrow \mathbb{R};$$

it is called the *(absolute) energy* of (M, g).

3.9. The regularity conditions of R. Miron.

Definition 1. A Finsler metric $g \in \mathcal{T}^0_2(\overset{\circ}{\tau})$ is said to be *regular in Miron's sense*, or briefly *regular*, if its first Cartan tensor has the following properties:

M.reg.1. $\mathcal{C}_\flat(\widetilde{X}, \delta, \delta) = 0$ *for all* $\widetilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})$.

M.reg.2. *The map* $\widetilde{A} : \widetilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \widetilde{A}(\widetilde{X}) := \widetilde{X} + \mathcal{C}(\widetilde{X}, \delta) \in \mathfrak{X}(\overset{\circ}{\tau})$ *is injective.*

Coordinate description. Let $(\mathcal{U}, (u^i)_{i=1}^n)$ be a chart on M and consider the induced chart $(\overset{\circ}{\tau}{}^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ on $\overset{\circ}{T}M$. Taking into account 2.50, Remark 3(c), for all $i \in \{1, \dots, n\}$ we have

$$\mathcal{C}_\flat\left(\widehat{\frac{\partial}{\partial u^i}}, y^j \widehat{\frac{\partial}{\partial u^j}}, y^k \widehat{\frac{\partial}{\partial u^k}}\right) = y^j y^k \frac{\partial g_{jk}}{\partial y^i}; \tag{1}$$

$$\begin{aligned}\widetilde{A}\left(\widehat{\frac{\partial}{\partial u^i}}\right) &= \widehat{\frac{\partial}{\partial u^i}} + \mathcal{C}\left(\widehat{\frac{\partial}{\partial u^i}}, y^j \widehat{\frac{\partial}{\partial u^j}}\right) = \widehat{\frac{\partial}{\partial u^i}} + y^j \frac{\partial g_{jk}}{\partial y^i} g^{\ell k} \widehat{\frac{\partial}{\partial u^\ell}} \\ &= \left(\delta^\ell_i + y^j \frac{\partial g_{jk}}{\partial y^i} g^{\ell k}\right) \widehat{\frac{\partial}{\partial u^\ell}}. \end{aligned}\tag{2}$$

Thus the regularity conditions take the following local form:

$$\boxed{\left.\begin{array}{l} \textbf{M.reg.1} \iff y^j y^k \dfrac{\partial g_{jk}}{\partial y^i} = 0 \quad (1 \leqq i \leqq n) \\[2ex] \textbf{M.reg.2} \iff \det\left(\delta^\ell_i + y^j \dfrac{\partial g_{jk}}{\partial y^i} g^{\ell k}\right) \neq 0 \end{array}\right\} \text{ over any induced chart}}\,.$$

Note. Conditions **M.reg.1, 2** were introduced by R. Miron in [57], in the above coordinate form.

Lemma 1. *If the Finsler metric g satisfies the condition* **M.reg.1**, *then the energy $\frac{1}{2}g(\delta,\delta)$ is positive-homogeneous of degree 2.*

Proof. $CE = \frac{1}{2}C(g(\delta,\delta)) = \frac{1}{2}\big((\nabla_C g)(\delta,\delta) + 2g(\nabla_C\delta,\delta)\big) = \frac{1}{2}\mathcal{C}_\flat(\delta,\delta,\delta) + g(\delta,\delta) \overset{\text{M.reg.1}}{=} g(\delta,\delta) = 2E.$ □

Definition 2. Let (M,g) be a generalized Finsler manifold.

(1) By the *canonical one-form* of (M,g) we mean the one-form

$$\theta_g : \xi \in \mathfrak{X}(\overset{\circ}{T}M) \mapsto \theta_g(\xi) := g(\mathbf{j}\,\xi,\delta) \in C^\infty(\overset{\circ}{T}M) \quad \text{on } \overset{\circ}{T}M,$$

or the one-form

$$\widetilde{\theta}_g : \widetilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \widetilde{\theta}_g(\widetilde{X}) := g(\widetilde{X},\delta) \in C^\infty(\overset{\circ}{T}M) \quad \text{along } \overset{\circ}{\tau}.$$

(2) The differential $\omega_g := d\theta_g$ of the canonical one-form on $\overset{\circ}{T}M$ is said to be the *fundamental two-form* of (M,g).

Lemma 2. *If ω_g is the fundamental two-form of the generalized Finsler manifold (M,g), then*

$$\omega_g(J\xi,\eta) = g(\widetilde{A}(\mathbf{j}\,\xi),\mathbf{j}\,\eta) \quad \text{for all } \xi,\eta \in \mathfrak{X}(\overset{\circ}{T}M),$$

where $\widetilde{A}$ is the $(1,1)$ tensor defined in **M.reg.2**.

Proof. We may assume that $\xi = X^c$, $X \in \mathfrak{X}(M)$. Then $J\xi = X^{\mathrm{v}}$, $\mathbf{j}\,\xi = \widehat{X}$, and a straightforward calculation leads to the result:

$$\begin{aligned}
&\omega_g(X^{\mathrm{v}},\eta) = (d\theta_g)(X^{\mathrm{v}},\eta) = X^{\mathrm{v}}\theta_g(\eta) - \eta\theta_g(X^{\mathrm{v}}) - \theta_g([X^{\mathrm{v}},\eta]) = X^{\mathrm{v}}g(\mathbf{j}\,\eta,\delta)\\
&\quad - g(\mathbf{j}[X^{\mathrm{v}},\eta],\delta) = (\nabla_{X^{\mathrm{v}}}g)(\mathbf{j}\,\eta,\delta) + g(\nabla_{X^{\mathrm{v}}}\mathbf{j}\,\eta,\delta) + g(\mathbf{j}\,\eta,\nabla_{X^{\mathrm{v}}}\delta)\\
&\quad - g\big(\mathbf{j}[X^{\mathrm{v}},\mathcal{H}\circ\mathbf{j}\,\eta],\delta\big) = \mathcal{C}_\flat(\widehat{X},\mathbf{j}\,\eta,\delta) + g(\nabla_{X^{\mathrm{v}}}\mathbf{j}\,\eta,\delta) + g(\mathbf{j}\,\eta,\widehat{X})\\
&\quad - g(\nabla_{X^{\mathrm{v}}}\mathbf{j}\,\eta,\delta) = \mathcal{C}_\flat(\widehat{X},\delta,\mathbf{j}\,\eta) + g(\widehat{X},\mathbf{j}\,\eta) = g(\mathcal{C}(\widehat{X},\delta),\mathbf{j}\,\eta) + g(\widehat{X},\mathbf{j}\,\eta)\\
&\quad = g(\widetilde{A}(\widehat{X}),\mathbf{j}\,\eta).
\end{aligned}$$

□

Corollary 1. *The fundamental two-form of a generalized Finsler manifold is non-degenerate if, and only if, the metric satisfies the regularity condition* **M.reg.2**.

Proof. Necessity. Suppose the assertion is false, i.e. ω_g is non-degenerate, but the map $\widetilde{A} : \mathfrak{X}(\overset{\circ}{\tau}) \longrightarrow \mathfrak{X}(\overset{\circ}{\tau})$ is not injective. Then there is a non-zero vector field X on M such that $\widetilde{A}(\widehat{X}) = 0$, and so for all $\eta \in \mathfrak{X}(\overset{\circ}{T}M)$ we have $0 = g(\widetilde{A}(\widehat{X}), \mathbf{j}\,\eta) = \omega_g(X^{\mathrm{v}}, \eta)$. Since $X^{\mathrm{v}} \neq 0$, this is a contradiction.

Sufficiency. Arguing locally, choose an induced chart $(\overset{\circ}{\tau}{}^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ on $\overset{\circ}{T}M$, as usual. Let $\widetilde{A}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = A_i^j \widehat{\frac{\partial}{\partial u^j}}$ $(1 \leqq i \leqq n)$. Then, by Lemma 2,

$$\omega_g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = -\omega_g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^i}\right) = -g\left(\widetilde{A}\left(\widehat{\frac{\partial}{\partial u^j}}\right), \mathbf{j}\,\frac{\partial}{\partial x^i}\right)$$

$$= -A_j^k g\left(\widehat{\frac{\partial}{\partial u^k}}, \widehat{\frac{\partial}{\partial u^i}}\right) = -A_j^k g_{ki},$$

$$\omega_g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g\left(\widetilde{A}\left(\widehat{\frac{\partial}{\partial u^i}}\right), \mathbf{j}\left(\frac{\partial}{\partial y^j}\right)\right) = 0 \quad (1 \leqq i, j \leqq n).$$

Thus, with respect to the given chart, ω_g can by represented by a $2n \times 2n$ matrix of form

$$\begin{pmatrix} (B_{ij}) & (-g_{ik}A_j^k) \\ (g_{kj}A_i^k) & 0 \end{pmatrix}.$$

Since $\det(A_i^k) \neq 0$ by **M.reg.2**, this matrix is regular, and hence ω_g is non-degenerate. $\square$

Lemma 3. *Let E be the energy of the generalized Finsler manifold (M, g), and let $g_E := \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E$. Under the condition* **M.reg.1** *we have:*

(i) $$g(\widetilde{X}, \delta) = (\mathbf{i}\,\widetilde{X})E = g_E(\widetilde{X}, \delta) \quad \textit{for all } \widetilde{X} \in \mathfrak{X}(\overset{\circ}{\tau}).$$

(ii) $$g_E(\widehat{X}, \widehat{Y}) = g(\widetilde{A}(\widehat{X}), \widehat{Y}) \quad \textit{for all } X, Y \in \mathfrak{X}(M).$$

Proof. Starting with the condition **M.reg.1**, $0 = \mathcal{C}_\flat(\widetilde{X}, \delta, \delta) = (\nabla_{\mathbf{i}\,\widetilde{X}} g)(\delta, \delta) = \mathbf{i}\,\widetilde{X}(g(\delta, \delta)) - 2g(\nabla_{\mathbf{i}\,\widetilde{X}}\delta, \delta) = 2(\mathbf{i}\,\widetilde{X})E - 2g(\widetilde{X}, \delta)$; hence $g(\widetilde{X}, \delta) = (\mathbf{i}\,X)E$. Moreover, since E is positive-homogeneous of degree 2 by Lemma 1, $g_E(\widetilde{X}, \delta) = (\nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E)(\widetilde{X}, \delta) = (\nabla_{\mathbf{i}\,\widetilde{X}}(\nabla^{\mathrm{v}}E))(\delta) = \nabla_{\mathbf{i}\,\widetilde{X}}\nabla_C E - \nabla^{\mathrm{v}}E(\nabla_{\mathbf{i}\,\widetilde{X}}\delta) = 2(\mathbf{i}\,\widetilde{X})E - \nabla^{\mathrm{v}}E(\widetilde{X}) = (\mathbf{i}\,\widetilde{X})E$, as desired. Thus part (i) is proved.

Relation (ii) can also easily be deduced. We have on the one hand $g_E(\widehat{X},\widehat{Y}) = (\nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E)(\widehat{X},\widehat{Y}) \overset{2.50,\,\text{Remark 1}}{=} X^{\mathrm{v}}(Y^{\mathrm{v}}E) \overset{\text{(i)}}{=} X^{\mathrm{v}}g(\widehat{Y},\delta)$. On the other hand, $g(\widetilde{A}(\widehat{X}),\widehat{Y}) \overset{\text{Lemma 2}}{=} \omega_g(X^{\mathrm{v}},Y^c) = d\theta_g(X^{\mathrm{v}},Y^c) = X^{\mathrm{v}}\theta_g(Y^c) - Y^c\theta_g(X^{\mathrm{v}}) - \theta_g([X^{\mathrm{v}},Y^c]) = X^{\mathrm{v}}\theta_g(Y^c) = X^{\mathrm{v}}g(\widehat{Y},\delta)$, whence (ii). □

Lemma 4. *The following properties are equivalent for a generalized Finsler manifold* (M,g)*:*

(1) (M,g) *satisfies the regularity condition* **M.reg.1**.

(2) *The canonical one-form can be expressed by the energy function* $E = \frac{1}{2}g(\delta,\delta)$ *as follows:*

$$\boxed{\widetilde{\theta}_g = d^{\mathrm{v}}E} \quad \textit{or} \quad \boxed{\theta_g = (d^{\mathrm{v}}E)_0 = d^{\mathrm{v}}E\circ\mathbf{j} = d_JE}.$$

Proof. We have learnt in the proof of Lemma 3 that

$$\mathcal{C}_\flat(\mathbf{j}\,\xi,\delta,\delta) = 2(J\xi)E - 2g(\mathbf{j}\,\xi,\delta) \qquad \text{for all } \xi\in\mathfrak{X}(\overset{\circ}{T}M).$$

Thus

$$\begin{aligned}
\textbf{M.reg.1} \quad &\Longleftrightarrow \quad \forall\xi\in\mathfrak{X}(\overset{\circ}{T}M) : \mathcal{C}_\flat(\mathbf{j}\,\xi,\delta,\delta) = 0\\
&\Longleftrightarrow \quad \forall\xi\in\mathfrak{X}(\overset{\circ}{T}M) : g(\mathbf{j}\,\xi,\delta) = (J\xi)E\\
&\Longleftrightarrow \quad \forall\xi\in\mathfrak{X}(\overset{\circ}{T}M) : \theta_g(\xi) = (d_JE)(\xi) \Longleftrightarrow \theta_g = d_JE\\
&\overset{2.38,\,\text{Prop. 1}}{\Longleftrightarrow} \widetilde{\theta}_g = d^{\mathrm{v}}E.
\end{aligned}$$

□

Definition 3. (1) A one-form α on TM (or on $\overset{\circ}{T}M$) is said to be a *Hilbert one-form* (other terms: *Poincaré* or *Poincaré-Cartan one-form*), if it is semibasic and d_J-closed, i.e. if $i_J\alpha = 0$ and $d_J\alpha = 0$.

(2) A one-form $\widetilde{\alpha}$ along τ (or along $\overset{\circ}{\tau}$) is called a Hilbert one-form, if $(\widetilde{\alpha})_0$ is a Hilbert one-form.

Remark. The vertical lift α^{v} of a one-form α on M (or, what is essentially the same, a basic one-form $\widetilde{\alpha}\in\mathcal{A}^1(\tau)$) is a trivial example of Hilbert one-forms. A one-form $\widetilde{\alpha}$ along τ is a Hilbert one-form if, and only if, $d^{\mathrm{v}}\widetilde{\alpha} = 0$. Indeed, $(d^{\mathrm{v}}\widetilde{\alpha})_0 = d_J(\widetilde{\alpha})_0$ by 2.38, Proposition 1.

Corollary 2. *If a generalized Finsler manifold satisfies the condition* **M.reg.1***, then its canonical one-form is a Hilbert one-form.*

Proof. This is an immediate consequence of Lemma 4 and the property $d^{\mathrm{v}} \circ d^{\mathrm{v}} = 0$ (see 2.34). □

3.10. Semi-Finsler manifolds.

Lemma 1. *Let g be a variational Finsler metric, namely $g = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}} L$, $L \in C^\infty(\overset{\circ}{T}M)$. If $E_L := CL - L$, then the canonical one-form of the generalized Finsler manifold (M, g) can be expressed as follows:*

$$\boxed{\theta_g = d_J E_L = (d^{\mathrm{v}} E_L)_0}\,.$$

Thus θ_g is a Hilbert one-form. The fundamental two-form $\omega_g := d\theta_g$ has the property

$$\boxed{i_J \omega_g = 0}\,.$$

Proof. (1) For any vector field ξ on TM we have

$$\begin{aligned}\theta_g(\xi) := g(\mathbf{j}\,\xi, \delta) = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}} L(\mathbf{j}\,\xi, \delta) = \big(\nabla_{J\xi}(\nabla^{\mathrm{v}} L)\big)(\delta) = \nabla_{J\xi}\nabla_C L \\ - \nabla^{\mathrm{v}} L(\nabla_{J\xi}\delta) = \nabla_{J\xi} CL - \nabla^{\mathrm{v}} L(\mathbf{j}\,\xi) = J\xi(CL) - (J\xi)L \\ = J\xi(E_L) = (d_J E_L)(\xi).\end{aligned}$$

(2) $$\begin{aligned}i_J\omega_g = i_J d d_J E_L \overset{2.27}{=} -i_J d_J d E_L \overset{2.31,\,\mathrm{Cor.}\,2}{=} d_J i_J d E_L \overset{2.31,\,\mathbf{5}(2)}{=} \\ = d_J^2 E_L \overset{2.31,\,\mathbf{2}}{=} 0.\end{aligned}$$
□

Remark 1. With the notations of the lemma, the function E_L is said to be the *energy function associated to L.*

Corollary. *If g is a variational Finsler metric, namely $g = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}} L$, and satisfies* **M.reg.1***, then the absolute energy $E = \frac{1}{2} g(\delta, \delta)$ and the energy function E_L associated to L differ only in a vertical lift.*

Proof. By the preceding lemma and by 3.9, Lemma 4 we obtain $\theta_g = d_J E = d_J E_L$, hence $d_J(E - E_L) = 0$. This implies by 2.31, Lemma (i) that $E - E_L$ is the vertical lift of a smooth function on M. □

Definition. A generalized Finsler manifold (M, g) is said to be a *semi-Finsler manifold* if g is variational and satisfies the condition **M.reg.2**. If $g = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}} L$, then the function $\widetilde{E}_L := CE_L - E_L$ is called *principal energy.*

Note. The attribute 'semi' refers to the fact that homogeneity properties concerning the metric are not required.

Lemma 2. *For the principal energy we have* $d_J\widetilde{E}_L \neq 0$, *and hence* $d\widetilde{E}_L \neq 0$.

Proof. Let g be the given variational Finsler metric. Since **M.reg.2** is assumed, the fundamental two-form ω_g is non-degenerate. So we have

$$\begin{aligned} 0 \neq i_C\omega_g &\overset{\text{Lemma 1}}{=} i_C dd_J E_L \overset{2.27}{=} -i_C d_J dE_L \overset{2.31,\,\text{Cor. 1}}{=} -i_J dE_L \\ &+ d_J i_C dE_L = -d_J E_L + d_J(CE_L) = d_J\widetilde{E}_L, \end{aligned}$$

as we claimed.

Proposition 1 and definition. *Suppose that* (M, g) *is a semi-Finsler manifold;* $g = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}L$, $E_L := CL - L$, $\widetilde{E}_L = CE_L - E_L$. *Then there is a unique second-order vector field* ξ *on* $\overset{\circ}{T}M$ *such that*

$$i_\xi\omega_g = -d\widetilde{E}_L.$$

ξ *is called the* canonical second-order vector field *for the semi-Finsler manifold* (M, g) *(or simply for* g*).*

Proof. Due to the non-degeneracy of ω_g and Lemma 2, there is a unique vector field ξ on $\overset{\circ}{T}M$ satisfying $i_\xi\omega_g = -d\widetilde{E}_L$. So our only task is to show that ξ is a second-order vector field, i.e. $J\xi = C$. This can be done by a quite immediate calculation:

$$\begin{aligned} i_{J\xi}\omega_g &\overset{2.29,\ \text{Cor.(i)}}{=} i_\xi i_J\omega_g - i_J i_\xi\omega_g \overset{\text{Lemma 1}}{=} -i_J i_\xi\omega_g = i_J d\widetilde{E}_L \\ &= d_J d_C E_L - d_J E_L \overset{2.31,\ \text{Cor. 1}}{=} d_C d_J E_L + d_J E_L - d_J E_L \\ &\overset{1.40(3)}{=} i_C dd_J E_L + di_C d_J E_L = i_C\omega_g; \end{aligned}$$

thus, again by the non-degeneracy of ω_g, $J\xi = C$. □

Remark 2. Keeping the conditions and notations of the Proposition, we have $d_\xi\omega_g = 0$. Indeed, $d_\xi\omega_g \overset{1.40(3)}{=} i_\xi d\omega_g + di_\xi\omega_g = -dd\widetilde{E}_L = 0$.

Lemma 3. *Let* (M, g) *be a generalized Finsler manifold,* ω_g *the fundamental two-form of* (M, g). *Assume that* ξ *is a second-order vector field on* $\overset{\circ}{T}M$ *and*

h *is the horizontal projector arising from* ξ *according to* 3.3*, Theorem 1. If* $\Gamma := [J, \xi]$*, then the following properties are equivalent:*

(1) $i_\Gamma \omega_g = 0$.

(2) $\omega_g(\mathbf{h}\,\eta, \mathbf{h}\,\zeta) = 0$ *for all* $\eta, \zeta \in \mathfrak{X}(\overset{\circ}{T}M)$.

Proof. From the theorem quoted we know that $\Gamma = 2\mathbf{h} - 1_{TTM}$, hence $\Gamma \circ \mathbf{h} = \mathbf{h}$, $\Gamma \circ \mathbf{v} = -\mathbf{v}$. Thus, applying part (4) of the Theorem in 2.26,

$$i_\Gamma \omega_g(\mathbf{h}\,\eta, \mathbf{h}\,\zeta) = \omega_g(\Gamma \circ \mathbf{h}\,\eta, \mathbf{h}\,\zeta) + \omega_g(\mathbf{h}\,\eta, \Gamma \circ \mathbf{h}\,\zeta) = 2\omega_g(\mathbf{h}\,\eta, \mathbf{h}\,\zeta),$$

$$i_\Gamma \omega_g(\mathbf{h}\,\eta, \mathbf{v}\,\zeta) = \omega_g(\mathbf{h}\,\eta, \mathbf{v}\,\zeta) - \omega_g(\mathbf{h}\,\eta, \mathbf{v}\,\zeta) \overset{3.9,\,\text{Lemma 2}}{=} 0,$$

$$i_\Gamma \omega_g(\mathbf{v}\,\eta, \mathbf{v}\,\zeta) = -2\omega_g(\mathbf{v}\,\eta, \mathbf{v}\,\zeta) = 0,$$

whence the assertion.

Remark 3. Property (2) in Lemma 3 is usually expressed as follows: 'the horizontal subbundle $\operatorname{Im}\mathbf{h}$ is *Lagrangian* for ω_g'.

Proposition 2. *Let* (M, g) *be a semi-Finsler manifold, and let* ξ *be the canonical second-order vector field for* g*. Then the horizontal subbundle* $\operatorname{Im}\mathbf{h} \subset T\overset{\circ}{T}M$ *is Lagrangian for the fundamental two-form* ω_g*., i.e.*

$$\omega_g(\mathbf{h}\,\eta, \mathbf{h}\,\zeta) = 0$$

for all $\eta, \zeta \in \mathfrak{X}(\overset{\circ}{T}M)$.

Proof. As above, we assume that $g = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}L$. Let

$$\alpha := i_\xi \omega_g + d\widetilde{E}_L = i_\xi dd_J E_L + d(CE_L) - dE_L.$$

According to 3.3, Corollary 2,

$$i_\xi dd_J E_L + d(CE_L) - dE_L = d_J i_\xi dE_L - 2d_{\mathbf{h}} E_L.$$

Thus, since $0 = d_{[J,\mathbf{h}]} = d_J \circ d_{\mathbf{h}} + d_{\mathbf{h}} \circ d_J$, it follows that

$$\begin{aligned} d_J \alpha &= -2d_J d_{\mathbf{h}} E_L = 2d_{\mathbf{h}} d_J E_L = 2i_{\mathbf{h}} dd_J E_L - 2di_{\mathbf{h}} d_J E_L \\ &= 2i_{\mathbf{h}}\omega_g - 2\omega_g \overset{2.26,\,\text{Th.}}{=} i_{2\mathbf{h}-1_{TTM}}\omega_g = i_\Gamma \omega_g \end{aligned}$$

(taking into account that $d_J E_L$ is semibasic, and so $i_{\mathbf{h}} d_J E_L = d_J E_L$). But $\alpha = 0$, therefore $i_\Gamma \omega_g = 0$, which implies by Lemma 3 that $\operatorname{Im}\mathbf{h}$ is a Lagrangian subbundle for ω_g. $\square$

Proposition 3. *Hypothesis as in Proposition 2. For any basic vector fields $\widehat{X}$, $\widehat{Y}$ in $\mathfrak{X}(\overset{\circ}{\tau})$ we have*

$$(\mathcal{L}_\xi^h g)(\widehat{X},\widehat{Y}) + (\mathcal{L}_\xi^h \mathcal{C}_\flat)(\widehat{X},\widehat{Y},\delta) + \mathcal{C}_\flat(\widehat{X},\widehat{Y},\mathcal{V}\xi) = 0.$$

Proof. By Remark 2, the fundamental two-form ω_g has the property $d_\xi\omega_g = 0$. Thus

$$0 = (d_\xi\omega_g)(X^{\mathrm{v}},Y^h) = \xi\omega_g(X^{\mathrm{v}},Y^h) - \omega_g([\xi,X^{\mathrm{v}}],Y^h) - \omega_g(X^{\mathrm{v}},[\xi,Y^h]).$$

Next, we express the three terms on the right-hand side with the help of the metric tensor g.

(1) $\omega_g(X^{\mathrm{v}},Y^h) \overset{3.9,\ \text{Lemma 2}}{=} g(\widetilde{A}(\widehat{X}),\widehat{Y}) = g(\widehat{X},\widehat{Y}) + g(\mathcal{C}(\widehat{X},\delta),\widehat{Y})$
$= g(\widehat{X},\widehat{Y}) + \mathcal{C}_\flat(\widehat{X},\widehat{Y},\delta)$.

(2) $[\xi,X^{\mathrm{v}}] \overset{3.6,\,\text{Remark}}{=} -X^h + \mathbf{i}\,\mathcal{L}_\xi^h\widehat{X}$, hence $\omega_g([\xi,X^{\mathrm{v}}],Y^h) = -\omega_g(X^h,Y^h)$
$+\,\omega_g(\mathbf{i}\,\mathcal{L}_\xi^h\widehat{X},Y^h) \overset{\text{Prop. 2}}{=} \omega_g(\mathbf{i}\,\mathcal{L}_\xi^h\widehat{X},Y^h) \overset{3.9,\,\text{Lemma 2}}{=} g(\widetilde{A}(\mathcal{L}_\xi^h\widehat{X}),\widehat{Y})$
$= g(\mathcal{L}_\xi^h\widehat{X},\widehat{Y}) + \mathcal{C}_\flat(\mathcal{L}_\xi^h\widehat{X},\widehat{Y},\delta)$.

(3) Also by the box-formula in 3.6, Remark, $[\xi,Y^h] = \mathcal{H}(\mathcal{L}_\xi^h\widehat{Y}) + \mathbf{i}\,\widetilde{\Phi}(\widehat{Y})$;
therefore $\omega_g(X^{\mathrm{v}},[\xi,Y^h]) = \omega_g(X^{\mathrm{v}},\mathcal{H}(\mathcal{L}_\xi^h\widehat{Y}) + \omega_g(X^{\mathrm{v}},\mathbf{i}\,\widetilde{\Phi}(\widehat{Y}))$
$= g(\widetilde{A}(\widehat{X}),\mathcal{L}_\xi^h\widehat{Y}) = g(\widehat{X},\mathcal{L}_\xi^h\widehat{Y}) + \mathcal{C}_\flat(\widehat{X},\mathcal{L}_\xi^h\widehat{Y},\delta)$.

Thus

$$0 = \xi g(\widehat{X},\widehat{Y}) - g(\mathcal{L}_\xi^h\widehat{X},\widehat{Y}) - g(\widehat{X},\mathcal{L}_\xi^h\widehat{Y}) + \xi\mathcal{C}_\flat(\widehat{X},\widehat{Y},\delta) - \mathcal{C}_\flat(\mathcal{L}_\xi^h\widehat{X},\widehat{Y},\delta)$$
$$- \mathcal{C}_\flat(\widehat{X},\mathcal{L}_\xi^h\widehat{Y},\delta) \overset{3.2,\,\text{Lemma 3}}{=} (\mathcal{L}_\xi^h g)(\widehat{X},\widehat{Y}) + (\mathcal{L}_\xi^h\mathcal{C}_\flat)(\widehat{X},\widehat{Y},\delta) + \mathcal{C}_\flat(\widehat{X},\widehat{Y},\mathcal{V}\xi),$$

which concludes the proof.

3.11. Finsler manifolds.

Definition 1. A covariant or a $\overset{\circ}{\tau}{}^*\tau$-valued covariant tensor field $\widetilde{T}$ along $\overset{\circ}{\tau}$ is said to be *homogeneous* of degree $k \in \mathbb{Z}$ if

$$\nabla_\delta^{\mathrm{v}}\widetilde{T} = k\widetilde{T},$$

where ∇^{v} is the canonical v-covariant derivative, or equivalently, the v-part of an arbitrary Berwald derivative in $\tau^*\tau$. In particular, a *Finsler metric* will be called *homogeneous,* if it is homogeneous of degree 0.

Lemma 1. *Let g be a Finsler metric in $\overset{\circ}{\tau}{}^*\tau$. g is homogeneous if, and only if, its components $g_{ij} := g\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right)$ are positive-homogeneous functions of degree 0, i.e.*

$$y^k \frac{\partial g_{ij}}{\partial y^k} = 0 \qquad (1 \leqq i, j \leqq k)$$

holds over any induced chart $(\overset{\circ}{\tau}{}^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ on $\overset{\circ}{T}M$. If g is homogeneous, then its first Cartan $\mathcal{C}$ tensor has the following properties:

(1) $\mathcal{C}(\delta, \widetilde{X}) = 0$*, for all* $\widetilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})$.

(2) $\nabla^{\mathrm{v}}_{\delta}\mathcal{C} = -\mathcal{C}$*, i.e.* $\mathcal{C}$ *is homogeneous of degree* -1.

Proof. For any vector fields X, Y on M we have

$$(\nabla^{\mathrm{v}}_{\delta} g)(\widehat{X}, \widehat{Y}) = (\nabla_C g)(\widehat{X}, \widehat{Y}) = C(g(\widehat{X}, \widehat{Y})),$$

since e.g. $\nabla_C \widehat{X} = \mathbf{j}[C, X^h] = 0$, being $[C, X^h]$ vertical. Thus, in particular,

$$(\nabla^{\mathrm{v}}_{\delta} g)\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) = C g_{ij} = y^k \tfrac{\partial g_{ij}}{\partial y^k};$$

this leads immediately to the local criterion of the homogeneity of g. Next we assume that g is homogeneous. Then

$$g(\mathcal{C}(\delta, \widetilde{X}), \widetilde{Y}) := (\nabla^{\mathrm{v}}_{\delta} g)(\widetilde{X}, \widetilde{Y}) = 0 \quad \text{for all } \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}),$$

which implies by the non-degeneracy of g the relation $\mathcal{C}(\delta, \,.\,) = 0$.

Finally we show that $\mathcal{C}$ is homogeneous of degree -1. Let $\widehat{X}$, $\widehat{Y}$, $\widehat{Z}$ be arbitrary basic vector fields along $\overset{\circ}{\tau}$. The homogeneity of g yields

$$0 = (\nabla_C g)(\mathcal{C}(\widehat{X}, \widehat{Y}), \widehat{Z}) = C g(\mathcal{C}(\widehat{X}, \widehat{Y}), \widehat{Z}) - g(\nabla_C(\mathcal{C}(\widehat{X}, \widehat{Y})), \widehat{Z}),$$

hence

$$g((\nabla_C \mathcal{C})(\widehat{X}, \widehat{Y}), \widehat{Z}) = C(g(\mathcal{C}(\widehat{X}, \widehat{Y}), \widehat{Z}) = C((\nabla_{X^{\mathrm{v}}} g)(\widehat{Y}, \widehat{Z})) = C(X^{\mathrm{v}} g(\widehat{Y}, \widehat{Z}))$$
$$= [C, X^{\mathrm{v}}] g(\widehat{Y}, \widehat{Z}) + X^{\mathrm{v}}(C g(\widehat{Y}, \widehat{Z})) = -X^{\mathrm{v}} g(\widehat{Y}, \widehat{Z}) = -g(\mathcal{C}(\widehat{X}, \widehat{Y}), \widehat{Z}),$$

and so, again by the non-degeneracy of g, we conclude the desired relation $\nabla_C \mathcal{C} = -\mathcal{C}$. □

Definition 2. A Finsler metric g in $\overset{\circ}{\tau}{}^*\tau$ is said to be *normal*, if its first Cartan tensor satisfies the condition

$$\boxed{\mathcal{C}(\widetilde{X}, \delta) = 0 \quad \text{for all } \widetilde{X} \in \mathfrak{X}(\overset{\circ}{\tau})}\,.$$

Coordinate expression. Given a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M, let the local expression of g be $g_{ij}\widehat{du^i} \otimes \widehat{du^j}$. If $\mathcal{C}\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) = \mathcal{C}_{ij}^{\ell}\, \widehat{\frac{\partial}{\partial u^\ell}}$ then, as we have learnt in 2.50, $\mathcal{C}_{ij}^{\ell} = g^{\ell k}\, \frac{\partial g_{jk}}{\partial y^i}$ $(1 \leqq i, j, k \leqq n)$. Thus

$$\mathcal{C}\left(\widehat{\frac{\partial}{\partial u^i}}, \delta\right) = \mathcal{C}\left(\widehat{\frac{\partial}{\partial u^i}}, y^j\, \widehat{\frac{\partial}{\partial u^j}}\right) = y^j\, \frac{\partial g_{jk}}{\partial y^i}\, g^{\ell k}\, \widehat{\frac{\partial}{\partial u^\ell}} \quad (1 \leqq i \leqq n),$$

which yields the following local criterion:

$$\boxed{g \text{ is normal} \iff y^j\, \frac{\partial g_{jk}}{\partial y^i} = 0 \ (1 \leqq i, k \leqq n)}\,.$$

Definition 3. By a **Finsler manifold** we mean a manifold endowed with a **normal** Finsler metric.

Example. Suppose that (M, g) is a generalized Finsler manifold satisfying **M.reg.1, 2**. Consider the absolute energy $E := \frac{1}{2}g(\delta, \delta)$, and let $g_E = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E$. Then (M, g_E) is a Finsler manifold, called *associated to* (M, g). Indeed, due to **M.reg.1**, by 3.9, Lemma 3 we get

$$g_E(\widehat{X}, \widehat{Y}) = g(\widetilde{A}(\widehat{X}), \widehat{Y}) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

According to **M.reg.2**, $\widetilde{A}$ is injective, so it follows that g_E is non-degenerate. Let $\mathcal{C}_E$ be the Cartan tensor of g_E. Applying 3.9, Lemma 3(i) twice and taking into account 2.45, Example 2, for any vector fields X, Y on M we have

$$\begin{aligned} g_E(\mathcal{C}_E(\widehat{X}, \delta), \widehat{Y}) &:= (\nabla_{X^{\mathrm{v}}} g_E)(\delta, \widehat{Y}) = X^{\mathrm{v}} g_E(\delta, \widehat{Y}) - g_E(\nabla_{X^{\mathrm{v}}}\delta, \widehat{Y}) \\ &= X^{\mathrm{v}}(Y^{\mathrm{v}}E) - g_E(\widehat{X}, \widehat{Y}) = X^{\mathrm{v}}(Y^{\mathrm{v}}E) - (\nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E)(X^{\mathrm{v}}, Y^{\mathrm{v}}) \\ &= X^{\mathrm{v}}(Y^{\mathrm{v}}E) - X^{\mathrm{v}}(Y^{\mathrm{v}}E) = 0, \end{aligned}$$

hence, by the non-degeneracy of g_E, $\mathcal{C}_E(\widehat{X}, \delta) = 0$. Thus the metric g_E is normal, as we claimed.

Elementary properties.

1. *Every Finsler manifold satisfies the regularity conditions* **M.reg.1, 2**. *In particular, if (M, g) is a Finsler manifold, then it coincides with its associated Finsler manifold (M, g_E).*

Indeed, if $\mathcal{C}(\,\cdot\,, \delta) = 0$, then $\mathcal{C}_\flat(\,\cdot\,, \delta, \delta)$ holds automatically. The tensor $\widetilde{A}$ in **M.reg.2** reduces to the identity map $1_{\mathfrak{X}(\overset{\circ}{\tau})}$, so 3.9, Lemma 3(ii) leads to the relation $g_E = g$.

2. *If (M, g) is a Finsler manifold, then g is variational, namely $g = g_E = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E$, where $E := \frac{1}{2}g(\delta, \delta)$ is the absolute energy.*

This is obvious by Property 1.

3. *The absolute energy of a Finsler manifold is positive-homogeneous of degree 2; the metric tensor and the first Cartan tensor are homogeneous of degree 0 and degree -1, respectively.*

Proof. We have shown in 3.9, Lemma 2 that under the condition **M.reg.1** $E := \frac{1}{2}g(\delta, \delta)$ is positive-homogeneous of degree 2. Then g is also homogeneous: for any vector fields X, Y on M we have

$$\begin{aligned}(\nabla^{\mathrm{v}}_{\delta} g)(\widehat{X}, \widehat{Y}) &= (\nabla_C g)(\widehat{X}, \widehat{Y}) = C(g(\widehat{X}, \widehat{Y})) \overset{\mathbf{2}}{=} C(g_E(\widehat{X}, \widehat{Y})) = C(X^{\mathrm{v}}(Y^{\mathrm{v}}E)) \\ &= [C, X^{\mathrm{v}}](Y^{\mathrm{v}}E) + X^{\mathrm{v}}(C(Y^{\mathrm{v}}E)) = -X^{\mathrm{v}}Y^{\mathrm{v}}E + X^{\mathrm{v}}([C, Y^{\mathrm{v}}]E + Y^{\mathrm{v}}(CE)) \\ &= -2X^{\mathrm{v}}(Y^{\mathrm{v}}E) + 2X^{\mathrm{v}}(Y^{\mathrm{v}}E) = 0;\end{aligned}$$

hence $\nabla^{\mathrm{v}}_{\delta} g = 0$. The homogeneity of g implies by Lemma 1 that $\mathcal{C}$ is homogeneous of degree -1. $\square$

4. *The absolute energy $E : \overset{\circ}{T}M \longrightarrow \mathbb{R}$ can be uniquely extended to a C^1 function $\widetilde{E} : TM \longrightarrow \mathbb{R}$.*

By the second-degree positive-homogeneity of E this is a consequence of the next

Observation. *Let $\mathcal{U} \subset \mathbb{R}^n$ be a non-empty open set and $f : \mathcal{U} \times (\mathbb{R}^k \setminus \{0\}) \longrightarrow \mathbb{R}$ a smooth function. If f is positive-homogeneous of degree 2 in its $\mathbb{R}^k$-variables, then f can be uniquely prolonged to a C^1 function $\widetilde{f} : \mathcal{U} \times \mathbb{R}^n \longrightarrow \mathbb{R}$.*

In view of the conceptual importance of this fact, we present here the elementary proof.

We are obviously forced to define $\widetilde{f}$ by $\widetilde{f}(p, 0) := 0$ for all $p \in \mathcal{U}$. We shall show that all partial derivatives $D_i\widetilde{f}$, $1 \leqq i \leqq n + k$ exist and are continuous at every point $(p, 0) \in \mathcal{U} \times \mathbb{R}^k$. Two cases are distinguished.

(1) $1 \leqq i \leqq n$. Then $D_i\widetilde{f}(p, 0) = \lim\limits_{t \longrightarrow 0} \dfrac{\widetilde{f}(p + te_i, 0) - \widetilde{f}(p, 0)}{t} = 0$. On the other hand, $D_i\widetilde{f}$ is also positive-homogenous of degree 2. Indeed, for all $\lambda \in \mathbb{R}^*_+$ we have

$$\begin{aligned}D_i\widetilde{f}(p, \lambda v) &= \lim_{t \longrightarrow 0} \frac{\widetilde{f}(p + te_i, \lambda v) - \widetilde{f}(p, \lambda v)}{t} \\ &= \lambda^2 \lim_{t \longrightarrow 0} \frac{\widetilde{f}(p + te_i, v) - \widetilde{f}(p, v)}{t} = \lambda^2 D_i\widetilde{f}(p, v).\end{aligned}$$

Next we check that for every sequence $\big((p_j, v_j)\big)$ of points of $\mathcal{U} \times \mathbb{R}^k$ such that $\lim_{j \longrightarrow \infty} (p_j, v_j) = (p, 0)$, the sequence $\big((D_i\widetilde{f})(p_j, v_j)\big)$ tends to 0. We may suppose that all of the vectors v_j differ from zero, since $(D_i\widetilde{f})(p_j, 0) = 0$ $(j \in \mathbb{N}^*)$. Then

$$(D_i\widetilde{f})(p_j, v_j) = \|v_j\|^2 (D_i\widetilde{f}) \left(p_j, \frac{v_j}{\|v_j\|} \right), \quad j \in \mathbb{N}^*.$$

Here $\|v_j\|^2$ tends zero as $(p_j, v_j) \longrightarrow (p, 0)$, while $(D_i\widetilde{f}) \left(p_j, \frac{v_j}{\|v_j\|} \right)$ is bounded. Therefore $(D_i\widetilde{f})(p_j, v_j)$ tends to zero as $(p_j, v_j) \longrightarrow (p, 0)$.

(2) $n + 1 \leqq i \leqq n + k$. Then

$$\begin{aligned} D_i\widetilde{f}(p, 0) &= \lim_{t \longrightarrow 0} \frac{\widetilde{f}(p, te_i) - \widetilde{f}(p, 0)}{t} = \lim_{t \longrightarrow 0+} \frac{t^2 \widetilde{f}(p, e_i) - \widetilde{f}(p, 0)}{t} \\ &= \lim_{t \longrightarrow 0+} t\widetilde{f}(p, e_i) = 0. \end{aligned}$$

In this case the functions $D_i\widetilde{f}$ are positive-homogeneous of degree 1 since for all $\lambda \in \mathbb{R}^*_+$ we have

$$\begin{aligned} D_i\widetilde{f}(p, \lambda v) &= \lim_{t \longrightarrow 0} \frac{\widetilde{f}(p, \lambda v + te_i) - \widetilde{f}(p, \lambda v)}{t} \\ &= \lambda \lim_{t \longrightarrow 0} \frac{\widetilde{f}(p, v + \frac{t}{\lambda} e_i) - \widetilde{f}(p, v)}{\frac{t}{\lambda}} = \lambda (D_i\widetilde{f})(p, v). \end{aligned}$$

Now the proof can be completed by the above argument.

In what follows E will mean the extended energy function.

5. *The canonical one-form of a Finsler manifold (M, g) is the Hilbert one-form $\theta_g = d_J E$, the fundamental two-form is $\omega_g = dd_J E$. Both θ_g and ω_g are homogenous of degree 1, i.e.*

$$d_C \theta_g = \theta_g, \quad d_C \omega_g = \omega_g.$$

We also have

$$i_C \omega_g = \theta_g, \quad i_J \omega_g = 0.$$

Proof. Since (M, g) satisfies **M.reg.1**, $\theta_g = d_J E$ by 3.9, Lemma 4. Thus

$$d_C \theta_g = d_C d_J E \overset{2.31,\,\text{Cor. 1}}{=} d_J d_C E - d_J E \overset{\mathbf{3}}{=} 2 d_J E - d_J E = \theta_g,$$

$$d_C \omega_g = d_C d\theta_g \overset{1.40(5)}{=} dd_C \theta_g = d\theta_g = \omega_g,$$

$$i_C \omega_g = i_C d\theta_g \overset{1.40(3)}{=} d_C \theta_g - di_C \theta_g = \theta_g - di_C d_J E = \theta_g;$$

while the relation $i_J\omega_g = 0$ is an immediate consequence of Property **2** and 3.10, Lemma 1. $\square$

6. *The canonical second-order vector field ξ of a Finsler manifold (M, g) is given by the 'Euler-Lagrange equation'*

$$\boxed{i_\xi\omega_g = -dE; \quad \omega_g = dd_JE,\ E = \frac{1}{2}g(\delta,\delta)},$$

and ξ is homogeneous of degree 2, i.e. $[C,\xi] = \xi$.

Proof. Due to the second-degree positive-homogeneity of E, the principal energy $\widetilde{E}_L = \widetilde{E}_E$ reduces to E, and 3.10, Proposition 1 leads to the given form of the Euler-Lagrange equation. Now

$$\begin{aligned} i_{[C,\xi]}\omega_g &\overset{1.40(2)}{=} d_Ci_\xi\omega_g - i_\xi d_C\omega_g \overset{\mathbf{5}}{=} -d_CdE - i_\xi\omega_g = -d_CdE + dE \\ &\overset{1.40(5)}{=} -dd_CE + dE = -dE = i_\xi\omega_g, \end{aligned}$$

hence $[C,\xi] = \xi$. $\square$

7. *The canonical second-order vector field of a Finsler manifold can be prolonged to a C^1 map $\widetilde{\xi} : TM \longrightarrow TTM$ such that $\widetilde{\xi} \restriction TM \setminus \overset{\circ}{T}M = 0$, therefore $\widetilde{\xi}$ is a spray over M.*

This is a consequence of the second-degree homogeneity of the canonical second-order vector field and the above Observation. $\widetilde{\xi}$ is called the *canonical spray* of the Finsler manifold and will be denoted simply by ξ.

8. *The first Cartan tensor of any Finsler manifold has the following properties:*

(i) $\mathcal{C}_\flat$ *is totally symmetric;*

(ii) $i_\delta\mathcal{C} = 0, \quad i_\delta\mathcal{C}_\flat = 0.$

Indeed, the metric tensor is variational, as we pointed out in **2**. Furthermore, by the Corollary in 2.50, the lowered Cartan tensor of any variational Finsler metric is totally symmetric, so (i) is true. This implies (ii), since $\mathcal{C}(\,\cdot\,,\delta) = 0$.

9. *If ξ is the canonical spray of the Finsler manifold (M, g) and ∇ is the Berwald derivative generated by ξ, then*

$$\boxed{\nabla_\xi g = 0}.$$

Proof. Since ξ is homogeneous of degree 2, the horizontal map $\mathcal{H}$ generated by ξ is homogeneous according to 3.3, Theorem 1(b). Thus the tension of $\mathcal{H}$ vanishes, and 3.4, Lemma 1 yields the relation $\nabla_\xi = \mathcal{L}^h_\xi$. Taking into account Property **8**(ii), this relation and 3.10, Proposition 3 give the result. □

10. *Under the preceding conditions, let us consider the second Cartan tensor $\mathcal{C}^h := \nabla^h g$ of g with respect to the horizontal map generated by the canonical spray ξ. Then:*

(i) $\mathcal{C}^h$ *is homogeneous of degree 0, i.e.* $\nabla^{\mathrm{v}}_\delta \mathcal{C}^h = 0$.

(ii) $\mathcal{C}^h = -\nabla_\xi \mathcal{C}$.

(iii) $\mathcal{C}^h$ *is symmetric,* $\mathcal{C}^h_\flat$ *is totally symmetric.*

(iv) $i_\delta \mathcal{C}^h = 0,\ i_\delta \mathcal{C}^h_\flat = 0$.

Proof. (i) Since g and $\mathcal{H}$ are both homogeneous, for any vector fields X, Y, Z on M we have

$$g(\nabla_C \mathcal{C}^h(\widehat{X},\widehat{Y}),\widehat{Z}) = Cg(\mathcal{C}^h(\widehat{X},\widehat{Y}),\widehat{Z}) = C((\nabla_{X^h} g)(\widehat{Y},\widehat{Z})) = C(X^h g(\widehat{Y},\widehat{Z}))$$
$$= [C, X^h] g(\widehat{Y},\widehat{Z}) + X^h(Cg(\widehat{Y},\widehat{Z})) = X^h(\nabla_C g(\widehat{Y},\widehat{Z})) = 0,$$

therefore $\nabla_C \mathcal{C}^h = \nabla^{\mathrm{v}}_\delta \mathcal{C}^h = 0$, as we claimed.

(ii) By 3.5, Proposition, for any vector field X on M we have $0 = R^\nabla(\xi, X^{\mathrm{v}}) = \nabla_\xi \circ \nabla_{X^{\mathrm{v}}} - \nabla_{X^{\mathrm{v}}} \circ \nabla_\xi - \nabla_{[\xi, X^{\mathrm{v}}]}$. Since $\mathcal{L}^h_\xi = \nabla_\xi$ as we have just seen, 3.3, Corollary 5 yields $[\xi, X^{\mathrm{v}}] = -X^h + \mathbf{i}\,\nabla_\xi \widehat{X}$. Hence

$$\nabla_\xi \circ \nabla_{\mathfrak{X}^{\mathrm{v}}} - \nabla_{X^{\mathrm{v}}} \circ \nabla_\xi - \nabla_{\mathbf{i}\,\nabla_\xi \widehat{X}} = -\nabla_{X^h}.$$

Now we operate by both sides of this relation on the metric tensor g. Since $\nabla_\xi g = 0$ by **9**, this yields

$$\nabla_\xi \nabla_{X^{\mathrm{v}}} g - \nabla_{\mathbf{i}\,\nabla_\xi \widehat{X}} g = -\nabla_{X^h} g.$$

For any vector fields Y, Z on M, we have on the one hand

$$-(\nabla_{X^h} g)(\widehat{Y},\widehat{Z}) =: -\mathcal{C}^h_\flat(\widehat{X},\widehat{Y},\widehat{Z}).$$

On the other hand,

$$(\nabla_\xi \nabla_{X^{\mathrm{v}}} g)(\widehat{Y},\widehat{Z}) - (\nabla_{\mathbf{i}\,\nabla_\xi \widehat{X}} g)(\widehat{Y},\widehat{Z}) = \xi\big((\nabla_{X^{\mathrm{v}}} g)\big)(\widehat{Y},\widehat{Z})\big) - (\nabla_{X^{\mathrm{v}}} g)(\nabla_\xi \widehat{Y},\widehat{Z})$$
$$- (\nabla_{X^{\mathrm{v}}} g)(\widehat{Y},\nabla_\xi \widehat{Z}) - \mathcal{C}_\flat(\nabla_\xi \widehat{X},\widehat{Y},\widehat{Z}) = \xi \mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) - \mathcal{C}_\flat(\nabla_\xi \widehat{X},\widehat{Y},\widehat{Z})$$
$$- \mathcal{C}_\flat(\widehat{X},\nabla_\xi \widehat{Y},\widehat{Z}) - \mathcal{C}_\flat(\widehat{X},\widehat{Y},\nabla_\xi \widehat{Z}) = (\nabla_\xi \mathcal{C}_\flat)(\widehat{X},\widehat{Y},\widehat{Z}),$$

therefore

$$g(\mathcal{C}^h(\widehat{X},\widehat{Y}),\widehat{Z}) = \mathcal{C}^h_\flat(\widehat{X},\widehat{Y},\widehat{Z}) = -(\nabla_\xi \mathcal{C}_\flat)(\widehat{X},\widehat{Y},\widehat{Z}).$$

Finally, using $\nabla_\xi g = 0$ again, we get

$$(\nabla_\xi \mathcal{C}_\flat)(\widehat{X},\widehat{Y},\widehat{Z}) = g\big((\nabla_\xi \mathcal{C})(\widehat{X},\widehat{Y}),\widehat{Z}\big),$$

whence

$$\mathcal{C}^h(\widehat{X},\widehat{Y}) = -(\nabla_\xi \mathcal{C})(\widehat{X},\widehat{Y}) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

This concludes the proof of (ii).

(iii) Since $\mathcal{C}$ is symmetric and $\nabla_\xi \mathcal{C}$ remains obviously symmetric, it follows from (ii) that $\mathcal{C}^h$ is symmetric, and hence $\mathcal{C}^h_\flat$ is totally symmetric.

(iv) The homogeneity of $\mathcal{H}$ implies by 3.3, Theorem 1 that $\mathcal{H} \circ \delta = \xi$. Applying this, for any vector fields X, Y on M we have

$$g(\mathcal{C}^h(\delta,\widehat{X}),\widehat{Y}) = (\nabla^h_\xi g)(\widehat{X},\widehat{Y}) = (\nabla_{\mathcal{H}\circ\delta} g)(\widehat{X},\widehat{Y}) = (\nabla_\xi g)(\widehat{X},\widehat{Y}) \overset{\mathbf{9}}{=} 0.$$

Thus $\mathcal{C}^h(\delta,\,\cdot\,) = 0$ and (iii) leads to the conclusion $i_\delta \mathcal{C}^h = 0,\ i_\delta \mathcal{C}^h_\flat = 0.$ $\square$

11. *Hypothesis as above. Choose a chart $(\mathcal{U},(u^i)_{i=1}^n)$ on M, and let $(\tau^{-1}(\mathcal{U}),((x^i)_{i=1}^n,(y^i)_{i=1}^n))$ be the induced chart on TM. Then the forces $G^i := -\frac{1}{2}\xi(y^i)$ defined by the canonical spray of (M,g) with respect to the chart $(\mathcal{U},(u^i)_{i=1}^n)$ can be expressed as follows:*

$$\boxed{G^i = g^{ij}G_j, \quad G_j = \frac{1}{2}\left(y^k\,\frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j}\right) \quad (1 \leqq i,j \leqq n)}\,.$$

We have the relations

$$\frac{\partial E}{\partial x^i} = y^j\,\frac{\partial G_j}{\partial y^i} \qquad (1 \leqq i \leqq n).$$

The coordinate expression for ξ can be deduced by a straightforward but lengthy calculation. As for the remaining, using repeatedly the second-degree positive-homogeneity of E, we get:

$$2y^j\frac{\partial G_j}{\partial y^i} = y^j\,\frac{\partial}{\partial y^i}\left(y^k\,\frac{\partial^2 E}{\partial x^k\partial y^j} - \frac{\partial E}{\partial x^j}\right) = y^j\,\frac{\partial^2 E}{\partial x^i \partial y^j} + y^j y^k\,\frac{\partial^3 E}{\partial y^i \partial x^k \partial y^j}$$
$$-\,y^j\,\frac{\partial^2 E}{\partial y^i\partial x^j} = y^j\,\frac{\partial^2 E}{\partial x^i\partial y^j} + y^k\,\frac{\partial^2 E}{\partial x^k \partial y^i} - y^j\,\frac{\partial^2 E}{\partial y^i \partial x^j} = y^j\,\frac{\partial^2 E}{\partial x^i\partial y^j} = 2\frac{\partial E}{\partial x^i},$$

whence the result.

Comment. The approach to Finsler to manifolds sketched here was strongly inspired by M. Hashiguchi's paper [38]. More usually, the concept of Finsler manifolds is built on the notion of the *fundamental function* ([53], [67], or on the *Minkowski norm* ([7]) or on the *energy function* [36]. For example, the definition accepted by Grifone in [36] sounds as follows:

'Let a function $E : TM \longrightarrow \mathbb{R}$ be given. Assume:

(1) $\forall v \in TM : E(v) \geqq 0,\ E(0) = 0.$

(2) *E is of class C^1 on TM, smooth on $\overset{\circ}{T}M$.*

(3) *E is positive-homogeneous of degree 2.*

(4) *The two-form $dd_J E$ is non-degenerate.*

Then (M, E) is said to be a *Finsler manifold* with the energy E.'

According to Property **4**, our Definition 3 leads to this concept, if we assume that $g(\delta, \delta)$ is non-negative over $\overset{\circ}{T}M$. Finsler manifolds defined by a fundamental function, called *Finsler–Lagrangian*, will appear in 3.13.

D. Covariant derivative operators on a Finsler manifold

3.12. The fundamental lemma of Finsler geometry.

Definition. A horizontal map $\mathcal{H} : TM \times_M TM \longrightarrow TTM$ is called *conservative* with respect to a C^1 function $F : TM \longrightarrow \mathbb{R}$ if

$$X^h F = 0 \quad \text{for all } X \in \mathfrak{X}(M);$$

more concisely, if

$$\boxed{d^h F = 0 \quad \text{or} \quad d_{\mathbf{h}} F = 0}$$

(d^h is the h-exterior derivative on $\mathcal{A}(\tau)$ with respect to $\mathcal{H}$, see 2.37, Lemma 1; $d_{\mathbf{h}}$ is the Lie derivative with respect to the horizontal projector $\mathbf{h}$, see 2.26.) A horizontal map on a Finsler manifold is said to be conservative, if it is conservative with respect to the absolute energy.

Theorem. (Fundamental lemma of Finsler geometry.) *Given a Finsler manifold (M, g), there exists a* unique *horizontal map*

$$\mathcal{H} : TM \times_M TM \longrightarrow TTM$$

called the canonical horizontal map *(of g) such that*

CH 1. *The torsion of $\mathcal{H}$ vanishes.*

CH 2. *$\mathcal{H}$ is homogeneous.*

CH 3. *$\mathcal{H}$ is conservative.*

The canonical horizontal map is generated by the canonical spray of the Finsler manifold.

Proof. (a) *Existence.* Let ξ be the canonical spray of (M, g) and $\mathcal{H}$ the horizontal map generated by ξ. Then the horizontal projector belonging to $\mathcal{H}$ is

$$\mathbf{h} = \frac{1}{2}(1_{TTM} + [J, \xi]).$$

Theorem 1 in 3.3 guarantees that $\mathcal{H}$ satisfies **CH 1** and **CH 2**, so we have only to check that $\mathcal{H}$ is conservative.

Let ∇ be the Berwald derivative induced by $\mathcal{H}$. For every $X \in \mathfrak{X}(M)$ we have

$$X^h E = \frac{1}{2} X^h(g(\delta, \delta)) = \frac{1}{2}\big((\nabla_{X^h} g)(\delta, \delta) + 2g(\nabla_{X^h}\delta, \delta)\big) = \frac{1}{2}\mathcal{C}_\flat^h(\widehat{X}, \delta, \delta)$$
$$+ g(\mathcal{V}[X^h, C], \delta) \overset{\text{homogeneity}}{=} \frac{1}{2}\mathcal{C}_\flat^h(\widehat{X}, \delta, \delta) \overset{3.11, \mathbf{10}(\text{iv})}{=} 0.$$

This proves the existence.

(b) *Uniqueness.* Assume that $\widetilde{\mathcal{H}} : TM \times_M TM \longrightarrow TTM$ is a horizontal map satisfying **CH 1**–**CH 3**. Let $\widetilde{\mathbf{h}}$, $\widetilde{\mathbf{T}}$ and $X^{\widetilde{h}}$ be the corresponding horizontal projector, the torsion of $\widetilde{\mathcal{H}}$ and the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ by $\widetilde{\mathcal{H}}$, respectively. Next we show that $\widetilde{\mathbf{h}} = \mathbf{h}$. The proof is done in several steps.

(1) We first prove that $\operatorname{Im}\widetilde{\mathcal{H}}$ *is a Lagrangian subbundle for the fundamental two-form* ω_g, i.e.

$$\omega_g(X^{\widetilde{h}}, Y^{\widetilde{h}}) = 0 \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

$$\begin{aligned}
&\text{Indeed, } \omega_g(X^{\widetilde{h}}, Y^{\widetilde{h}}) = d\theta_g(X^{\widetilde{h}}, Y^{\widetilde{h}}) = X^{\widetilde{h}}\theta_g(Y^{\widetilde{h}}) - Y^{\widetilde{h}}\theta_g(X^{\widetilde{h}}) - \theta_g([X^{\widetilde{h}}, Y^{\widetilde{h}}]) \\
&\overset{3.11,\mathbf{5}}{=} X^{\widetilde{h}} d_J E(Y^{\widetilde{h}}) - Y^{\widetilde{h}} d_J E(X^{\widetilde{h}}) - d_J E\big([X^{\widetilde{h}}, Y^{\widetilde{h}}]\big) = X^{\widetilde{h}}(Y^{\mathrm{v}}E) - Y^{\widetilde{h}}(X^{\mathrm{v}}E) \\
&- J[X^{\widetilde{h}}, Y^{\widetilde{h}}]E = [X^{\widetilde{h}}, Y^{\mathrm{v}}]E + Y^{\mathrm{v}}(X^{\widetilde{h}}E) - [Y^{\widetilde{h}}, X^{\mathrm{v}}]E - X^{\mathrm{v}}(Y^{\widetilde{h}}E) - [X, Y]^{\mathrm{v}}E \\
&\overset{\mathbf{CH\,3}}{=} \big([X^{\widetilde{h}}, Y^{\mathrm{v}}] - [Y^{\widetilde{h}}, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}}\big) \overset{2.32,\,\text{Lemma}}{=} (\widetilde{\mathbf{T}}(X^c, Y^c))E \overset{\mathbf{CH\,1}}{=} 0.
\end{aligned}$$

(2) *The second-order vector field* $\xi_{\widetilde{\mathcal{H}}} := \widetilde{\mathcal{H}} \circ \delta$ *associated to* $\widetilde{\mathcal{H}}$ *is a spray.*

According to 3.3, Lemma 1, $\xi_{\widetilde{\mathcal{H}}} = \widetilde{\mathbf{h}}[C, \xi_{\widetilde{\mathcal{H}}}]$. Since $\widetilde{\mathcal{H}}$ is homogeneous by the condition **CH 2**,

$$0 = [\widetilde{\mathbf{h}}, C]\xi_{\widetilde{\mathcal{H}}} = [\widetilde{\mathbf{h}}\,\xi_{\widetilde{\mathcal{H}}}, C] - \widetilde{\mathbf{h}}[\xi_{\widetilde{\mathcal{H}}}, C] = [\xi_{\widetilde{\mathcal{H}}}, C] + \xi_{\widetilde{\mathcal{H}}};$$

hence $[C, \xi_{\widetilde{\mathcal{H}}}] = \xi_{\widetilde{\mathcal{H}}}$. Thus $\xi_{\widetilde{\mathcal{H}}}$ is indeed a spray.

(3) We show that *the canonical spray* ξ *equals* $\xi_{\mathcal{H}}$. Let X be an arbitray vector field on M. Then

$$\begin{aligned}
(i_{\xi_{\widetilde{\mathcal{H}}}}\omega_g)(X^{\mathrm{v}}) &= \omega_g(\xi_{\widetilde{\mathcal{H}}}, X^{\mathrm{v}}) = -\omega_g(X^{\mathrm{v}}, \xi_{\widetilde{\mathcal{H}}}) \overset{3.9,\,\text{Lemma 2}}{=} -g(\widehat{X}, \mathbf{j}\,\xi_{\widetilde{\mathcal{H}}}) \\
&= -g(\widehat{X}, \delta) = -\theta_g(X^c) \overset{3.11,\mathbf{5}}{=} -d_J E(X^c) = -X^{\mathrm{v}}E \\
&= -(dE)(X^{\mathrm{v}}) \overset{3.11,\mathbf{6}}{=} (i_\xi \omega_g)(X^{\mathrm{v}}).
\end{aligned}$$

Since $\xi_{\widetilde{\mathcal{H}}}$ is horizontal with respect to $\widetilde{\mathcal{H}}$, in view (1) $i_{\xi_{\widetilde{\mathcal{H}}}}\omega_g$ is completely determined by its action over $\{X^{\mathrm{v}} \mid X \in \mathfrak{X}(M)\}$. The same is true for $i_\xi\omega_g$, so we conclude

$$i_{\xi_{\widetilde{\mathcal{H}}}}\omega_g = i_\xi\omega_g.$$

By the non-degeneracy of ω_g this implies the desired equality $\xi_{\widetilde{\mathcal{H}}} = \xi$.

(4) Conditions **CH 1**, **CH 2** imply by 3.3, Corollary 6 that $\widetilde{\mathcal{H}}$ may be generated by a *spray* $\widetilde{\xi}$, thus

$$\widetilde{h} = \frac{1}{2}(1_{TTM} + [J, \widetilde{\xi}]).$$

According to Theorem 1(b) in 3.3, this implies $\xi_{\widetilde{\mathcal{H}}} = \widetilde{\xi}$. Hence, by (3), $\widetilde{\xi} = \xi$ and we obtain $\widetilde{h} = h$.

This concludes the proof of the theorem.

Note. The fundamental lemma of Finsler geometry is due to J. Grifone [36]; the proof presented here differs essentially from Grifone's deduction of the theorem. The nonlinear connection determined by the canonical horizontal map is also mentioned as the *canonical nonlinear connection* of the Finsler manifold. The corresponding horizontal projector is called the *Barthel* endomorphism e.g. in [75].

3.13. The equations of A. Rapcsák.

Definition 1. By a *Finsler-Lagrangian* we mean a function $L : TM \longrightarrow \mathbb{R}$, having the following properties:

FL 1. *L is continuous.*

FL 2. *$L : \overset{\circ}{T}M \longrightarrow \mathbb{R}$ is smooth.*

FL 3. *L is positive-homogeneous of degree 1.*

FL 4. *If $E := \frac{1}{2}L^2$, then the two-form $\omega := dd_J E$ is non-degenerate.*

E is called the *energy* associated to L, ω is the *fundamental two-form* determined by L. A Finsler-Lagrangian L is said to be *positive*, if it satisfies

FL 5. $L(v) \geqq 0$ for all $v \in TM$; $L(v) = 0$ if, and only if, $v = 0$.

Lemma 1. *If $L : TM \longrightarrow \mathbb{R}$ is a Finsler-Lagrangian, then*

(1) *$d_J L$ is homogeneous of degree 0;*

(2) *$i_C dd_J L = 0$, therefore the two-form $dd_J L$ is degenerate.*

Proof. $d_C d_J L \overset{2.31,\,\mathrm{Cor.}\,1}{=} d_J d_C L - d_J L = d_J L - d_J L = 0$;

$i_C dd_J L = i_C dd_J L + d((d_J L)C) = i_C dd_J L + d i_C d_J L \overset{1.40(3)}{=} d_C d_J L \overset{(1)}{=} 0.$ □

Lemma 2. *Let $L : TM \longrightarrow \mathbb{R}$ be a Finsler-Lagrangian, E the energy associated to L. If $g := \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E$, then (M, g) is a Finsler manifold. The metric tensor g is related to the fundamental two-form ω by*

$$(g)_0(\xi, \eta) = \omega(J\xi, \eta) \quad \textit{for all } \xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M).$$

Proof. $(g)_0(\xi,\eta) \overset{2.22,\,\text{Lemma 1}}{:=} g(\mathbf{j}\,\xi,\mathbf{j}\,\eta) := \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E(\mathbf{j}\,\xi,\mathbf{j}\,\eta) = (\nabla_{J\xi}(\nabla^{\mathrm{v}}E))\mathbf{j}\,\eta = J\xi(J\eta(E)) - \nabla^{\mathrm{v}}E(\nabla_{J\xi}\mathbf{j}\,\eta) \overset{2.44,\ \text{Ex. 1}}{=} J\xi(J\eta(E)) - \nabla^{\mathrm{v}}E(\mathbf{j}[J\xi,\eta]) = J\xi(J\eta(E)) - J[J\xi,\eta]E$. An immediate calculation yields the same expression also for $\omega(J\xi,\eta)$, so we have

$$(g)_0(\xi,\eta) = \omega(J\xi,\eta) \quad \text{for all } \xi,\eta \in \mathfrak{X}(\overset{\circ}{T}M).$$

Due to the non-degeneracy of ω, from this it follows that g is non-degenerate. Finally, we obtain by the calculation of 3.11, Example that g is normal. □

Remark 1. (1) The Finsler manifold (M,g) obtained in this way is called the Finsler manifold determined by the Finsler-Lagrangian L. Then we speak simply of the *Finsler manifold* (M,L).

(2) The *canonical spray* of the Finsler manifold (M,L) is given by the Euler-Lagrange equation

$$\boxed{i_\xi\omega = -dE; \quad E := \frac{1}{2}L^2, \omega := dd_J E}\,.$$

Indeed, this follows from Lemma 2 and 3.11, **6**, since

$$\begin{aligned}
\frac{1}{2}g(\delta,\delta) &= \frac{1}{2}\nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E(\delta,\delta) = \frac{1}{2}(\nabla_C CE - \nabla^{\mathrm{v}}E(\nabla_C\delta)) \\
&\overset{2.44,\,\text{Ex. 1}}{=} \frac{1}{2}(4E - \nabla^{\mathrm{v}}E(\mathbf{j}[C,\mathcal{H}\circ\delta])) \\
&\overset{3.2,\,\text{Lemma 1(1)}}{=} \frac{1}{2}(4E - \nabla^{\mathrm{v}}E(\delta)) = \frac{1}{2}(4E-2E) = E.
\end{aligned}$$

(3) The non-degeneracy of the fundamental two-form ω guarantees that for any one-form $\alpha \in \mathcal{A}^1(\overset{\circ}{T}M)$ there exists a unique vector field $\alpha^\sharp$ (read: α *sharp*) on $\overset{\circ}{T}M$ such that $i_{\alpha^\sharp}\omega = \alpha$. Thus we obtain a module isomorphism from $\mathcal{A}^1(\overset{\circ}{T}M)$ onto $\mathfrak{X}(\overset{\circ}{T}M)$, called the *sharp operator* with respect to ω (cf. 1.30(**5**)). Using this language, the canonical spray of (M,L) is simply $\xi := -(dE)^\sharp$ over $\overset{\circ}{T}M$; $\xi(0) := 0$.

Lemma 3. *If* (M,L) *is a Finsler manifold and* ξ *is its canonical spray, then* $i_\xi dd_J L = 0$.

Proof. Let $\mathcal{H}$ be the canonical horizontal map for (M,L), i.e. the horizontal map generated by the canonical spray (3.12, Theorem). As usual, we denote by $\mathbf{h}$ the corresponding horizontal projector. Let X be an arbitrary vector field on M.

(i) $i_\xi dd_J L(X^{\mathrm{v}}) = dd_J L(\xi, X^{\mathrm{v}}) = -X^{\mathrm{v}} d_J L(\xi) - d_J L([\xi, X^{\mathrm{v}}]) =$
$-X^{\mathrm{v}} dL(C) - dL(J[\xi, X^{\mathrm{v}}]) \overset{3.2,\,\mathrm{Lemma}\,1(1)}{=} -X^{\mathrm{v}} L + X^{\mathrm{v}} L = 0.$

(ii) $i_\xi dd_J L(X^h) = dd_J L(\xi, X^h) = \xi(X^{\mathrm{v}} L) - X^h L - J[\xi, X^h] L \overset{3.3,\,\mathrm{Cor.}\,3}{=}$
$\xi(X^{\mathrm{v}} L) - X^h L + X^h L - X^c L = [\xi, X^{\mathrm{v}}] L + X^{\mathrm{v}}(\xi L) - X^c L \overset{3.3,\,\mathrm{Th.}\,1(\mathrm{a})}{=} -2X^h L +$ $X^{\mathrm{v}}(\xi L) = -2d_h L(X^h) + X^{\mathrm{v}} d_h L(\xi) = 0$, taking into account that $\mathcal{H}$ is conservative and ξ is horizontal by 3.3, Corollary 1.

This concludes the proof.

Lemma 4. *Let $\bar{L} : TM \longrightarrow \mathbb{R}$ be a positive Finsler-Lagrangian, and assume that ξ is a spray over M. Then, over $\overset{\circ}{T}M$, the canonical spray $\bar{\xi}$ of the Finsler manifold $(M, \bar{L})$ may be represented in the form*

$$\boxed{\bar{\xi} = \xi - \frac{\xi \bar{L}}{\bar{L}} C - \bar{L}(i_\xi dd_J \bar{L})^\sharp},$$

where the sharp operator is taken with respect to the fundamental two-form of $(M, \bar{L})$.

Proof. Let $\bar{E} := \frac{1}{2}\bar{L}^2$ be the energy associated to $\bar{L}$. Then

$$\begin{aligned} i_\xi dd_J \bar{E} &= i_\xi d(\bar{L} d_J \bar{L}) = i_\xi (d\bar{L} \wedge d_J \bar{L} + \bar{L} dd_J \bar{L}) = (\xi \bar{L}) d_J \bar{L} - d\bar{L}(C\bar{L}) \\ &+ \bar{L} i_\xi dd_J \bar{L} = \frac{\xi \bar{L}}{\bar{L}} d_J \bar{E} - d\bar{E} + \bar{L} i_\xi dd_J \bar{L} \overset{3.11,\,\mathbf{5}}{=} \frac{\xi \bar{L}}{\bar{L}} i_C dd_J \bar{E} \\ &+ i_{\bar{\xi}} dd_J \bar{E} + \bar{L} i_{(i_\xi dd_J \bar{L})^\sharp} dd_J \bar{E} = i_{\bar{\xi} + \frac{\xi \bar{L}}{\bar{L}} C + \bar{L}(i_\xi dd_J \bar{L})^\sharp} dd_J \bar{E}. \end{aligned}$$

Since $dd_J \bar{E}$ is non-degenerate, this leads to the desired relation. □

Lemma 5 and definition. *Let $\alpha \in \mathcal{A}_0(TM)$, $A \in \mathcal{B}_0(TM)$, i.e. let α be a semibasic, A a vector-valued semibasic form on TM. If ξ is a second-order vector field (or a semispray) over M, then*

$$\alpha^0 := i_\xi \alpha \quad \text{and} \quad A^0 := i_\xi A$$

are also semibasic forms, which do not depend on the choice of ξ. α^0 and A^0 are called the potential *of α and A, respectively.*

Proof. Since the difference of two second-order vector fields is vertical, the potential is well-defined. α^0 and A^0 remain obviously semibasic. □

Lemma 6. *If a semibasic form is homogeneous of degree $r \in \mathbb{Z}$, then its potential is homogeneous of degree $r+1$.*

Proof. Let $\alpha \in \mathcal{A}_0^{k+1}(TM)$ ($k \in \mathbb{N}$) be homogeneous of degree $r+1$. By Lemma 5, in forming α^0 we may take a spray ξ. Then for any vector fields $X_1, \dots, X_k$ on M we obtain:

$$(d_C(i_\xi\alpha))(X_1^c, \dots, X_k^c) \overset{1.33(2)}{=} C[(i_\xi\alpha)(X_1^c, \dots, X_k^c)]$$
$$-\sum_{i=1}^{k}(i_\xi\alpha)(X_1^c, \dots, [C, X_i^c], \dots, X_k^c) \overset{2.20,\,\text{Lemma 2}}{=} C(\alpha(\xi, X_1^c, \dots, X_k^c))$$
$$= (d_C\alpha)(\xi, X_1^c, \dots, X_k^c) + \alpha([C,\xi], X_1^c, \dots, X_k^c) = r\alpha(\xi, X_1^c, \dots, X_k^c)$$
$$+ \alpha(\xi, X_1^c, \dots, X_k^c) = (r+1)(i_\xi\alpha)(X_1^c, \dots, X_k^c).$$

Since a semibasic form is fully determined by its action on the complete lifts of vector fields, it follows that $d_C\alpha^0 = (r+1)\alpha^0$. □

Lemma 7. *Let α be a semibasic k-form on TM. Assume that α is homogeneous of degree r and $k + r \neq 0$. Then we have*

$$\alpha = \frac{1}{k+r}((d_J\alpha)^0 + d_J\alpha^0).$$

If, in particular, α is d_J-closed i.e. $d_J\alpha = 0$, then α may be reconstructed from its potential, namely

$$\alpha = \frac{1}{k+r}d_J\alpha^0.$$

For a proof see [24], p. 196.

Lemma 8. *Let $\alpha \in \mathcal{A}^1(TM)$ be a Hilbert one-form. If α is homogeneous of degree 0, then $(d\alpha)^0$ is semibasic.*

Proof. Choose a spray ξ over M. Let $\mathcal{H}$ be the (homogenous) horizontal map generated by ξ according to 3.3, Theorem 1. As usual, we denote by $\mathbf{h}$ the corresponding horizontal projector.

For any vector field X on M we have $(d\alpha)^0(X^\mathrm{v}) = (i_\xi d\alpha)(X^\mathrm{v}) = d\alpha(\xi, X^\mathrm{v}) \overset{(*)}{=} -X^\mathrm{v}\alpha(\xi) - \alpha([\xi, X^\mathrm{v}]) \overset{(*)}{=} X^\mathrm{v} i_\xi\alpha +$ $\alpha(\mathbf{h}[X^\mathrm{v}, \xi]) \overset{3.3,\,\text{Th. 1(a)}}{=} -X^\mathrm{v}\alpha^0 + \alpha(X^h) \overset{\text{Lemma 7}}{=} -X^\mathrm{v}\alpha^0 + d_J\alpha^0(X^h) =$ $-X^\mathrm{v}\alpha^0 + d\alpha^0(X^\mathrm{v}) = 0$ (at the steps denoted by an asterisk we used that α is semibasic). This proves the assertion.

Lemma 9. *Let $\mathcal{H}$ be a horizontal map,* $\mathbf{h}$ *the horizontal projector belonging to $\mathcal{H}$. If α is a semibasic one-form on TM, then $d_{\mathbf{h}}\alpha$ is also semibasic and we have*

$$d_{\mathbf{h}}\alpha = \mathbf{h}^* d\alpha,$$

where $\mathbf{h}^*$ *is the adjoint operator of* $\mathbf{h}$ *in the sense of 2.31, Definition* (1)*.*

Proof. Since α is semibasic, $d_{\mathbf{h}}\alpha := i_{\mathbf{h}}d\alpha - di_{\mathbf{h}}\alpha = i_{\mathbf{h}}d\alpha - d\alpha$. It may immediately be checked that the pairs of the form

$$(X^{\mathrm{v}}, Y^{\mathrm{v}}), (X^{\mathrm{v}}, Y^h), (X^h, Y^{\mathrm{v}}); \quad X, Y \in \mathfrak{X}(M)$$

kill the two-form $i_{\mathbf{h}}d\alpha - d\alpha$, so $d_{\mathbf{h}}\alpha$ is indeed semibasic. On the other hand, for any vector fields X, Y on M we have

$$d_{\mathbf{h}}\alpha(X^h, Y^h) = (i_{\mathbf{h}}d\alpha - d\alpha)(X^h, Y^h) = d\alpha(X^h, Y^h) = (\mathbf{h}^* d\alpha)(X^h, Y^h),$$

therefore $d_{\mathbf{h}}\alpha = \mathbf{h}^* d\alpha$. □

Definition 2. Let $\varphi : \overset{\circ}{T}M \longrightarrow \mathbb{R}$ be a function. Assume:

(i) φ is smooth on $\overset{\circ}{T}M$.

(2) φ is positive-homogeneous of degree 1, i.e. $C\varphi = \varphi$.

If ξ is a spray, then $\bar{\xi} := \xi + \varphi C$ is called a *projective change* of ξ. Two sprays, ξ and $\bar{\xi}$ are said to be *projectively equivalent* if there is a smooth function $\varphi : \overset{\circ}{T}M \longrightarrow \mathbb{R}$ such that $\bar{\xi} = \xi + \varphi C$ holds over $\overset{\circ}{T}M$.

Elementary properties.

(1) *If* $\bar{\xi} := \xi + \varphi C$ *is a projective change of* ξ*, then* $\bar{\xi}$ *is a spray.*

Indeed, $J\bar{\xi} = C$ is obviously valid. $\bar{\xi}$ is automatically smooth on $\overset{\circ}{T}M$. Since

$$[C, \bar{\xi}] = [C, \xi] + [C, \varphi C] = \xi + (C\varphi)C = \xi + \varphi C = \bar{\xi},$$

$\bar{\xi}$ is homogeneous of degree 2. Now, using the Observation in $3.11, \mathbf{4}$, $\bar{\xi}$ can be extended to a C^1 map from TM into TTM, so it is indeed a spray (in this sense).

(2) *If* ξ *and* $\bar{\xi}$ *are projectively equivalent sprays, i.e.* $\bar{\xi} = \xi + \varphi C$ $(\varphi \in C^\infty(\overset{\circ}{T}M))$*, then* φ *is positive-homogeneous of degree 1.*

Indeed, $\xi + \varphi C = \bar{\xi} = [C, \bar{\xi}] = [C, \xi] + (C\varphi)C$, hence $C\varphi = \varphi$.

(3) *The projective equivalence of sprays is an equivalence relation, the equivalence classes are called* projective sprays.

(4) *Two sprays over the same manifold are projectively equivalent if, and only if, their geodesics differ only in a strictly increasing parameter transformation.*

Definition 3. (1) A manifold endowed with a spray is said to be a *spray manifold*. Two spray manifolds (M, ξ) and $(M, \bar{\xi})$ are called *projectively equivalent* if ξ and $\bar{\xi}$ represent the same projective spray.

(2) A spray manifold (M, ξ) is said to be *projectively equivalent* to a Finsler manifold $(M, \bar{L})$ if ξ is projectively equivalent to the canonical spray of $(M, \bar{L})$. In this case we also say that (M, ξ) is *Finsler-metrizable in a broad sense* or *projectively Finsler*.

Remark 2. A spray manifold is called *Finsler metrizable in a natural sense* if there exists a Finsler-Lagrangian on the manifold whose canonical spray is the given spray. Sprays with this property are also called *variational*; this is another important concept of metrizability.

Theorem. *Suppose* (M, ξ) *is a spray manifold. Let* $\bar{L} : TM \longrightarrow \mathbb{R}$ *be a positive Finsler-Lagrangian. The following conditions are equivalent to* (M, ξ) *being projectively equivalent to the Finsler manifold* $(M, \bar{L})$.

RAP 1. $i_\xi dd_J \bar{L} = 0$.

RAP 2. $d_{\mathbf{h}} d_J \bar{L} = 0$, *where* $\mathbf{h}$ *is the horizontal projector of the nonlinear connection generated by* ξ.

Proof. (a) Let $\bar{\xi}$ be the canonical spray of $(M, \bar{L})$. Assume that **RAP 1** holds. Then Lemma 4 implies $\bar{\xi} = \xi - \frac{\xi\bar{L}}{\bar{L}}C$, therefore $(M, \bar{L})$ is projectively equivalent to (M, ξ). Conversely, if $\bar{\xi} = \xi + \varphi C$ ($\varphi \in C^\infty(\overset{\circ}{T}M)$), then

$$i_\xi dd_J\bar{L} = i_{\bar{\xi}} dd_J L - \varphi i_C dd_J \bar{L} \overset{\text{Lemmas 1(2), 3}}{=} 0.$$

Thus **RAP 1** *holds if, and only if,* (M, ξ) *is projectively equivalent to* $(M, \bar{L})$.

(b) Next we show that the conditions **RAP 1**, **RAP 2** are equivalent. Assume **RAP 1**. To prove **RAP 2**, we first notice that due to the homogeneity of $\mathbf{h}$,

$$d_{\mathbf{h}} \circ d_C - d_C \circ d_{\mathbf{h}} = d_{[\mathbf{h}, C]} = 0,$$

hence

$$d_C d_{\mathbf{h}} d_J \bar{L} = d_{\mathbf{h}} d_C d_J \bar{L} \overset{\text{Lemma 1(1)}}{=} 0,$$

therefore the semibasic two-form $d_{\mathbf{h}} d_J \bar{L}$ is homogeneous of degree 0. Applying Lemma 7, it follows that

$$(*) \qquad d_{\mathbf{h}} d_J \bar{L} = \frac{1}{2}[(d_J d_{\mathbf{h}} d_J \bar{L})^0 + d_J(d_{\mathbf{h}} d_J \bar{L})^0].$$

Since the torsion of the nonlinear connection generated by ξ vanishes, we have

$$d_J \circ d_{\mathbf{h}} + d_{\mathbf{h}} \circ d_J = [d_J, d_{\mathbf{h}}] =: d_{[J,\mathbf{h}]} = 0.$$

Using this, we get

$$d_J d_{\mathbf{h}} d_J \bar{L} = -d_{\mathbf{h}} d_J^2 \bar{L} \overset{2.31, \mathbf{1}}{=} 0,$$

thus the first term on the right-hand side of $(*)$ vanishes. Now we turn to the second term. The Hilbert one-form $d_J\bar{L}$ is homogeneous of degree 0 by Lemma 1(1), so Lemma 8 assures that $i_\xi dd_J \bar{L}$ is semibasic. Hence

$$(d_{\mathbf{h}} d_J \bar{L})^0 = i_\xi d_{\mathbf{h}} d_J \bar{L} \overset{\text{Lemma 9}}{=} i_\xi \mathbf{h}^* dd_J \bar{L} = i_\xi dd_J \bar{L} \overset{\text{Lemma 3}}{=} 0.$$

This concludes the proof that **RAP 1** implies **RAP 2**.

Conversely, assume **RAP 2**. Then, by the preceding argument,

$$i_\xi dd_J \bar{L} = i_\xi \mathbf{h}^* dd_J \bar{L} \overset{\text{Lemma 9}}{=} i_\xi d_{\mathbf{h}} d_J \bar{L} \overset{\text{RAP 2}}{=} 0,$$

thus **RAP 2** implies **RAP 1**. $\square$

Note. Both **RAP 1** and **RAP 2** provide, in the form of second-order partial differential equations, necessary and sufficient conditions for the Finsler-metrizability of a spray in a broad sense. Their coordinate versions were discovered by the Hungarian geometer András Rapcsák in the early 1960s, so we refer to them as the *Rapcsák equations of metrizability.* The index-free deduction presented here is from [78]. For further information the reader is referred to [78], Rapcsák's papers [64]–[66], and Z. Shen's book [69].

3.14. The Finslerian Berwald derivative. Berwald manifolds and locally Minkowski manifolds.

Theorem 1. *Let (M, g) be a Finsler manifold, $E := \frac{1}{2}g(\delta, \delta)$ the absolute energy of (M, g). There exist a unique horizontal map $\mathcal{H}$ over M and a unique covariant derivative operator D in $\tau^*\tau$ satisfying the following conditions:*

B.covd.1. *The Finsler torsion $\mathcal{S}$ of D vanishes.*

B.covd.2. *The v-mixed torsion $\mathbf{P}^1$ of D vanishes.*

B.covd.3. *D is associated to $\mathcal{H}$, i.e. $D\delta = \mathcal{V}$.*

B.covd.4. *The (h-horizontal) torsion $\mathcal{T}$ of D vanishes.*

B.covd.5. *$(\mathcal{H}\widehat{X})E = 0$ for all $X \in \mathfrak{X}(M)$, i.e. $\mathcal{H}$ is conservative.*

The horizontal map $\mathcal{H}$ is the canonical horizontal map of g, and D the Berwald derivative generated by $\mathcal{H}$.

Proof. The existence is clear: if $\mathcal{H}$ is the canonical horizontal map of g and ∇ is the Berwald derivative generated by $\mathcal{H}$, then ∇ and $\mathcal{H}$ have obviously the properties **B.covd.1**–**B.covd.5**.

Conversely, suppose that a horizontal map $\mathcal{H}$ over M and a covariant derivative operator D in $\tau^*\tau$ satisfy the conditions **B.covd.1**–**B.covd.5**. Then, in view of 2.45, Corollary, the first two of these conditions imply that D is the Berwald derivative induced by $\mathcal{H}$. According to 2.44, Example 2, **B.covd.3** yields the homogeneity of $\mathcal{H}$. Due to **B.covd.4** and 2.47,**1**, the torsion of $\mathcal{H}$ vanishes. Thus, taking into account **B.covd.5**, we conclude that $\mathcal{H}$ satifies **CH 1**–**CH 3**, therefore, by the uniqueness statement of the fundamental lemma of Finsler geometry, $\mathcal{H}$ is the canonical horizontal map of g. □

Note. The covariant derivative operator characterized by Theorem 1 may rightly be mentioned as the 'Finslerian Berwald derivative' on (M, g). The first axiomatic description of the Finslerian Berwald derivative is due T. Okada [60].

Theorem 2 and definition. *Let (M, g) be a Finsler manifold. The following conditions are equivalent:*

Berw. 1. *The Finslerian Berwald derivative on (M, g) is h-basic, i.e. there is a covariant derivative operator D on M such that*
$$[X^h, Y^{\mathrm{v}}] = (D_X Y)^{\mathrm{v}} \quad \textit{for all } X, Y \in \mathfrak{X}(M)$$
(the horizontal lift is taken with respect to the canonical horizontal map).

Berw. 2. *The Berwald curvature of the Finslerian Berwald derivative vanishes.*

Berw. 3. *The canonical spray of (M, g) is everywhere smooth.*

If one, and hence all, of these conditions are satisfied then (M, g) is said to be a **Berwald manifold**.

Proof. The equivalence of **Berw. 1** and **Berw. 2** has already been proved, see 2.49, Example.

Berw. 1 $\Longrightarrow$ **Berw. 3** Let ξ be the canonical spray of (M, g), and let $\widetilde{\xi}$ be an arbitrary (everywhere) smooth spray on TM. Such a spray certainly exists: the spray arising from a covariant derivative operator on M is smooth. If $\mathbf{h}$ is the horizontal projector belonging to the canonical horizontal map, then $\xi = \mathbf{h}\widetilde{\xi}$. Since $\mathbf{h}$ is everywhere smooth by **Berw. 1**, we conclude that the canonical spray of (M, g) is also smooth everywhere.

Berw. 3 $\Longrightarrow$ **Berw. 1** Let X and Y be arbitrary vector fields on M. Observe that $[X^h, Y^{\mathrm{v}}]$ is homogeneous of degree 0. Indeed,
$[C, [X^h, Y^{\mathrm{v}}]] = -[X^h, [Y^{\mathrm{v}}, C]] - [Y^{\mathrm{v}}, [C, X^h]] = [X^h, [C, Y^{\mathrm{v}}]] = -[X^h, Y^{\mathrm{v}}]$.
As ξ is everywhere smooth by the condition **Berw. 3**, so are the horizontal projector $\mathbf{h} = \frac{1}{2}(1_{TTM} + [J, \xi])$ and hence the vector field $[X^h, Y^{\mathrm{v}}]$ as well. Thus by 2.6, Proposition(1), $[X^h, Y^{\mathrm{v}}]$ is a vertical lift. Now, as in the proof of Lemma 2 in 2.49, it follows that there is a covariant derivative operator D on M such that

$$[X^h, Y^{\mathrm{v}}] = (D_X Y)^{\mathrm{v}} \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

This concludes the proof.

Note. Due to Z. I. Szabó's activity, the positive definite Berwald manifolds are completely classified [72].

Theorem 3 and definition. *Let (M, g) be a Finsler manifold. The following conditions are equivalent:*

Mink. 1. *The Finslerian Berwald derivative on (M, g) is h-basic and the curvature of the base covariant derivative vanishes.*

Mink. 2. *The Finslerian Berwald derivative has vanishing curvature, i.e. both its Riemann curvature and Berwald curvature vanish.*

Mink. 3. *The canonical spray of (M, g) is v-linearizable.*

Mink. 4. *The geodesics with respect to the canonical spray are 'rectilinear' in the sense that around each point of M there is a chart such that over the induced chart on TM the differential equation of the geodesics takes the form*

$$\boxed{x^{i\prime\prime} = 0 \quad (1 \leqq i \leqq n)}.$$

Mink. 5. *Around each point of M there is a chart such that over the induced chart we have*

$$\boxed{\frac{\partial E}{\partial x^i} = 0 \quad (1 \leqq i \leqq n)};$$

roughly speaking, the absolute energy of the Finsler manifold 'does not depend on the position'.

If one, and hence all, of these conditions are satisfied then (M, g) is said to be a **locally Minkowski manifold**.

Proof. In view of the v-linearizability theorem of Martínez and Cariñena (3.7), conditions **Mink. 1**–**Mink. 3** are equivalent.

Mink. 3 $\Longleftrightarrow$ **Mink. 4** Assume that the canonical spray ξ of (M, g) is v-linearizable. Then around any point of M there is a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ such that over the induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ the forces $G^i := -\frac{1}{2}\xi(y^i)$ are of the form

$$G^i = (A^i_j \circ \tau) y^j + b^i \circ \tau; \quad A^i_j, b^i \in C^\infty(\mathcal{U}) \quad (1 \leqq i, j \leqq n).$$

Since ξ is a spray, the functions G^i are positive-homogeneous of degree 2. The relation $CG^i = 2G^i$ $(1 \leqq i \leqq n)$ leads to

$$-(A^i_j \circ \tau) y^j = 2b^i \circ \tau \qquad (1 \leqq i \leqq n),$$

hence $A^i_j \circ \tau = 0$ and $b^i \circ \tau = 0$ $(1 \leqq i, j \leqq n)$. Therefore all of the forces vanish over $\tau^{-1}(\mathcal{U})$, which proves **Mink. 4**. The converse is clear.

Mink. 3 $\Longleftrightarrow$ **Mink. 5** If ξ is v-linearizable then, as we have just seen, around any point of M there is a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ such that the forces $G^i = -\frac{1}{2}\xi((u^i)^c)$ $(1 \leqq i \leqq n)$ vanish. This implies by 3.11, **11** that over $\tau^{-1}(\mathcal{U})$

$$\frac{\partial E}{\partial x^i} = 0, \qquad 1 \leqq i \leqq n.$$

Thus **Mink. 3** implies **Mink. 5**.

Conversely, assume **Mink. 5**. Then around each point of M there is a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ such that over $\tau^{-1}(\mathcal{U})$ we have $\frac{\partial E}{\partial x^i} = 0$ $(1 \leqq i \leqq n)$. Applying again 3.11, **11**, it follows that

$$G^i = g^{ij} G_j = \frac{1}{2} g^{ij} \left(y^k \frac{\partial}{\partial y^j} \frac{\partial E}{\partial x^k} - \frac{\partial E}{\partial x^j} \right) = 0,$$

therefore ξ is v-linearizable. □

3.15. The Cartan, the Chern–Rund and the Hashiguchi derivative.

Lemma 1. *Let $\bar{\mathcal{H}}$ be a horizontal map for τ, with vanishing torsion. Let $\bar{\nabla}$ be the Berwald derivative induced by $\bar{\mathcal{H}}$ (other objects defined by $\bar{\mathcal{H}}$ will also be distinguished by a 'bar'). Assume that*

$$\psi : \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \longrightarrow \mathfrak{X}(\tau)$$

is a $C^\infty(TM)$-bilinear map, and let the tensors ψ^{v}, $\bar{\psi}^h$ be given by

$$\psi^{\mathrm{v}}(\widetilde{X}, \widetilde{Y}) := \psi(\mathbf{i}\,\widetilde{X}, \widetilde{Y}), \ \bar{\psi}^h(\widetilde{X}, \widetilde{Y}) := \psi(\mathcal{H}\widetilde{X}, \widetilde{Y}) \quad \text{for all } \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\tau).$$

(1) *The map*

$$\bar{D} : (\xi, \widetilde{Y}) \in \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \mapsto \bar{D}_\xi \widetilde{Y} := \bar{\nabla}_\xi \widetilde{Y} + \psi(\xi, \widetilde{Y}) \in \mathfrak{X}(\tau)$$

is a covariant derivative operator in $\tau^*\tau$. For the $\bar{h}$-horizontal, the Finsler, the v-mixed and the v-vertical torsion of $\bar{D}$ we have:

$$\bar{\mathcal{T}}(\widetilde{X}, \widetilde{Y}) = \bar{\psi}^h(\widetilde{X}, \widetilde{Y}) - \bar{\psi}^h(\widetilde{Y}, \widetilde{X}),$$

$$\bar{\mathcal{S}}(\widetilde{X}, \widetilde{Y}) = -\psi^{\mathrm{v}}(\widetilde{Y}, \widetilde{X}),$$

$$\bar{\mathbf{P}}^1(\widetilde{X}, \widetilde{Y}) = \bar{\psi}^h(\widetilde{X}, \widetilde{Y}),$$

$$\mathbf{Q}^1(\widetilde{X}, \widetilde{Y}) = \psi^{\mathrm{v}}(\widetilde{X}, \widetilde{Y}) - \psi^{\mathrm{v}}(\widetilde{Y}, \widetilde{X})$$

$(\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\tau))$. $\bar{\mathcal{T}}$ vanishes if, and only if, $\bar{\psi}^h$ is symmetric; $\mathbf{Q}^1$ vanishes if, and only if, ψ^{v} is symmetric.

(2) *If g is a (not necessarily normal) Finsler metric in $\overset{\circ}{\tau}{}^*\tau$, then for any vector fields ξ on $\overset{\circ}{T}M$; $\widetilde{Y}$, $\widetilde{Z}$ in $\mathfrak{X}(\overset{\circ}{\tau})$ we have*

$$(\bar{D}_\xi g)(\widetilde{Y},\widetilde{Z}) = \mathcal{C}_\flat(\bar{\mathcal{V}}\xi,\widetilde{Y},\widetilde{Z}) + \bar{\mathcal{C}}^h_\flat(\mathbf{j}\,\xi,\widetilde{Y},\widetilde{Z}) - g(\psi(\xi,\widetilde{Y}),\widetilde{Z}) \\ - g(\widetilde{Y},\psi(\xi,\widetilde{Z})),$$

where $\mathcal{C}$ is the first Cartan tensor of g, and $\bar{\mathcal{C}}$ is the second Cartan tensor of g with respect to $\bar{\mathcal{H}}$.

Proof. $\bar{D}$ is obviously a covariant derivative operator (cf. 1.41(3)). Since the corresponding torsions of $\bar{\nabla}$ vanish by 2.47; **1**, **2**, we get immediately the formulae for $\bar{\mathcal{T}}$, $\bar{\mathcal{S}}$, $\bar{\mathbf{P}}^1$ and $\mathbf{Q}^1$. The expression for $\bar{D}_\xi g$ may also be obtained by an easy calculation. □

Lemma 2. *Let g be a variational Finsler metric, namely $g = \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}L$ ($L \in C^\infty(\overset{\circ}{T}M)$). Suppose that $\bar{\mathcal{H}}$ is a conservative horizontal map for L, with vanishing torsion. Then the second Cartan tensor $\bar{\mathcal{C}}^h_\flat$ of g with respect to $\bar{\mathcal{H}}$ is totally symmetric.*

Proof. According to 2.50, Lemma 5, we have

$$\bar{\mathcal{C}}^h_\flat(\widehat{X},\widehat{Y},\widehat{Z}) = -(\mathbf{i}\,\bar{\mathbf{P}}(\widehat{X},\widehat{Y})\widehat{Z})L + Z^{\mathrm{v}}(Y^{\mathrm{v}}(X^{\bar{h}}L)) \quad \text{for all } X,Y,Z \in \mathfrak{X}(M)$$

where $\bar{\mathbf{P}}$ is the Berwald curvature of the Berwald derivative induced by $\bar{\mathcal{H}}$. Since $\bar{\mathcal{H}}$ is conservative, $X^{\bar{h}}L = 0$. On the other hand, the vanishing of the torsion of $\bar{\mathcal{H}}$ implies by 2.33, **8** that $\bar{\mathbf{P}}$ is totally symmetric, whence the statement. □

Proposition. *Let (M,g) be a Finsler manifold. Assume that $\bar{\mathcal{H}}$ is a conservative horizontal map with vanishing torsion and $\bar{\nabla}$ is the Berwald derivative induced by $\bar{\mathcal{H}}$. Let $\psi : \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \longrightarrow \mathfrak{X}(\tau)$ be a $C^\infty(TM)$-bilinear map and define the tensors $\psi^{\mathrm{v}}, \bar{\psi}^h \in \mathcal{T}^1_2(\tau)$ as above. Consider the covariant derivative operator*

$$\bar{D} : (\xi,\widetilde{Y}) \in \mathfrak{X}(\overset{\circ}{T}M) \times \mathfrak{X}(\overset{\circ}{\tau}) \mapsto \bar{D}_\xi\widetilde{Y} := \bar{\nabla}_\xi\widetilde{Y} + \psi(\xi,\widetilde{Y}) \in \mathfrak{X}(\overset{\circ}{\tau}).$$

(1) *$\bar{D}$ is $\bar{h}$-metrical, i.e. $\bar{D}^{\bar{h}}g = 0$, and D has vanishing h-horizontal torsion if, and only if, $\bar{\psi}^h = \frac{1}{2}\bar{\mathcal{C}}^h$.*

(2) *$\bar{D}$ is v-metrical, i.e. $\bar{D}^{\mathrm{v}}g = 0$, and $\bar{D}$ has vanishing v-vertical torsion if, and only if, $\psi^{\mathrm{v}} = \frac{1}{2}\mathcal{C}$.*

Proof. Assume that $\bar{D}^{\bar{h}} g = 0$ and $\bar{\mathcal{T}} = 0$. Let $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ be arbitrary vector fields along $\overset{\circ}{\tau}$. Taking into account Lemma 1(2), the first condition yields

$$g(\bar{\mathcal{C}}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) = g(\bar{\psi}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) + g(\widetilde{Y}, \bar{\psi}^h(\widetilde{X}, \widetilde{Z})).$$

Permuting the letters cyclically, we get

$$g(\bar{\mathcal{C}}^h(\widetilde{Y}, \widetilde{Z}), \widetilde{X}) = g(\bar{\psi}^h(\widetilde{Y}, \widetilde{Z}), \widetilde{X}) + g(\widetilde{Z}, \bar{\psi}^h(\widetilde{Y}, \widetilde{X})),$$

$$-g(\bar{\mathcal{C}}^h(\widetilde{Z}, \widetilde{X}), \widetilde{Y}) = -g(\bar{\psi}^h(\widetilde{Z}, \widetilde{X}), \widetilde{Y}) - g(\widetilde{X}, \bar{\psi}^h(\widetilde{Z}, \widetilde{Y})).$$

Since $\bar{\mathcal{C}}^h_\flat$ is totally symmetric according to Lemma 2, and $\bar{\psi}^h$ is also symmetric by the vanishing of $\bar{\mathcal{T}}$ (see Lemma 1(1)), adding these three equations we obtain

$$\frac{1}{2} g(\bar{\mathcal{C}}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) = g(\bar{\psi}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}).$$

By the non-degeneracy of g, this leads to the desired relation $\bar{\psi}^h = \frac{1}{2}\bar{\mathcal{C}}^h$.

Conversely, if $\bar{\psi}^h = \frac{1}{2}\bar{\mathcal{C}}^h$, then $\bar{\mathcal{T}} = 0$ by Lemma 1. Furthermore, applying the total symmetry of $\bar{\mathcal{C}}^h_\flat$, for any vector fields $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ along $\overset{\circ}{\tau}$ we have

$$(\bar{D}^h g)(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = \big(\bar{D}_{\bar{\mathcal{H}}\widetilde{X}} g\big)(\widetilde{Y}, \widetilde{Z}) = \bar{\mathcal{H}}\widetilde{X}(g(\widetilde{Y}, \widetilde{Z})) - g\big(\nabla_{\bar{\mathcal{H}}\widetilde{X}} \widetilde{Y}, \widetilde{Z}\big)$$
$$- g\big(\widetilde{Y}, \nabla_{\bar{\mathcal{H}}\widetilde{X}} \widetilde{Z}\big) - \frac{1}{2} g(\bar{\mathcal{C}}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) - \frac{1}{2} g(\widetilde{Y}, \bar{\mathcal{C}}^h(\widetilde{X}, \widetilde{Z})) = (\bar{\nabla}^h g)(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$$
$$- g(\bar{\mathcal{C}}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) = g(\bar{\mathcal{C}}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) - g(\bar{\mathcal{C}}^h(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) = 0.$$

This concludes the proof of (1). Assertion (2) may be verified similarly; note, that in this case the necessary symmetry properties of $\mathcal{C}$ are guaranteed by the structure (see 3.11, **8**). □

Corollary 1. *Let (M, g) be a Finsler manifold, $\mathcal{H}$ the canonical horizontal map of g. There is a unique covariant derivative operator D in $\overset{\circ}{\tau}{}^*\tau$ satisfying the following conditions:*

C.covd.1. *D is v-metrical, i.e. $D^{\mathrm{v}} g = 0$.*

C.covd.2. *D is h-metrical, i.e. $D^h g = 0$.*

C.covd.3. *The (h-horizontal) torsion of D vanishes.*

C.covd.4. *The v-vertical torsion of D vanishes.*

The covariant derivative operator D acts by the following rules for calculation:

$$\boxed{\begin{aligned} D_{\mathbf{i}\widetilde{X}} \widetilde{Y} &= \nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} + \frac{1}{2}\mathcal{C}(\widetilde{X}, \widetilde{Y}) = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}] + \frac{1}{2}\mathcal{C}(\widetilde{X}, \widetilde{Y}) \\ D_{\mathcal{H}\widetilde{X}} \widetilde{Y} &= \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} + \frac{1}{2}\mathcal{C}^h(\widetilde{X}, \widetilde{Y}) = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] + \frac{1}{2}\mathcal{C}^h(\widetilde{X}, \widetilde{Y}) \end{aligned}}$$

(∇ is the Berwald derivative induced by $\mathcal{H}$, i.e. the Finslerian Berwald derivative; $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$). In particular, for any basic vector fields $\widehat{X}$, $\widehat{Y}$ along τ we have

$$\boxed{D_{X^{\mathrm{v}}}\widehat{Y} = \frac{1}{2}\mathcal{C}(\widehat{X}, \widehat{Y}), \quad D_{X^h}\widehat{Y} = \mathcal{V}[X^h, Y^{\mathrm{v}}] + \frac{1}{2}\mathcal{C}^h(\widehat{X}, \widehat{Y})}.$$

D is associated to $\mathcal{H}$:

$$\boxed{D\delta = \mathcal{V}}.$$

Proof. Both the existence and the uniqueness of D is an immediate consequence of the Proposition. Since $\mathcal{H}$ is homogeneous, according to 2.44, Example 2, ∇ is associated to $\mathcal{H}$, i.e. $\nabla\delta = \mathcal{V}$. So, for any vector field ξ on $\overset{\circ}{T}M$, we have

$$(D\delta)\xi = D_\xi\delta = \nabla_\xi\delta + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi, \delta) + \frac{1}{2}\mathcal{C}^h(\mathbf{j}\,\xi, \delta) = \mathcal{V}(\xi) + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi, \delta)$$
$$+ \frac{1}{2}\mathcal{C}^h(\mathbf{j}\,\xi, \delta) \overset{3.11,\,\mathbf{10}}{=} \mathcal{V}(\xi).$$

This concludes the proof.

Note. The covariant derivative operator given by the Corollary is said to be the *Cartan derivative.* The first axiomatic description of this derivative is due to M. Matsumoto. The present treatment is based on [75], [76] and Crampin's paper [20].

Remark. Let (M, g) be a *generalized* Finsler manifold satisfying the regularity conditions **M.reg.1, 2**. Then, as we have seen in 3.11, Example, (M, g_E) is a Finsler manifold, if $g_E := \nabla^{\mathrm{v}}\nabla^{\mathrm{v}}E$, $E := \frac{1}{2}g(\delta, \delta)$. Let $\mathcal{H}_{g_E}$ be the canonical horizontal map of g_E. With the help of $\mathcal{H}_{g_E}$ we may construct the covariant derivative operator described in 2.51, Proposition. The covariant derivative obtained in this way is called the *Miron derivative* or the *Miron-Cartan derivative* on the generalized Finsler manifold (M, g). If, in particular, (M, g) is a Finsler manifold, then the Miron derivative reduces to the Cartan derivative.

Corollary 2. *Hypothesis as in Corollary 1. There exists a unique covariant derivative operator D in $\overset{\circ}{\tau}{}^*\tau$ with the following properties:*

Ch.-R.covd.1. *$D^{\mathrm{v}} := \nabla^{\mathrm{v}}$ = the canonical v-covariant derivative.*

Ch.-R.covd.2. *D is h-metrical, i.e. $D^h g = 0$.*

Ch.-R.covd.3. *The h-horizontal torsion of D vanishes.*

The rules of calculation for this derivative are

$$\boxed{\begin{array}{rcl} D_{\mathbf{i}\,\widetilde{X}}\widetilde{Y} & = & \nabla_{\mathbf{i}\,\widetilde{X}}\widetilde{Y} = \mathbf{j}[\mathbf{i}\,\widetilde{X}, \mathcal{H}\widetilde{Y}], \\ D_{\mathcal{H}\widetilde{X}}\widetilde{Y} & = & \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} + \dfrac{1}{2}\mathcal{C}^h(\widetilde{X}, \widetilde{Y}) = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\,\widetilde{Y}] + \dfrac{1}{2}\mathcal{C}^h(\widetilde{X}, \widetilde{Y}) \end{array}}$$

$(\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau}))$. *In particular,*

$$\boxed{D_{X^{\mathrm{v}}}\widehat{Y} = 0, \quad D_{X^h}\widehat{Y} = \mathcal{V}[X^h, Y^{\mathrm{v}}] + \frac{1}{2}\mathcal{C}^h(\widehat{X}, \widehat{Y}) \quad \textit{for all } X, Y \in \mathfrak{X}(M)}\,.$$

D is associated to $\mathcal{H}$, i.e. $D\delta = \mathcal{V}$. □

Corollary 3. *Hypothesis as above. There exists a unique covariant derivative oparator D in $\overset{\circ}{\tau}{}^*\tau$ with the following properties:*

H.covd.1. *D is v-metrical, i.e. $D^{\mathrm{v}}g = 0$.*

H.covd.2. *The v-vertical torsion of D vanishes.*

H.covd.3. *$D^h := \nabla^h$.*

The rules of calculation with respect to D are the following:

$$\boxed{\begin{array}{rcl} D_{\mathbf{i}\,\widetilde{X}}\widetilde{Y} & = & \nabla_{\mathbf{i}\,\widetilde{X}}\widetilde{Y} + \dfrac{1}{2}\mathcal{C}(\widetilde{X}, \widetilde{Y}) = \mathbf{j}[\mathbf{i}\,\widetilde{X}, \mathcal{H}\widetilde{Y}] + \dfrac{1}{2}\mathcal{C}(\widetilde{X}, \widetilde{Y}), \\ D_{\mathcal{H}\widetilde{X}}\widetilde{Y} & = & \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\,\widetilde{Y}] \end{array}}\,;$$

in particular,

$$\boxed{D_{X^{\mathrm{v}}}\widehat{Y} = \frac{1}{2}\mathcal{C}(\widetilde{X}, \widehat{Y}), \quad D_{X^h}\widehat{Y} = \mathcal{V}[X^h, Y^{\mathrm{v}}]}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})$; $X, Y \in \mathfrak{X}(M)$.

D is associated to the canonical horizontal map, i.e. $D\delta = \mathcal{V}$. □

Note. The covariant derivative operators given by these corollaries are called the *Chern–Rund* and the *Hashiguchi derivative*, respectively. The former was discovered by S.S. Chern in 1948, and independently, by H. Rund in 1951. The two operators were indentified by M. Anastasiei in 1996 (!), see [4].

Appendix

A.1. Basic conventions.

(**1**) To shorten statements, we sometimes use the symbols $\forall$ (read: 'for all', 'for any', etc.), $\Longrightarrow$ (read: 'implies', 'if…, then'), $\Longleftrightarrow$ (read: 'logically equivalent', 'if, and only if'). The notation $a = b$ means that the objects denoted by the symbols a and b are the same; its negation is written $a \neq b$. The symbol ':=' means that its left-hand side is defined by the right-hand side, the symbol '=:' is used analogously.

(**2**) Throughout our study the language and symbolism of naive set theory will be applied. The meaning of the symbols $\in$, $\cap$, $\cup$, $\emptyset$ is common. If A and B are sets, we use the symbol $A \subset B$ to mean that A is contained in B but may be equal to B. We have products (or Descartes products) of sets, say finite products $A \times B$, or $A_1 \times \cdots \times A_n$, and products of families (see below) of sets.

(**3**) If $f : A \longrightarrow B$ is a map of one set into another, we write $a \mapsto f(a)$ to denote the effect of f on an element $a \in A$. In order to save parentheses, sometimes we write fa rather than $f(a)$. For every set S, 1_S denotes the identity map of S onto itself. When it is clear which set we mean, we write simply 1.

Let $f : A \longrightarrow B$ be a map, and A' a subset of A. The *restriction* f to A' is the map $f \restriction A' : A' \longrightarrow B$ defined by $(f \restriction A')(a) := f(a)$ for all $a \in A'$.

(**4**) Let A and I be two sets. By a *family of elements of* A, having I as the set of indices we mean a map $f : I \longrightarrow A$, which is written as $(a_i)_{i \in I}$, or simply (a_i) when no confusion can arise. In particular, if $I = \{1, \ldots, n\}$, we also write $(a_i)_{i=1}^n$ and speak of a finite sequence of elements of A. A 'double family' (a_{ij}) (with $1 \leqq i \leqq m$, $1 \leqq j \leqq n$; m, n are positive integers) is called a *matrix* with m rows and n columns (or $m \times n$ matrix) over the set A.

We may also speak of *a family* $(A_i)_{i \in I}$ *of subsets of* A, then $A_i \subset A$ for all $i \in I$. The union, intersection and the product of the family $(A_i)_{i \in I}$ is denoted by

$$\bigcup_{i \in I} A_i, \quad \bigcap_{i \in I} A_i \quad \text{and} \quad \mathop{\times}_{i \in I} A_i,$$

respectively. For an extremely clear axiomatic treatment of these subtle concepts the reader is referred to the book of M. Eisenberg [32].

(**5**) Let S be a set, $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ two families of subsets of S. $(B_j)_{j\in J}$ is said to be a *refinement* of $(A_i)_{i\in I}$ if for every $j \in J$ there is an index $i \in I$ (depending on j) such that $B_j \subset A_i$.

(**6**) If $f : A \longrightarrow S$ and $g : B \longrightarrow S$ are maps, then $A \times_S B$ denotes the *fibre product* of A and B over S relative to f and g. The elements of $A \times_S B$ are the ordered pairs $(a, b) \in A \times B$ such that $f(a) = g(b)$; briefly

$$A \times_S B := \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

(**7**) Following Bourbaki, we say that a map $f : A \longrightarrow B$ is *injective*, if $a \neq b \implies f(a) \neq f(b)$, *surjective*, if given an element $b \in B$ there is at least one $a \in A$ such that $f(a) = b$, *bijective* if both injective and surjective. A bijective map of a set S onto itself is also called a *permutation* of S. The set of all permutations of S is denoted by $\mathfrak{S}(S)$. If $S = \{1, \dots, n\}$, we write $\mathfrak{S}_n$ in place of $\mathfrak{S}(S)$.

(**8**) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are two maps, then we have a composition map $g \circ f$ such that $g \circ f(a) := g[f(a)]$ for all $a \in A$. A *diagram*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

of maps is said to be *commutative* if $g \circ f = h$. Similarly, a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is commutative, if $g \circ f = k \circ h$.

(**9**) The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural numbers, integers, rationals, reals, and complexes. Again as Bourbaki, we denote by $\mathbb{N}^*$, $\mathbb{R}^*$, $\mathbb{R}_+$, $\mathbb{R}_+^*$ etc. the positive integers, the non-zero reals, the non-negative reals and the set of positive reals, respectively. A mapping into $\mathbb{R}$ or $\mathbb{C}$ will be called a *function*.

A.2. Topology.

A *topological space* is a set S in which a family $\mathcal{T}$ of subsets, called *open sets*, has been specified with the following properties: S and $\emptyset$ are open, the intersection of any two open sets is open, the union of every family of open sets is open. Such a family $\mathcal{T}$ is called a *topology on S*.

The reader is expected to be familiar with the basics of point set topology, but, for convenience, we present here some of the vocabulary we use.

Let S be a topological space with a topology $\mathcal{T}$.

(**1**) A set $H \subset S$ is *closed*, if its complement is open. A *neighbourhood* of a point $p \in S$ is any set that contains an open set containing p. $(S, \mathcal{T})$ is a *Hausdorff space*, if distinct points of S have disjoint neighbourhoods. S is *connected* if it is not possible to express S as a union of two disjoint non-empty open sets. $\mathcal{T}' \subset \mathcal{T}$ is a *base* for $\mathcal{T}$ if every member of $\mathcal{T}$ is a union of members of $\mathcal{T}'$. S is *second countable* if it has a countable base.

(**2**) An *open cover* of S is a subfamily of $\mathcal{T}$ whose union is S. S is *compact* if every open cover of S has a finite subcover.

(**3**) A family $(A_i)_{i\in I}$ of subsets of S is *locally finite* if for every $a \in S$ there is a neighbourhood $\mathcal{U}$ of a such that

$$\{i \in I \mid \mathcal{U} \cap A_i \neq \emptyset\}$$

is finite.

(**4**) If T is another topological space, a map $f : S \longrightarrow T$ is said to be *continuous* if the inverse image of an open set (in T) is open in S. f is *continuous at a point* $a \in S$ if given a neighbourhood $\mathcal{V}$ of $f(a)$ there exists a neighbourhood $\mathcal{U}$ of a such that $f(\mathcal{U}) \subset \mathcal{V}$. A continuous map which admits a continuous inverse map is called a *homeomorphism*. A continuous map $f : S \longrightarrow T$ is said to be a *local homeomorphism* if any point $a \in S$ has an open neighbourhood which f maps homeomorphically onto an open neighbourhood of $f(a)$ in T.

A.3. The Euclidean n-space $\mathbb{R}^n$.

(**1**) Let $n \in \mathbb{N}^*$. $\mathbb{R}^n$ is defined as the set of all 'n-tuples' $a = (\alpha^1, \dots, \alpha^n)$ of real numbers α^i, i.e. its elements are the sequences $(\alpha^i)_{i=1}^n$ of n real numbers; $\mathbb{R}^1 := \mathbb{R}$. With the law of composition

$$a + b = (\alpha^i)_{i=1}^n + (\beta^i)_{i=1}^n := (\alpha^1 + \beta^1, \dots, \alpha^n + \beta^n)$$

and with the map

$$\mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ (\lambda, a) \mapsto (\lambda\alpha^1, \dots, \lambda\alpha^n)$$

$\mathbb{R}^n$ is an n-dimensional real vector space. The *usual* or *canonical basis* of $\mathbb{R}^n$ is $(e_i)_{i=1}^n$, where $e_i := (0, \dots, 1, \dots, 0)$, with the 1 in the ith place. The linear functions

$$e^i : a \in \mathbb{R}^n \mapsto e^i(a) := \alpha^i \in \mathbb{R} \quad (1 \leqq i \leqq n)$$

constitute the *dual basis* $(e^i)_{i=1}^n$ of the basis (e_i). Then

$$e^i(e_j) = \delta^i_j := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases};$$

δ^i_j is called the *Kronecker symbol.*

(**2**) In $\mathbb{R}^n$ the *standard inner product* is defined by

$$\langle a, b\rangle := \sum_{i=1}^{n} \alpha^i \beta^i, \quad \text{if } a = (\alpha^i)_{i=1}^n,\ b = (\beta^i)_{i=1}^n;$$

thus $\mathbb{R}^n$ becomes a Euclidean vector space. The *norm* of a vector $v \in \mathbb{R}^n$ is $\|v\| := \langle v, v\rangle^{1/2}$. By an *open ball* in $\mathbb{R}^n$ centered at a point a and of radius $\varrho \in \mathbb{R}^*_+$, we mean the set

$$B(a;\varrho) := \{p \in \mathbb{R}^n \mid \|p - a\| < \varrho\}.$$

We define a set $\mathcal{U} \subset \mathbb{R}^n$ to be *open* if for each point $a \in \mathcal{U}$ there is an open ball $B(a;\varrho)$ such that $B(a;\varrho) \subset \mathcal{U}$. It is easy to verify that this defines a topology on $\mathbb{R}^n$, called the *ordinary topology.* We agree once and for all that

$$\boxed{\mathbb{R}^n \text{ is endowed with the ordinary topology}}\,.$$

A.4. Smoothness.

Let a non-empty open set $\mathcal{U} \subset \mathbb{R}^n$ be given, and let f be a function with domain $\mathcal{U}$.

(**1**) The ith *partial derivative* of f at a point $p \in \mathcal{U}$ is

$$D_i f(p) := \lim_{t \longrightarrow 0} \frac{f(p + te_i) - f(p)}{t},$$

if the limit exists. (In the particular case $n = 1$ e_i is nothing but the $1 \in \mathbb{R}$, and we get the derivative $f'(p)\cdot)$. If the partial derivatives $D_1 f, \dots, D_n f$ exist for every $p \in \mathcal{U}$, and the functions

$$D_i f : \mathcal{U} \longrightarrow \mathbb{R}, \quad p \mapsto D_i f(p) \qquad (1 \leqq i \leqq n)$$

are continuous, then f is called *continuously differentiable* or *of class* C^1 on $\mathcal{U}$, denoted by $f \in C^1(\mathcal{U})$. We agree that the continuous functions are *of class* C^0.

(**2**) Now we may define inductively the notion of a *k-fold continuously differentiable function*: f is *of class* C^k ($k \in \mathbb{N}^*$) on $\mathcal{U}$ if $D_i f$ exists and is of class C^{k-1} for all $i \in \{1, \dots, n\}$. We say that f is *smooth* on $\mathcal{U}$ if f is of class C^k for every $k \in \mathbb{N}$. As in the case C^1, we denote these classes of functions on $\mathcal{U}$ by $C^k(\mathcal{U})$ and $C^\infty(\mathcal{U})$.

(**3**) Let finally a map $F : \mathcal{U} \longrightarrow \mathbb{R}^m$ be given. If $(\ell^j)_{j=1}^m$ is the dual of the canonical basis of $\mathbb{R}^m$, then the functions $F^j := \ell^j \circ F$ ($1 \leqq j \leqq m$) are called the (Euclidean) coordinate functions of F. F is said to be of class C^k (or C^∞) on $\mathcal{U}$, if each of the coordinate functions of F belongs to the class $C^k(\mathcal{U})$ (or $C^\infty(\mathcal{U})$).

A.5. Modules and exact sequences.

In the sequel K *will mean a commutative ring with unit element 1.*

(**1**) A commutative group $(V,+)$ (or simply V) is said to be a *module over* K, or a K-*module*, if a map

$$\mathsf{K} \times V \longrightarrow V,\ (\alpha, v) \mapsto \alpha v$$

is given, having the following properties:

$$\alpha(v+w) = \alpha v + \alpha w, \quad (\alpha+\beta)v = \alpha v + \beta v,$$

$$(\alpha\beta)v = \alpha(\beta v), \quad 1v = v$$

(in these conditions v and w are arbitrary elements of V, sometimes called *vectors*; α and β are arbitrary elements of K, called *scalars*).

If, in particular, K is a field, then we obtain the notion of a *vector space* over K.

A non-empty subset W of a K-module V is called a *submodule* of V, if

$$v, w \in W \implies v + w \in W; \quad \alpha \in \mathsf{K},\ v \in V \implies \alpha v \in V.$$

When V is a vector space, its submodules are called *vector* subspaces (or simple *subspaces* if no confusion can arise).

(**2**) There is a strong analogy between K-modules and vector spaces over a field, at least at the level of basic concepts (*linear dependence* and *independence, system of generators, basis*, etc.), so we do not repeat here these common definitions. However, we have to emphasize that most of the results concerning these concepts do not generalize to K-modules.

A K-module V possessing a basis is said to be *free*. If V is a finitely-generated free module, then any two bases of V have the same cardinality, called the *dimension* of V and denoted by $\dim_{\mathsf{K}} V$ or simply $\dim V$.

(**3**) Let V and W be K-modules. A (K-) *homomorphism* or a K-*linear map* of V into W is a map $f : V \longrightarrow W$ such that

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \quad \text{for all } u, v \in V \text{ and for all } \alpha, \beta \in \mathsf{K}.$$

If $W = V$ then f is called an *endomorphism* of V. An *isomorphism* of V onto W is a bijective homomorphism of V onto W.

We denote by $\mathrm{Hom}_{\mathsf{K}}(V, W)$ the set of all K-homomorphisms of V into W; $\mathrm{Hom}_{\mathsf{K}}(V, W)$ can immediately be made into a K-module. If $f \in \mathrm{Hom}_{\mathsf{K}}(V, W)$, then the *kernel*

$$\operatorname{Ker} f := f^{-1}(0) := \{v \in V \mid f(v) = 0\} \subset V$$

of f is a submodule of V. Clearly, f is injective if, and only if, $\operatorname{Ker} f = \{0\}$. Similarly, the *image*

$$\operatorname{Im} f := f(V) := \{f(v) \in W \mid v \in V\} \subset W$$

of f is a submodule of W.

(**4**) Let V be a K-module. $\mathrm{End}_{\mathsf{K}}(V) := \mathrm{Hom}_{\mathsf{K}}(V, V)$ with the laws of composition

$$(f, g) \mapsto f + g, \quad (f, g) \mapsto f \circ g$$

is a ring with unit element 1_V, called the *ring of endomorphisms of the module* V. An endomorphism $f \in \mathrm{End}_{\mathsf{K}}(V)$ is said to be a *projection* (or *projector*) if $f^2 := f \circ f = f$.

The bijective endomorphisms of V are called *automorphisms.* The set $\mathrm{GL}(V)$ of all automorphisms of V is a subgroup of the group $\mathfrak{S}(V)$ of all permutations of the set V. This group $\mathrm{GL}(V)$ is said to be the *general linear group* of the module V.

(**5**) Let us denote by $\mathrm{Mat}_n(\mathsf{K})$ the ring of $n \times n$ matrices

$$(\alpha^i_j) =: \begin{pmatrix} \alpha^1_1 & \alpha^1_2 & \dots & \alpha^1_n \\ \alpha^2_1 & \alpha^2_2 & \dots & \alpha^2_n \\ \vdots & \vdots & & \vdots \\ \alpha^n_1 & \alpha^n_2 & \dots & \alpha^n_n \end{pmatrix}$$

over K, and let $\mathrm{GL}(n, \mathsf{K})$ be the set of all invertible $n \times n$ matrices with elements in K:

$\mathrm{GL}(n, \mathsf{K}) := \{A \in \mathrm{Mat}_n(\mathsf{K}) \mid \text{there is } B \in \mathrm{Mat}_n(\mathsf{K}) \text{ such that } AB = BA = 1_n\}$,

$1_n := (\delta^i_j) \in \mathrm{Mat}_n(\mathsf{K})$ is the unit matrix of order n.

Now let V be a *finitely generated free module* over K, and let $\mathcal{B} = (b_i)_{i=1}^n$ be a basis of V. Then the map

$$\begin{cases} M_{\mathcal{B}} : f \in \mathrm{End}_{\mathsf{K}}(V) \mapsto M_{\mathcal{B}}(f) =: (\alpha^i_j) \in \mathrm{Mat}_n(\mathsf{K}), \\ f(b_j) = \displaystyle\sum_{i=1}^n \alpha^i_j b_i, \quad 1 \leqq j \leqq n \end{cases}$$

is a ring-isomorphism of the ring of endomorphisms of V onto the ring of $n \times n$ matrices over K. It follows immediately that the relations

$$f \in \mathrm{GL}(V) \quad \text{and} \quad M_{\mathcal{B}}(f) \in \mathrm{GL}(n, \mathsf{K})$$

are equivalent.

Note. If $\mathsf{K} := \mathbb{R}$, then we shall write $\mathrm{GL}(n)$ rather then $\mathrm{GL}(n, \mathsf{K})$.

(**6**) If V is a K-module, the K-module $V^* := \mathrm{Hom}_{\mathsf{K}}(V, \mathsf{K})$ of *linear forms* or *functionals* on V is called the *dual* of V; $V^{**} := (V^*)^*$ is the *bidual* of V, etc. The map

$$\begin{cases} c_V : V \longrightarrow V^{**}, \ v \mapsto c_V(v) \\ \forall \ell \in V^* : c_V(v)(\ell) := \ell(v) \end{cases}$$

is a K-linear injection, called the *canonical injection* of Vinto V^{**}.

Lemma 1. *Assume that V is a* finitely generated free *module over* K. *Then:*

(i) *The dual V^* of V is also a free module. If $(b_i)_{i=1}^n$ is a basis a for V, and b^i is the functional such that*

$$b^i(b_j) = \delta^i_j = \textit{the Kronecker symbol} \quad (1 \leqq i, j \leqq n),$$

then $(b^i)_{i=1}^n$ is a basis for V^*.

(ii) *The canonical injection c_V is an* isomorphism *of V onto V^{**}.* □

Given a basis $(b_i)_{i=1}^n$ for V as in the lemma, we call the basis $(b^i)_{i=1}^n$ the *dual basis* to $(b_i)_{i=1}^n$. If $v = \sum_{j=1}^{n} \nu^j b_j$, then $b^i(v) = \nu^i$, hence $v = \sum_{i=1}^{n} b^i(v) b_i$. Thus b^i may rightly be called the ith *coordinate function* with respect to the basis $(b_i)_{i=1}^n$. On the other hand, if $\ell = \sum_{i=1}^{n} \lambda_i b^i$, then

$$\ell(v) = \sum_{i=1}^{n} \lambda_i b^i \left(\sum_{j=1}^{n} \nu^j b_j \right) = \sum_{i=1}^{n} \lambda_i \nu^i,$$

so $\ell(v)$ can be obtained as the usual dot product of n-tuples, cf. A.3(2).

(**7**) Let $V_1, \dots, V_k$ $(k \geqq 2)$ be submodules of a given K-module V. The sum of these submodules is

$$V_1 + \cdots + V_k := \{v_1 + \cdots + v_k \in V \mid v_i \in V_i,\ i \leqq i \leqq k\}.$$

The submodules $V_1, \dots, V_k$ are said to be *linearly independent* if each vector $v \in V_1 + \cdots + V_k$ has a *unique* expression in the form

$$v = v_1 + \cdots + v_k \quad \text{with} \quad v_1 \in V_1, \dots, v_k \in V_k.$$

In this case we say that $V_1, + \cdots + V_k$ is the *direct sum* of the sequence (V_i) and we write

$$V_1 \oplus \cdots \oplus V_k \quad \text{or} \quad \bigoplus_{i=1}^{k} V_i.$$

It may immediately be seen that

$$V = V_1 \oplus V_2 \iff V = V_1 + V_2 \quad \text{and} \quad V_1 \cap V_2 = \{0\};$$

then the submodules V_1 and V_2 are called *complementary.*

Proposition 1. *Let V be a K-module, and let H be a submodule of V. The following conditions are equivalent:*

$\oplus_1$ *H is a* direct summand *in V,* i.e. *there is a submodule N of V such that $H \oplus N = V$.*

$\oplus_2$ *There is a projection* $f \in \operatorname{End}(V)$ *such that* $f(V) = H$.

$\oplus_3$ *There is a homomorphism* $h : V \longrightarrow H$ *such that* $h(v) = v$ *for all* $v \in H$.

The proof is easy and may be found e.g. in [34], §17, no. 4.

(**8**) Let V_1, V_2, V_3 be three K-modules, $f_1 : V_1 \longrightarrow V_2$, $f_2 : V_2 \longrightarrow V_3$ be homomorphisms. The pair (f_1, f_2) is said to be an *exact sequence* if $\operatorname{Im} f_1 = \operatorname{Ker} f_2$. Then we also say that the *diagram*

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$$

is an *exact sequence at* V_2. Similarly, a *diagram*

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} V_4$$

consisting of four K-modules and three homomorphisms is called *exact* if the diagrams

$$V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1}, \quad 2 \leqq i \leqq 3$$

are exact at V_i. Then we also speak of an *exact sequence*; 'longer' exact sequences are defined analogously.

Now let V and W be K-modules, $f : V \longrightarrow W$ a K-homomorphism. Let the module consisting of one element (the zero element) be denoted by 0. Obviously, there is only one homomorphism from 0 to V, and from V onto 0, so it is unimportant to give a name or a special symbol to these 'trivial' homomorphisms in the exact sequences where they appear. We also recall that the *factor module* V/H of V by a submodule H consists of the cosets $v + H$ $(v \in V)$, endowed with a 'natural' K-module structure.

After these preparations, we have the following

Lemma. (a) *For* $0 \longrightarrow V \xrightarrow{f} W$ *to be an exact sequence, it is necessary and sufficient that* f *be injective.*

(b) *For* $V \xrightarrow{f} W \longrightarrow 0$ *to be exact, it is necessary and sufficient that* f *be surjective.*

(c) *Assume that* V *is a submodule of* W*;* $\mathbf{i} : V \longrightarrow W$ *is the canonical injection (i.e.* $\mathbf{i}(v) := v$ *for all* $v \in V$*);* $\mathbf{j} : W \longrightarrow W/V$ *is the canonical surjection (i.e.* $\mathbf{j}(w) := w + V$ *for all* $w \in W$*). Then the diagram*

$$0 \longrightarrow V \xrightarrow{\mathbf{i}} W \xrightarrow{\mathbf{j}} W/V \longrightarrow 0$$

is an exact sequence.

(d) *The diagram*

$$0 \longrightarrow \operatorname{Ker} f \xrightarrow{\mathbf{i}} V \xrightarrow{f} W \xrightarrow{\mathbf{j}} W/\operatorname{Im} f \longrightarrow 0$$

is an exact sequence (again, $\mathbf{i}$ *and* $\mathbf{j}$ *are the canonical injection and surjection, respectively).* □

Proposition 2. *Let the diagram*

SEQ $$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

be an exact sequence. The following conditions are equivalent:

(**i**) *There exists a homomorphism* $s : W \longrightarrow V$ *such that* $g \circ s = 1_W$*.*

(**ii**) *There exists a homomorphism* $r : V \longrightarrow U$ *such that* $r \circ f = 1_U$*.*

When this is so, we have

$$V = \operatorname{Im} f \oplus \operatorname{Ker} r, \quad V = \operatorname{Ker} g \oplus \operatorname{Im} s,$$

and the map

$$f \oplus s : U \oplus W \longrightarrow V, \quad (u, w) \mapsto f(u) + s(w)$$

is an isomorphism.

This is an important fact, whose proof is easy, see e.g. [12] or/and [46]. The map s is called a *linear section*, while r is called a *linear retraction* associated with f. **SEQ** is said to be *split*, if it satisfies the equivalent conditions (**i**), (**ii**). Then s and r are also mentioned as *a right* and a *left splitting* of **SEQ**, respectively; and sometimes we write

$$0 \longrightarrow U \underset{r}{\overset{f}{\rightleftarrows}} V \underset{s}{\overset{g}{\rightleftarrows}} W \longrightarrow 0.$$

(**9**) **Proposition 3.** *Let P be a* K*-module. The following conditions are equivalent:*

(**i**) *Given a* K*-homomorphism* $f : P \longrightarrow W$ *and a surjective* K*-homomorphism* $g : V \longrightarrow W$*, there is a homomorphism* $h : P \longrightarrow V$ *making the following diagram commutative:*

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "f"] & \\ V \arrow[r, "g"'] & W \arrow[r] & 0\cdot \end{tikzcd}$$

(**ii**) *Every exact sequence* $0 \longrightarrow V \longrightarrow W \longrightarrow P \longrightarrow 0$ *splits.*

(**iii**) *P is a direct summand of a free module, i.e. there is a* K*-module V such that* $P \oplus V$ *is free.*

For a proof the reader is referred to S. Lang's Algebra [46]. If one – and hence all – of the conditions (**i**)–(**iii**) are satisfied then P is said to be a *projective module.*

A.6. Algebras and derivations.

(**1**) Let $V_1, \dots, V_n$ $(n \geqq 2)$ and W be K-modules. A map

$$f : V_1 \times \cdots \times V_n \longrightarrow W$$

is said to be K-*multilinear (n-linear, multilinear)* if it is linear in each variable. If $V_1 = \cdots = V_n := V$, we also say that f is a *multilinear map on* V; in particular, a multilinear map $V^n \longrightarrow \mathsf{K}$ is called a *multilinear* (more precisely, n-*linear*) *function* or *form* on V.

Notation. We denote by $L(V_1, \dots, V_n; W)$, $L^n(V, W)$ and $L^n(V)$ the sets of multilinear maps $V_1 \times \cdots \times V_n \longrightarrow W$, $V^n \longrightarrow W$ and n-linear forms $V^n \longrightarrow \mathsf{K}$, respectively.

(**2**) By an *algebra over* K (or a K-*algebra*) we mean a K-module A endowed with a K-bilinear map $\mathsf{A} \times \mathsf{A} \longrightarrow \mathsf{A}$ called *multiplication* and written by juxtaposition. An algebra A is said to be *associative* if

$$a(bc) = (ab)c \quad \text{for all } a, b, c \in \mathsf{A}$$

and *commutative* if

$$ab = ba \quad \text{for all } a, b \in \mathsf{A}.$$

$e \in \mathsf{A}$ is a *unit element* of the algebra A if

$$ea = ae = a \quad \text{for all } a \in \mathsf{A}.$$

If A has a unit element, then it is clearly unique.

Suppose A and B are K-algebras. A K-linear map $f : \mathsf{A} \longrightarrow \mathsf{B}$ is called a *homomorphism of algebras* if f preserves products, i.e. $f(ab) = f(a)f(b)$, for all $a, b \in \mathsf{A}$. If $\mathsf{B} = \mathsf{A}$, we speak of an *endomorphism* (of algebras). A bijective homomorphism of algebras is said to be an *isomorphism.*

(**3**) A K-algebra A with the multiplication

$$[\,,\,] : \mathsf{A} \times \mathsf{A} \longrightarrow \mathsf{A}, \quad (a, b) \mapsto [a, b]$$

is said to be a *Lie algebra* over K if:

LIE 1. $[a, a] = 0$ for all $a \in \mathsf{A}$.

LIE 2. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for all $a, b, c \in \mathsf{A}$.

The identity **LIE 2** is called the *Jacobi identity.* **LIE 1** implies immediately that

$$[a, b] = -[b, a] \quad \text{for all } a, b \in \mathsf{A}.$$

If B is an associative K-algebra with the multiplication $(a, b) \mapsto ab$, then the new multiplication

$$(a, b) \in \mathsf{B} \times \mathsf{B} \mapsto [a, b] := ab - ba \in \mathsf{B}$$

makes B into a Lie algebra.

(**4**) Let A be a K-algebra. A map $\theta : \mathsf{A} \longrightarrow \mathsf{A}$ is said to be a *derivation* of A if:

Der 1. θ is K-linear.

Der 2. $\theta(ab) = (\theta a)b + a(\theta b)$ for all $a, b \in \mathsf{A}$.

An immediate example: if A is the algebra of smooth functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and θ is the map defined by $\theta(f) := f'$ for all $f \in C^\infty(\mathbb{R})$ (cf. A.4), then θ is a derivation in the above sense.

Lemma. *Let A be a K-algebra and θ a derivation of A.*

(**i**) *If A has a unit element $e \neq 0$, then $\theta(e) = 0$.*

(**ii**) *If A is associative, then*

$$\theta(a_1 a_2 \dots a_n) = \sum_{i=1}^{n} a_1 \dots a_{i-1} \theta(a_i) a_{i+1} \dots a_n \quad \textit{for all } a_1, \dots, a_n \in \mathsf{A}.$$

(**iii**) *If θ_1 and θ_2 are derivations of A, then $[\theta_1, \theta_2] := \theta_1 \circ \theta_2 - \theta_2 \circ \theta_1$ is again a derivation.*

Proof. From $e^2 = e$ we obtain $\theta(e) = \theta(e^2) = (\theta e)e + e(\theta e) = \theta e + \theta e$, hence $\theta e = 0$. (ii) follows from **Der 2** by induction on n. (iii) may be checked by an easy calculation. □

A.7. Graded algebras and derivations.

(**1**) A K-algebra A is said to be a *graded algebra of type* $\mathbb{Z}$ if A is the direct sum of a sequence $(\mathsf{A}_n)_{n\in\mathbb{Z}}$ of submodules such that

$$\mathsf{A}_m \mathsf{A}_n \subset \mathsf{A}_{m+n} \quad \text{for all } m, n \in \mathbb{Z}.$$

The elements of A_n ($n \in \mathbb{Z}$) are called *homogeneous of degree* n. If A admits a unit element e, it is always understood that e is of degree 0. The zero element of $\mathsf{A} = \bigoplus_{n\in\mathbb{Z}} \mathsf{A}_n$ is homogeneous of any degree, but a homogeneous element $a \neq 0$ belongs to only one A_n. Then the notation $\deg(a) := n$ may correctly be used.

Analogously, a K-algebra A is said to a *graded algebra of type* $\mathbb{N}$ if it has a direct decomposition $\mathsf{A} = \bigoplus_{n\in\mathbb{N}} \mathsf{A}_n$ such that

$$\mathsf{A}_m \mathsf{A}_n \subset \mathsf{A}_{m+n} \quad \text{for all } m, n \in \mathbb{N}.$$

Then A can be identified with a graded K-algebra of type $\mathbb{Z}$ by setting $\mathsf{A}_n := 0$ for $n \leqq -1$. Whenever we refer to a graded algebra $\mathsf{A} = \bigoplus_{n\in\mathbb{N}} \mathsf{A}_n$ we shall mean this particular gradation of type $\mathbb{Z}$.

We call a graded K-algebra A of type $\mathbb{Z}$ (*graded*) *commutative*, respectively (*graded*) *anticommutative* if for all non-zero homogeneous elements a, b of A we have

$$ab = (-1)^{\deg(a)\deg(b)} ba, \quad \text{respectively} \quad ab = -(-1)^{\deg(a)\deg(b)} ba.$$

(**2**) Let $\mathsf{A} = \bigoplus_{n\in\mathbb{N}} \mathsf{A}_n$ be a graded K-algebra (in the above sense). A derivation θ of A is said to be of *degree* r, where $r \in \mathbb{Z}$, if

$$\theta(\mathsf{A}_n) \subset \mathsf{A}_{n+r} \quad \text{for all } n \in \mathbb{N}.$$

By a *graded derivation of degree* r ($r \in \mathbb{Z}$) of A we mean a derivation $\theta : \mathsf{A} \longrightarrow \mathsf{A}$ of degree r which satisfies the relation

$$\boxed{\theta(a_m a_n) = (\theta a_m) a_n + (-1)^{mr}(\theta a_n) \text{ for all } m, n \in \mathbb{N} \text{ and } a_m \in \mathsf{A}_m,\ a_n \in \mathsf{A}_n}\,.$$

(Then a graded derivation of *even* degree r is a derivation of degree r.)

Lemma 1. *Assume that* $\mathsf{A} = \bigoplus_{n\in\mathbb{N}} \mathsf{A}_n$ *is an associative graded* K*-algebra. If* θ *is a graded derivation of* A *of degree* r ($r \in \mathbb{Z}$)*, then*

$$\boxed{\theta(a_1 \dots a_n) = \sum_{i=1}^{n} (-1)^{r(m_1+\dots+m_{i-1})} a_1 \dots a_{i-1}(\theta a_i) a_{i+1} \dots a_n}\,,$$

where $a_j \in \mathsf{A}_{m_j}$, $1 \leqq j \leqq n$.

This can easily be verified by induction on n.

Corollary. *If two derivations (respectively graded derivations) of the same degree coincide on a set of homogeneous generators of an associative (respectively graded associative) algebra, then they are identical.*

Indeed, this is an immediate consequence of A.6, Lemma (ii), and the above Lemma 1.

Lemma 2. (a) *If* θ *is a graded derivation of odd degree* r *then* $\theta^2 := \theta \circ \theta$ *is a derivation of degree* $2r$.

(b) *Let* θ_r *and* θ_s *be graded derivations of degree* r *and* s*, respectively. Then their* graded commutator

$$\boxed{[\theta_r, \theta_s] := \theta_r \circ \theta_s - (-1)^{rs} \theta_s \circ \theta_r}$$

is a graded derivation of degree $r + s$.

Proof. Let a graded K-algebra $\mathsf{A} = \bigoplus_{n\in\mathbb{N}} \mathsf{A}_n$ be given. Assume that θ_r and θ_s are graded derivations of A of degree r and s, respectively. Choose $a \in \mathsf{A}$ so that $\deg(a) = n$. For every $b \in \mathsf{A}$ we have

$$\begin{aligned}\theta_r(\theta_s(ab)) = \theta_r((\theta_s a)b + (-1)^{ns} a\theta_s(b)) = (\theta_r(\theta_s a))b + (-1)^{(s+n)r}(\theta_s a)(\theta_r b)\\ + (-1)^{ns}(\theta_r a)(\theta_s b) + (-1)^{n(s+r)} a\theta_r(\theta_s b).\end{aligned}$$

If $\theta_r = \theta_s =: \theta$ and $r = s$ is odd, we obtain that $\theta \circ \theta$ is a derivation of degree $2r$. This proves (a).

If we interchange θ_r and θ_s in the above relation and subtract $(-1)^{rs}\theta_s(\theta_r(ab))$ from $\theta_r(\theta_s(ab))$ we get

$$[\theta_r, \theta_s](ab) = ([\theta_r, \theta_s]a)b + (-1)^{(r+s)n} a[\theta_r, \theta_s]b,$$

which proves (b).

(**3**) A graded K-algebra $\mathsf{A} = \bigoplus_{n\in\mathbb{N}} \mathsf{A}_n$ with the multiplication

$$[\,,\,] : \mathsf{A} \times \mathsf{A} \longrightarrow \mathsf{A}, \quad (a, b) \mapsto [a, b]$$

is said to be a *graded Lie algebra* if the multiplication satisfies the following two conditions:

Grad.Lie 1. $[a_m, a_n] = -(-1)^{mn}[a_n, a_m]$ for all $a_m \in \mathsf{A}_m,\ a_n \in \mathsf{A}_n$.

Grad.Lie 2. $(-1)^{mq}[a_m, [a_n, a_q]] + (-1)^{nm}[a_n, [a_q, a_m]]$
$+ (-1)^{qn}[a_q, [a_m, a_n]] = 0$ for all $a_m \in \mathsf{A}_m,\ a_n \in \mathsf{A}_n,\ a_q \in \mathsf{A}_q$.

The second relation is called the *graded Jacobi identity.*

Proposition. *The set of the graded derivations of a graded algebra becomes a graded Lie algebra with the graded commutator given by Lemma 2.*

The proof is a straightforward, but tedious computation. The graded Lie algebra of all graded derivations of A will be denoted by $\operatorname{Der}\mathsf{A}$.

A.8. Tensor algebras over a module.

We continue to let K *be a commutative ring with unit element. For the rest, we assume that*

$$\boxed{\text{all modules are finitely generated free modules over } \mathsf{K}}\,.$$

Some of the results do not depend on this assumption, these will be marked by an asterisk.

(**1**) Let V_1 and V_2 be K-modules. **There is** a K-module $V_1 \otimes V_2$ and a K-bilinear map

$$\otimes : V_1 \times V_2 \longrightarrow V_1 \otimes V_2, \quad (a, b) \mapsto \otimes(a, b) =: a \otimes b$$

with the following **universal property**: whenever W is a K-module and $f : V_1 \times V_2 \longrightarrow W$ a bilinear map, there exists a *unique* K-linear map $\widetilde{f} : V_1 \otimes V_2 \longrightarrow W$ such that the diagramm

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "\otimes"] \arrow[dr, "f"'] & V_1 \otimes V_2 \arrow[d, "\widetilde{f}"] \\ & W \end{tikzcd}$$

commutes. Then the pair $(V_1 \otimes V_2, \otimes)$ (or by an abuse of language, simply $V_1 \otimes V_2$) is said to be a **tensor product** of V_1 and V_2. $(V_1 \otimes V_2, \otimes)$ is **unique** up to a unique isomorphism in the sense that if $(V_1 \widetilde{\otimes} V_2, \widetilde{\otimes})$ is also a tensor product of V_1 and V_2, then there exists a unique isomorphism $\varphi : V_1 \otimes V_2 \longrightarrow V_1 \widetilde{\otimes} V_2$ of K-modules which makes the following diagram commutative:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "\otimes"] \arrow[dr, "\widetilde{\otimes}"'] & V_1 \otimes V_2 \arrow[d, "\varphi"] \\ & V_1 \widetilde{\otimes} V_2 \end{tikzcd}$$

For a careful proof of this not too deep, but fundamental result the reader is referred to [15]. Note that the existence and uniqueness of a tensor product is true without our box-assumption. As for the notations, when we want to emphasize the ring K we write $V_1 \otimes_{\mathsf{K}} V_2$ instead of $V_1 \otimes V_2$.

Basic properties.

(a)* $K \otimes V = V$ for every K-module V.

(b)* *The tensor product is associative:* if V_1, V_2, V_3 are K-modules, then there exists a unique isomorphism $(V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$ such that $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$ for all $a \in V_1$, $b \in V_2$ and $c \in V_3$.

(c)* For every integer $k \geqq 2$ and K-modules $V_1, \ldots, V_k$ there is a unique K-module $V_1 \otimes \cdots \otimes V_k$ and K-multilinear map

$$\otimes : V_1 \times \cdots \times V_k \longrightarrow V_1 \otimes \cdots \otimes V_k$$

with the above universal property, and we have the canonical identifications

$$V_1 \otimes (V_2 \otimes \cdots \otimes V_k) = (V_1 \otimes \cdots \otimes V_{k-1}) \otimes V_k = V_1 \otimes V_2 \otimes \cdots \otimes V_k.$$

(d)* *The tensor product is commutative:* $V_1 \otimes V_2$ is canonically isomorphic to $V_2 \otimes V_1$ via the map $a \otimes b \mapsto b \otimes a$ $(a \in V_1,\ b \in V_2)$.

(e)* For any three K-modules V_1, V_2, W we have the canonical isomorphisms

$$\boxed{\mathrm{Hom}_{\mathsf{K}}(V_1, \mathrm{Hom}_{\mathsf{K}}(V_2, W)) \cong L(V_1, V_2; W) \cong \mathrm{Hom}_{\mathsf{K}}(V_1 \otimes V_2, W)};$$

see e.g. [46], p. 607.

(f) If V and W are K-modules, then there is a canonical isomorphism

$$\boxed{V^* \otimes W \cong \mathrm{Hom}_{\mathsf{K}}(V, W)}.$$

Proof. Consider the map

$$f : V^* \times W \longrightarrow \mathrm{Hom}_{\mathsf{K}}(V, W), \quad (\ell, w) \mapsto f(\ell, w)$$

defined by

$$f(\ell, w)(v) := \ell(v)w \quad \text{for all } v \in V.$$

Then f is a obviously bilinear, so by the universal property there is a unique K-linear map $\widetilde{f} : V^* \otimes W \longrightarrow \mathrm{Hom}_{\mathsf{K}}(V, W)$ such that $f = \widetilde{f} \circ \otimes$. Then

$$\widetilde{f}(\ell \otimes w)(v) = f(\ell, w)(v) = \ell(v)w \quad \text{for all } \ell \in V^*,\ v \in V,\ w \in W.$$

Actually $\widetilde{f}$ is an isomorphism, the inverse isomorphism can be constructed as follows. Choose a basis $(b_i)_{i=1}^n$ for V, and let $(b^i)_{i=1}^n$ be the dual basis. If

$$g : \varphi \in \mathrm{Hom}_{\mathsf{K}}(V, W) \mapsto g(\varphi) := \sum_{i=1}^{n} b^i \otimes \varphi(b_i) \in V^* \otimes W,$$

then, as an easy calculation shows, $g = \widetilde{f}^{-1}$. □

Corollary. $\dim(V \otimes W) = (\dim V)(\dim W)$. □

(g) If $V_1, \ldots, V_k$ are K-modules with bases $(b_{i,1})_{i=1}^{n_1}, \ldots, (b_{i,k})_{i=1}^{n_k}$, then $V_1 \otimes \cdots \otimes V_k$ is a free K-module and the family

$$(b_{i_1,1} \otimes \cdots \otimes b_{i_k,k})_{1 \leqq i_1 \leqq n_1, \ldots, 1 \leqq i_k \leqq n_k}$$

is a basis for $V_1 \otimes \cdots \otimes V_k$.

(h) There is a unique K-module isomorphism

$$\iota : V_1^* \otimes \cdots \otimes V_k^* \longrightarrow (V_1 \otimes \cdots \otimes V_k)^*$$

such that

$$\iota(\ell^1 \otimes \cdots \otimes \ell^k)(v_1 \otimes \cdots \otimes v_k) = \prod_{i=1}^{n} \ell^i(v_i) \quad \text{for all } \ell^i \in V_i^*,\ v_i \in V_i \ (1 \leqq i \leqq k).$$

Proof. Let a map $f : V_1^* \times \cdots \times V_k^* \longrightarrow (V_1 \otimes \cdots \otimes V_k)^*$ be defined by

$$f(\ell^1, \ldots, \ell^k)(v_1 \otimes \cdots \otimes v_k) := \prod_{i=1}^{n} \ell^i(v_i); \quad \ell^i \in V_i^*,\ v_i \in V_i \quad (1 \leqq i \leqq k).$$

f is evidently K-multilinear, so Property (c) guarantees that there is a unique K-linear map $\iota : V_1^* \otimes \cdots \otimes V_k^* \longrightarrow (V_1 \otimes \cdots \otimes V_k)^*$ such that $f = \iota \circ \otimes$. Using bases, it may easily be checked that ι is an isomorphism, the desired isomorphism, between $V_1^* \otimes \cdots \otimes V_k^*$ and $(V_1 \otimes \cdots \otimes V_k)^*$.

(i) If $V_1, \ldots, V_k$ are K-modules, then we have a canonical K-module isomorphism

$$\boxed{V_1 \otimes \cdots \otimes V_k \cong L(V_1^*, \ldots, V_k^*;\ \mathsf{K})}.$$

This may be proved in the same way as the previous property.

(j) We have a canonical K-module isomorphism

$$\boxed{V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k;\ \mathsf{K})}.$$

Indeed, $V_1^* \otimes \cdots \otimes V_k^* \overset{(i)}{\cong} L(V_1^{**}, \ldots, V_k^{**}; \mathsf{K}) \overset{\text{A.5, Lemma 1}}{\cong} L(V_1, \ldots, V_k; \mathsf{K})$.

(**2**) Let V be a K-module. For every natural number r we define the rth *tensor power* of V by

$$\mathbf{T}^r(V) := \begin{cases} \mathsf{K}, & r = 0, \\ V, & r = 1, \\ \underbrace{V \otimes \cdots \otimes V}_{r}, & r \geqq 2. \end{cases}$$

The elements of $\mathbf{T}^r(V)$ are called *contravariant tensors of order* r on V. From the associativity of the tensor product, we obtain a bilinear map

$$\mathbf{T}^r(V) \times \mathbf{T}^q(V) \longrightarrow \mathbf{T}^{r+q}(V),$$

which makes the direct sum

$$\mathbf{T}^\bullet(V) := \bigoplus_{r \in \mathbb{N}} \mathbf{T}^r(V)$$

into an *associative, graded* K*-algebra*, called the **contravariant tensor algebra** of V.

Analogously, for every natural number s let

$$\mathbf{T}_s(V) := \begin{cases} \mathsf{K}, & s = 0, \\ V^*, & s = 1, \\ \underbrace{V^* \otimes \cdots \otimes V^*}_{s} & s \geqq 2. \end{cases}$$

$\mathbf{T}_s(V)$ is said to be the module of *covariant tensors of order* s on V. As before,

$$\mathbf{T}_\bullet(V) := \bigoplus_{s\in\mathbb{N}} \mathbf{T}_s(V)$$

is an associative, graded K-algebra, called the **covariant tensor algebra** of V.

Finally, we define the module of *type* (r,s) *(contravariant of order* r*, covariant of order* s*)* tensors on V as the tensor product

$$\mathbf{T}^r_s(V) := \mathbf{T}^r(V) \otimes \mathbf{T}_s(V).$$

We agree that

$$\mathbf{T}^r_0(V) := \mathbf{T}^r(V), \quad \mathbf{T}^0_s(V) := \mathbf{T}_s(V),\ \mathbf{T}^0_0(V) := \mathsf{K}.$$

If $r > 0$ and $s > 0$, the elements of $\mathbf{T}^r_s(V)$ are also called *mixed tensors of type* (r,s). We have a canonical isomorphism

$$\mathbf{T}^r_s(V) \otimes \mathbf{T}^{r'}_{s'}(V) \cong \mathbf{T}^{r+r'}_{s+s'}(V).$$

This enables us to make the direct sum

$$\mathbf{T}^\bullet_\bullet(V) = \bigoplus_{(r,s)\in\mathbb{N}\times\mathbb{N}} \mathbf{T}^r_s(V)$$

into an associative algebra over K, called the **mixed tensor algebra** of V.

Due to our considerations in (**1**), we have the following canonical K-module isomorphisms:

$$\boxed{\begin{array}{l} \mathbf{T}^r(V) \cong L^r(V^*),\ \mathbf{T}_s(V) \cong L^s(V), \\ \mathbf{T}^r(V^*) \cong (\mathbf{T}^r(V))^* \cong \mathbf{T}_r(V), \\ \mathbf{T}^r_s(V) \cong L((V^*)^r, V^s; \mathsf{K}), \\ \mathbf{T}^1_s(V) \cong \mathrm{Hom}_{\mathsf{K}}(V^s, V) \end{array}}\ .$$

Thus the mixed tensors of type (r,s) on V may be regarded as K-multilinear maps

$$\underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \longrightarrow \mathsf{K}.$$

The last isomorphism in the box provides a convenient and frequently used interpretation for the type $(1,s)$ $(s \neq 0)$ tensors on V: they may be considered as vector-valued K-multilinear maps

$$\underbrace{V \times \cdots \times V}_{s \text{ times}} \longrightarrow V.$$

A.9. The exterior algebra.

From now on we shall assume that the ring K *contains the field* $\mathbb{Q}$ *of rational numbers, so that for all* $\alpha \in \mathsf{K}$, $q \in \mathbb{Q}^*$ *we have* $q^{-1}\alpha \in \mathsf{K}$, *and* $q^{-1}\alpha$ *is the only element of* K *satisfying* $q(q^{-1}\alpha) = \alpha$.

(**1**) Let $n \in \mathbb{N}^*$ and consider the permuation group $\mathfrak{S}_n$. Among the elements of $\mathfrak{S}_n$ there are the *transpositions*, which interchange two consecutive integers without affecting the others, and which generate $\mathfrak{S}_n$. We denote by ε the *signature function*, the only homomorphism $\mathfrak{S}_n \longrightarrow \mathbb{Z}^*$ such that

$$\varepsilon(\sigma) = -1 \quad \text{for all transpositions } \sigma \in \mathfrak{S}_n.$$

Now let V be a K-module. If $f \in \mathbf{T}_n(V) \cong L^n(V)$ and $\sigma \in \mathfrak{S}_n$, then we define the function σf by

$$(\sigma f)(v_1, \ldots, v_n) := f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \quad \text{for all } v_1, \ldots, v_n \in V.$$

Cleary, then $\sigma f \in \mathbf{T}_n(V)$. $f \in \mathbf{T}_n(V)$ is said to be *symmetric*, respectively *skew-symmetric*, if

$$\sigma f = f, \text{ respectively } \sigma f = \varepsilon(\sigma) f \quad \text{for all } \sigma \in \mathfrak{S}_n.$$

The notion of a symmetric or skew-symmetric vector-valued covariant tensor is analogous. The symmetric, as well as the skew-symmetric covariant tensors constitute submodules of $\mathbf{T}_n(V)$, denoted sometimes by $L^n_{\text{sym}}(V)$ and $L^n_{\text{skew}}(V)$, respectively. For the module of skew-symmetric tensors of order n on V we prefer the notation $\mathbf{A}_n(V)$; the elements of $\mathbf{A}_n(V)$ are called *n-forms* on V. If W is also a K-module, $L^n_{\text{skew}}(V, W)$ denotes the K-module of W-valued skew-symmetric K-multilinear maps defined on V^n.

A symmetric or skew-symmetric bilinear form B on V is said to be *non-degenerate*, if there exists no vector $v \neq 0$ in V such that $B(v, w) = 0$ for all $w \in V$.

The map

$$\text{Alt} : \mathbf{T}_n(V) \longrightarrow \mathbf{T}_n(V), \quad f \mapsto \text{Alt}(f) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \sigma f$$

is a projection onto $\mathbf{A}_n(V)$, i.e. $\text{Alt} \circ \text{Alt} = \text{Alt}$ and $\text{Alt}(f) = f$ for all $f \in \mathbf{A}_n(V)$. Alt is called the *alternator* in $\mathbf{T}_n(V)$.

(**2**) Let V be a K-module.

(a) Assume that $k \in \mathbb{N}$, $k \geqq 2$. *There is a* K*-module* $\wedge^k(V)$ *and a skew-symmetric* K*-multilinear map*

$$\wedge : \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow \wedge^k(V), \; (v_1, \ldots v_k) \mapsto \wedge(v_1, \ldots, v_k) =: v_1 \wedge \cdots \wedge v_k$$

with the following **universal property**:

If W is a K*-module and f is a skew-symmetric multilinear map of $V \times \cdots \times V$ (k copies) into W, then there is a* unique K*-linear map $\widetilde{f} : \wedge^k(V) \longrightarrow W$ such that the diagram*

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\wedge"] \arrow[dr, "f"'] & \wedge^k(V) \arrow[d, "\widetilde{f}"] \\ & W \end{tikzcd}$$

commutes.

The pair $(\wedge^k(V), \wedge)$, or by abuse of language, $\wedge^k(V)$ is said to be a kth *exterior power of* V, the elements of $\wedge^k(V)$ are called *k-vectors.* As the tensor product of modules, $(\wedge^k(V), \wedge)$ is also *unique* up to a unique module-isomorphism, so we may speak of *the* kth exterior power of V.

(b) Let $\wedge^0(V) := \mathsf{K}$, $\wedge^1(V) := V$, $\wedge(V) := \bigoplus_{k\in\mathbb{N}} \wedge^k(V)$. *For each pair of positive integers (r, s) there exists a unique bilinear map*

$$\wedge^r(V) \times \wedge^s(V) \longrightarrow \wedge^{r+s}(V),$$

called the wedge *or* exterior product *and denoted also by $\wedge$, such that if $u_1, \ldots, u_r, v_1, \ldots, v_s \in V$ then*

$$(u_1 \wedge \cdots \wedge u_r,\ v_1 \wedge \cdots \wedge v_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge v_1 \wedge \cdots \wedge v_s.$$

The wedge product makes $\wedge(V)$ into a graded K*-algebra which is associative and (graded) commutative.*

This algebra is said to be the **exterior algebra**, or the **Grassmann algebra** of the K-module V.

A sketchy proof of this important result may be found e.g. in Lang's Algebra [46], for a detailed argument the reader is referred to [15].

(c) Let $\dim_{\mathsf{K}} V = n$. If $k > n$, then $\wedge^k(V) = 0$. Assume that $(b_i)_{i=1}^n$ is a basis of V. If $1 \leqq k \leqq n$, then $\wedge^k(V)$ is also a finitely generated free module, and the k-vectors

$$b_{i_1} \wedge \cdots \wedge b_{i_k}, \quad i_i < \cdots < i_k$$

form a basis of $\wedge^k(V)$. We have:

$$\dim \wedge^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad \dim \wedge(V) = 2^n.$$

(d) Let a map

$$f : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \longrightarrow \mathbf{A}_k(V), \quad (\ell^1, \dots, \ell^k) \mapsto f(\ell^1, \dots, \ell^k)$$

be defined by

$$\begin{aligned} f(\ell^1, \dots, \ell^k)(v_1, \dots, v_k) &:= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \ell^1(v_{\sigma(1)}) \dots \ell^k(v_{\sigma(k)}) \\ &= (\mathrm{Alt}(\ell^1 \otimes \cdots \otimes \ell^k))(v_1, \dots, v_k) \end{aligned}$$

($v_i \in V$, $1 \leqq i \leqq k$). f is obviously K-multilinear, so the universal property of the tensor power leads to a unique ('natural') K-linear map $\widetilde{f} : \wedge^k(V^*) \longrightarrow \mathbf{A}_k(V)$ which makes the diagram

$$\begin{tikzcd} V^* \times \cdots \times V^* \arrow[r] \arrow[dr, "f"'] & \wedge^k (V^*) \arrow[d, "\widetilde{f}"] \\ & \mathbf{A}_k(V) \end{tikzcd}$$

commutative. It may be readily shown that $\widetilde{f}$ is actually an isomorphism of K-modules, thus we have:

$$\boxed{\wedge^k(V^*) \cong \mathbf{A}_k(V)}\,.$$

From this it follows that

$$\boxed{\mathbf{A}(V) := \bigoplus_{k \in \mathbb{N}} \mathbf{A}_k(V) \cong \wedge(V^*)}\,.$$

In the text we shall exclusively use this interpretation of the exterior algebra of V^*. Then the wedge product describded by (b) has the following meaning:

$$f \wedge g = \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}(f \otimes g)$$

for all $f \in \wedge^k(V^*) \cong \mathbf{A}_k(V)$, $g \in \wedge^\ell(V^*) \cong \mathbf{A}_\ell(V)$. $f \otimes g$, as an element of $L^{k+\ell}(V)$, operates by the rule

$$(f \otimes g)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = f(v_1, \dots, v_k) g(v_{k+1}, \dots, v_{k+\ell})$$

($v_i \in V$, $1 \leqq i \leqq k + \ell$). A convenient way to compute wedge products is provided by the formula

$$(f \wedge g)(v_1, \dots, v_{k+\ell}) = \sum_{\sigma \in \mathfrak{S}^*_{k+\ell}} \varepsilon(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}),$$

where $\mathfrak{S}^*_{k+\ell}$ is the subset of $\mathfrak{S}_{k+\ell}$ consisting of permutations σ such that $\sigma(1) < \sigma(2) \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+\ell)$.

A.10. Categories and functors.

We tacitly accept the von Neumann-Bernays-Gödel axiom system of set theory. (A derivative of this system is presented in detail in the book of M. Eisenberg [32] cited before.) This enables us to speak legitimately of 'larger totalities' than sets, called *classes*. The axioms specify the behavior of classes. A *set* is by definition a class which is a member of some class.

(**1**) A **category** $\mathfrak{A}$ consists of

(i) a class $\mathrm{Ob}(\mathfrak{A})$ of *objects*;

(ii) for any two objects $A, B \in \mathrm{Ob}(\mathfrak{A})$ a set $\mathrm{Mor}(A, B)$ called the set of *morphisms* of A into B;

(iii) for each triple (A, B, C) of objects a law of composition (i.e. a map) $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \longrightarrow \mathrm{Mor}(A, C)$ which assigns to $\alpha \in \mathrm{Mor}(A, B)$ and $\beta \in \mathrm{Mor}(B, C)$ an elements $\beta \circ \alpha \in \mathrm{Mor}(A, C)$.

These obey three axioms:

CAT 1. Any two sets $\mathrm{Mor}(A, B)$ and $\mathrm{Mor}(A', B')$ are disjoint unless $A = A'$ and $B = B'$, in which case they are equal.

CAT 2. For each object A in $\mathrm{Ob}(\mathfrak{A})$ there is a morphism $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ such that if $\alpha \in \mathrm{Mor}(A, B)$ then $\alpha \circ \mathrm{id}_A = \alpha = \mathrm{id}_B \circ \alpha$.

CAT 3. The law of composition is associative (when defined), i.e. if $\alpha \in \mathrm{Mor}(A, B),\ \beta \in \mathrm{Mor}(B, C),\ \gamma \in \mathrm{Mor}(C, D),$ then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.$

Usually we write $\alpha : A \longrightarrow B$ or $A \xrightarrow{\alpha} B$ for $\alpha \in \mathrm{Mor}(A, B)$. By abuse of language, we sometimes refer to the class of objects as the category itself, if it is clear what the morphisms are meant to be.

(**2**) Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are categories. A **covariant functor** F of $\mathfrak{A}$ into $\mathfrak{B}$ is rule which associates with each object A in $\mathrm{Ob}(\mathfrak{A})$ an object $F(A)$ in $\mathrm{Ob}(\mathfrak{B})$, and with each morphism $\alpha \in \mathrm{Mor}(A, B)$ a morphism $F(\alpha) \in \mathrm{Mor}(F(A), F(B))$ satisfying the following conditions:

FUN 1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathrm{Ob}(\mathfrak{A})$.

FUN 2. If $\alpha \in \mathrm{Mor}(A, B),\ \beta \in \mathrm{Mor}(B, C)$ then $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.

The notion of a **contravariant functor** from $\mathfrak{A}$ into $\mathfrak{B}$ is analogous, only the arrows are reversed. Formally, a contravariant functor F associates with each morphism $\alpha \in \mathrm{Mor}(A, B)$ a morphism $F(\alpha) \in \mathrm{Mor}(F(B), F(A))$ satisfying

FUN 2'. If $\alpha \in \mathrm{Mor}(A, B),\ \beta \in \mathrm{Mor}(B, C)$ then $F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$.

Sometimes a functor is denoted by writing α_* instead of $F(\alpha)$ in the case of a covariant functor, and by writing α^*, if F is a contravariant functor (cf. e.g. 1.25 and 1.30, Example(3)).

In a similar way, one may define *functors of several variables* covariant in some of their variables and contravariant in others. For this procedure, the reader is referred to Mac Lane's book Homology [48].

Example. (a) Let $\mathfrak{A}$ be the category of K-modules: $\mathrm{Ob}(\mathfrak{A})$ is the class of K-modules and

$$\mathrm{Mor}(V, W) := \mathrm{Hom}_{\mathsf{K}}(V, W) \quad \text{for all } V, W \in \mathrm{Ob}(\mathfrak{A}).$$

Consider at the same time the category $\mathfrak{B}$ of graded K-algebras. Then $\mathrm{Ob}(\mathfrak{B})$ consists of the graded algebras over K and the morphisms are those K-algebra homomorphisms which respect the graduations, i.e. the *graded algebra homomorphisms* ([12], III, §3, no. 1). If

$$\wedge : V \in \mathrm{Ob}(\mathfrak{A}) \mapsto \wedge(V) := \text{the exterior algebra of } V,$$

and the map

$$f \in \mathrm{Mor}(V, W) \mapsto \wedge(f) \in \mathrm{Mor}(\wedge(V), \wedge(W))$$

is determined by

$$\wedge(f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k) \quad \text{for all } v_1, \ldots v_k \in V,$$

then $\wedge$ is a covariant functor from $\mathfrak{A}$ into $\mathfrak{B}$.

(b) In particular, let VS be the category of the finite-dimensional real vector spaces. In this case for any two objects $V, W \in \mathrm{Ob}(\mathrm{VS})$

$$\mathrm{Mor}(V, W) = \mathrm{Hom}_{\mathbb{R}}(V, W) =: L(V, W).$$

A covariant functor F from VS into VS is said to be *smooth* if the maps

$$L(V, W) \longrightarrow L(F(V), F(W))$$

are smooth as maps between smooth manifolds. (Recall that in view of 1.9 every finite-dimensional real vector space carries a natural smooth structure.) If

$$F(V) := \wedge^k(V) \text{ or } F(V) := \mathbf{T}^k(V) \quad (V \in \mathrm{Ob}(\mathrm{VS}))$$

then F is a smooth functor.

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