A GENERALIZATION OF WEYL'S THEOREM ON PROJECTIVELY RELATED AFFINE CONNECTIONS

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From a physical point of view, the geodesics in a four-dimensional Lorentzian spacetime are really significant only as point sets. In 1921 Weyl proved that two torsion-free covariant derivative operators D_M and $\bar{D}_M$ on a manifold M have the same geodesics with possibly different parametrizations if and only if there is a 1-form α on M such that $\bar{D} = D + \alpha \otimes 1 + 1 \otimes \alpha$, where 1 is the identity (1,1) tensor on M. By a theorem of Ambrose, Palais and Singer [1], torsion-free covariant derivative operators are generated by affine sprays, and *vice versa*. More generally, any (not necessarily affine) spray induces a number of covariant derivatives in the tangent bundle τ of M, or in the pull-back bundle $\tau^*\tau$. We show that in the context of sprays, similarly to Weyl's relation, a correspondence between the Yano derivatives can be detected.

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1. Introduction

The éclat of the general relativity theory of A. Einstein inspired an intensive development of Riemannian geometry and its generalizations in the twenties-thirties of the 20th century. H. Weyl was the first leading mathematician who devoted a whole monograph to the geometrical background of Einstein's gravitational theory and proposed at the same time a unified field theory [22]. Weyl also recognized the importance of a (torsion-free) covariant derivative operator, called an *affine connection* by him, independently of a metric [21]. We are inclined to think that Weyl's other outstanding work [23] from this period also has a physical motivation. Let us try to reconstruct a possible train of thoughts.

General relativity models aspects of the physical universe by (time-orientable, connected, non-flat) four-dimensional Lorentz manifolds. Thus 'time' is an integral part of the fundamental structure, therefore the parameter of a *freely falling particle* (mathematically: a *geodesic*) does not have a physical meaning as Newtonian time in classical mechanics. This suggests that the relevant transformations in general relativity are the *projective transformations* which respect geodesics only as point sets and not as parametrized curves (see also [11]). Now a quite natural question may be raised: to what extent is the geometric structure (more precisely, the affine connection) determined by the system of freely falling particles? In purely geometric terms, translated into our contemporary language, Weyl's answer is the following.

THEOREM 1 [23]. *Let D_M and $\bar{D}_M$ be two torsion-free covariant derivative operators on a manifold M. D_M and $\bar{D}_M$ have the same geodesics up to a (strictly increasing) parameter transformation if and only if there is a 1-form α on M such that*

$$\bar{D}_M = D_M + \alpha \otimes 1_{\mathfrak{X}(M)} + 1_{\mathfrak{X}(M)} \otimes \alpha.$$

Two covariant derivative operators connected by this *Weyl relation* are called *projectively equivalent*. (Weyl, of course, used the Christoffel symbols of $\bar{D}_M$ and D_M in his formulation.) For a proof of Weyl's theorem in the context of the classical tensor calculus we refer to [10]. Coordinate-free ('intrinsic') proofs are available e.g. in [13, 14 or 17]. It is also well known that there is a canonical bijective correspondence between the set of (torsion-free) covariant derivative operators on a manifold M and the set of some special vector fields, called *affine sprays*, on TM in such a way that the corresponding objects have the same geodesics. A modern formulation of this interplay is due to Ambrose, Palais and Singer [1], see also [5]. From this point of view, Weyl's theorem belongs to the theory of affine sprays. The purpose of the present work is to extend Weyl's result to the more general class of *non-affine* (i.e. not necessarily affine) *sprays*. To sketch a more precise preliminary picture, suppose that $(\mathcal{U}, \varphi)$ is an n-dimensional chart on the manifold M. Then, locally, the tangent manifold TM is identified with $\varphi(\mathcal{U}) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. An affine spray over M has a local expression

$$(p, v) \in \varphi(\mathcal{U}) \times \mathbb{R}^n \mapsto (p, v, v, f(p, v)) \in \varphi(\mathcal{U}) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n,$$

where the map $f : \varphi(\mathcal{U}) \times \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2, smooth on $\varphi(\mathcal{U}) \times (\mathbb{R}^n \setminus \{0\})$, and has the homogeneity property

$$\forall t \in \mathbb{R} : f(p, tv) = t^2 f(p, v), \qquad (p, v) \in \varphi(\mathcal{U}) \times \mathbb{R}^n.$$

Then it follows that f is *quadratic* in its second variable, namely

$$f(p, v) = \frac{1}{2} D_2 D_2 f(p, 0)(v, v) \qquad \text{for all } (p, v) \in \varphi(\mathcal{U}) \times \mathbb{R}^n$$

(see e.g. [9], p. 100). To avoid this strong consequence, we weaken the differentiability hypothesis and require only the 1-times continuous differentiability of f on its whole domain. Thus we get the concept of (non-affine) sprays. (An elegant intrinsic formulation of these ideas will be presented in Section 4.) This seemingly mild generalization leads us to another world. The appearance of non-affine sprays is typical in Finsler geometry (see e.g. [16]), while the geodesic spray in Riemannian geometry is always affine.

To obtain a Weyl-type relation in the general situation, we need an appropriate covariant derivative operator; it will be the so-called *Yano derivative*. The Yano derivative lives no longer on the base manifold M but in the pull-back bundle of the tangent bundle of M by its own projection. So we also need a machinery, the pull-back formalism, in a presentation, which is more or less known in Finsler geometry (see e.g. [3] or [18]), well known in recent investigations in the geometry of second-order ordinary differential equations or in the inverse problem of calculus of variations [4, 12], but quite new in the present context. For the convenience of the reader, we devote Sections 1–3 mainly to the necessary technicalities. Then, in Section 4, we examine how some basic geometric data transform under a projective change of the spray. Finally, in the last section, we find and prove the analogue of Weyl's theorem in the general framework of sprays.

We hope that our approach may be a stimulating initial step for a development of a new theory of 'projective connections' compared with the very elegant theory of 'Thomas-Whitehead projective connections' elaborated by C. W. Roberts [15] at the level of affine sprays. There exist, of course, indispensable classical sources as starting points for such a theory; we refer here only to J. Douglas's paper on the general geometry of paths [6].

CONVENTIONS. (1) Our base manifold is an n-dimensional ($n \geq 1$) smooth manifold M, whose topology is assumed to be Hausdorff, second-countable and connected. $C^\infty(M)$ stands for the ring of real-valued smooth functions on M. T_pM is the tangent space to M at $p \in M$, TM is the $2n$-dimensional tangent manifold of M, and $\overset{\circ}{T}M$ is the open submanifold of the non-zero tangent vectors to M. τ denotes the natural projection of TM onto M. The tangent bundle of M is the triple (TM, τ, M), the shorthand for this bundle will be τ. If $F : M \to N$ is a smooth map, then $F_* : TM \to TN$ denotes its induced tangent map. The velocity vector field of a smooth path $c : I \to M$ ($I \subset \mathbb{R}$ is an open interval) is the smooth path $\dot{c} : I \to TM, t \mapsto \dot{c}(t)$ given by $\dot{c}(t)f := (f \circ c)'(t)$ for all $f \in C^\infty(M)$.

(2) We denote by $\mathfrak{X}(M)$ the $C^\infty(M)$-module of (smooth) vector fields on M. $\mathcal{A}^1(M)$ is the dual of $\mathfrak{X}(M)$, i.e. the module of 1-forms on M. d is the exterior derivative on every manifold. The symbols $\otimes$ and $\odot$ are used for the tensor product and the symmetric product, respectively. Concerning the numerical factor to be employed in symmetric products, we adopt the convention of [7], p. 5.

(3) f^v and f^c stand for the vertical and the complete lift of a smooth function f on M into TM. $f^v := f \circ \tau$, while f^c is defined by

$$v \in TM \mapsto f^c(v) := (df)(v) := v(f) \in \mathbb{R}.$$

2. Tensor fields along the tangent bundle projection

2.1. The *pull-back bundle* $\tau^*\tau$ of τ by τ is the vector bundle (τ^*TM, p_1, TM), where the total manifold is defined by

$$TM \times_M TM := \{(v, w) \in TM \times TM \mid \tau(v) = \tau(w)\},$$

and the projection p_1 is the restriction of the first projection $TM \times TM \to TM$ to $TM \times_M TM$. The fibres of $\tau^*\tau$ are the n-dimensional real vector spaces $\{v\} \times$

$T_{\tau(v)}M \cong T_{\tau(v)}M$, $v \in TM$. Any section of $\tau^*\tau$ is of the form

$$\widetilde{X} : v \in TM \mapsto \widetilde{X}(v) = (v, \underline{X}(v)) \in TM \times_M TM,$$

where $\underline{X} : TM \to TM$ is a smooth map such that $\tau \circ \underline{X} = \tau$. $\widetilde{X}$ and $\underline{X}$ may be identified; we shall use this identification without any comment. The set $\mathfrak{X}(\tau)$ of all sections of $\tau^*\tau$ is a module over $C^\infty(TM)$ which will be mentioned as the module of *vector fields along* τ. An immediate example: if X is a vector field on M, then $\widehat{X} := X \circ \tau$ is a vector field along τ, called a *basic vector field*, or the *lift* of X into $\mathfrak{X}(\tau)$. Clearly, every vector field along τ can locally be combined from basic vector fields with coefficients in $C^\infty(TM)$. The dual of $\mathfrak{X}(\tau)$ is the $C^\infty(TM)$-module $A^1(\tau)$ of *1-forms along* τ. $A^1(\tau)$ is locally generated by the set $\{\widehat{\alpha} := \alpha \circ \tau \in A^1(\tau) \mid \alpha \in A^1(M)\}$ of *basic 1-forms*.

2.2. By a *tensor field of type* (k, l) $(\neq (0,0))$ *along* τ we mean a $C^\infty(TM)$-multilinear map $(A^1(\tau))^k \times (\mathfrak{X}(\tau))^l \to C^\infty(TM)$. The set $\mathcal{T}^k_l(\tau)$ of all tensor fields of type (k, l) along τ is a module over $C^\infty(TM)$. A tensor field of type $(0, 0)$ along τ is simply an element of $C^\infty(TM)$. In what follows we shall freely identify $\mathcal{T}^1_1(\tau)$ with $\mathrm{Hom}(\mathfrak{X}(\tau), \mathfrak{X}(\tau))$ and $\mathcal{T}^1_l(\tau)$ with $\mathrm{Mult}((\mathfrak{X}(\tau))^l, \mathfrak{X}(\tau))$ (the latter is the module of $C^\infty(TM)$-multilinear maps $(\mathfrak{X}(\tau))^l \to \mathfrak{X}(\tau)$).

CONVENTION. $X, Y, \ldots$ will denote vector fields on the base manifold M; $\widehat{X}, \widehat{Y}, \ldots$ stand for their lifts into $\mathfrak{X}(\tau)$. $\widetilde{X}, \widetilde{Y}, \ldots$ denote general vector fields along τ. We shall use Greek letters $\xi, \eta, \ldots$ for arbitrary vector fields on TM. $\alpha, \beta, \ldots$ denote 1-forms on M; $\widehat{\alpha}, \widehat{\beta}, \ldots$ are the corresponding basic 1-forms. $\widetilde{A}, \widetilde{B}, \ldots$ are written for arbitrary tensor fields along τ.

2.3. As in the case of any tensor algebra over a module, we also have a family of $C^\infty(TM)$-linear maps $c^i_j : \mathcal{T}^k_l(\tau) \to \mathcal{T}^{k-1}_{l-1}(\tau)$ $(1 \le i \le k, 1 \le j \le l)$ called (i, j)-contraction. An important particular example is the *trace* of a $(1, l)$ tensor: if $\widetilde{A} \in \mathcal{T}^1_l(\tau)$, then $\mathrm{tr}\,\widetilde{A} := c^1_1(\widetilde{A}) \in \mathcal{T}^0_{l-1}(\tau)$. For any (1,1) tensor field $\widetilde{K} \in \mathrm{Hom}(\mathfrak{X}(\tau), \mathfrak{X}(\tau))$ and type $(0, l)$ tensor field $\widetilde{A}$ $(l \ge 1)$, we define the tensor field $i_{\widetilde{K}}\widetilde{A} \in \mathcal{T}^0_l(\tau)$ by

$$i_{\widetilde{K}}\widetilde{A}(\widetilde{X}_1, \ldots, \widetilde{X}_l) := \sum_{j=1}^{l} \widetilde{A}(\widetilde{X}_1, \ldots, \widetilde{K}(\widetilde{X}_j), \ldots, \widetilde{X}_l); \qquad \widetilde{X}_j \in \mathfrak{X}(\tau),\ 1 \le j \le l.$$

If $\widetilde{Y}$ is a vector field along τ, $i_{\widetilde{Y}}$ will denote the usual substitution operator.

LEMMA 1. *Let* $\widetilde{A}$ *be a symmetric* $(0, l)$ $(l \ge 1)$ *tensor field and* $\widetilde{K}$ *a* $(1, 1)$ *tensor field along* τ. *Then we have*

$$\mathrm{tr}\,(\widetilde{A} \odot \widetilde{K}) = \mathrm{tr}\,(\widetilde{K})\widetilde{A} + i_{\widetilde{K}}\widetilde{A}; \tag{1}$$

$$\mathrm{tr}\,\widetilde{A} \otimes \widetilde{K}(\widetilde{X})) = i_{\widetilde{K}(\widetilde{X})}\widetilde{A}, \qquad \widetilde{X} \in \mathfrak{X}(\tau). \tag{2}$$

The proof is straightforward but lengthy, so we omit it.

3. Canonical objects and horizontal maps

3.1. Consider the canonical exact sequence

$$0 \longrightarrow \tau^*TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} \tau^*TM \longrightarrow 0 \tag{3}$$

of strong bundle maps, where $\mathbf{i}(v, w) := (t \in \mathbb{R} \mapsto tv + w)'(0)$ and $\mathbf{j}(z) := (z, (\tau_*)_v(z))$, if $z \in T_vTM$. Im($\mathbf{i}$) = Ker($\mathbf{j}$) is called the *vertical subbundle* of the tangent bundle of TM. The sections of the vertical subbundle form the $C^\infty(TM)$-module $\mathfrak{X}^v(TM)$ of *vertical vector fields* on TM. The sequence (3) gives rise to a short exact sequence

$$0 \longrightarrow \mathfrak{X}(\tau) \xrightarrow{\mathbf{i}} \mathfrak{X}(TM) \xrightarrow{\mathbf{j}} \mathfrak{X}(\tau) \longrightarrow 0 \tag{4}$$

of $C^\infty(TM)$-homomorphisms of modules, where, by a tolerable abuse of notation, the induced maps between the sections are also denoted by $\mathbf{i}$ and $\mathbf{j}$. The endomorphism $J := \mathbf{i} \circ \mathbf{j}$ is called the *vertical endomorphism.* We have a *canonical vector field along* τ, denoted by δ and defined by $\delta(v) := (v, v)$ for all $v \in TM$. Then $C := \mathbf{i} \circ \delta$ is another canonical object, the *Liouville vector field* (or radial vertical vector field).

3.2. It is well known that any vector field on TM is uniquely determined by its action on the set $\{f^c \in \mathfrak{X}(TM) \mid f \in C^\infty(M)\}$ of complete lifts of smooth functions on M (see e.g. [25] or [18]). This observation is very useful to define the *vertical lift* X^v and the *complete lift* X^c of a vector field X on M: $X^v f^c := (Xf)^v$ and $X^c f^c := (Xf)^c$, for all $X \in \mathfrak{X}(M)$. If $(X_i)_{i=1}^n$ is a local frame field on M, then $((X_i^c)_{i=1}^n, (X_i^v)_{i=1}^n)$ is a local frame field on TM. We shall need the following familiar relations:

$$X^v f^v = 0, \qquad \mathbf{i}\widehat{X} = X^v, \qquad \mathbf{j}X^c = \widehat{X}, \qquad JX^v = 0, \qquad JX^c = X^v; \tag{5}$$

$$[X^v, Y^v] = 0, \qquad [C, X^v] = -X^v, \qquad [C, X^c] = 0. \tag{6}$$

3.3. By a *horizontal map for* τ (called also a *nonlinear connection* on M) we mean a strong bundle map $\mathcal{H} : \tau^*TM \to TTM$, *smooth only on* $\mathring{T}M \times_M TM$, such that $\mathbf{j} \circ \mathcal{H} = 1_{\tau^*TM}$. Briefly: $\mathcal{H}$ is a smooth right splitting of (3) over $\mathring{T}M$. Specifying a horizontal map $\mathcal{H}$ for τ, we also get a unique strong bundle map $\mathcal{V} : TTM \to TM \times_M TM$, smooth on $T\mathring{T}M$, such that Ker$\mathcal{V}$ = Im$\mathcal{H}$. $\mathcal{V}$ is the *vertical map* belonging to $\mathcal{H}$. Then $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ and $\mathbf{v} := \mathbf{i} \circ \mathcal{V}$ are projection operators in TTM such that $\mathbf{h} \circ \mathbf{v} = 0, \mathbf{h} + \mathbf{v} = 1_{TTM}$ and hence TTM = Im$\mathbf{h}$ $\oplus$ Im$\mathbf{v}$; $\mathbf{h}$ and $\mathbf{v}$ are called the *horizontal* and the *vertical projectors* belonging to $\mathcal{H}$. The bundle maps $\mathcal{H}, \mathcal{V}, \mathbf{h}$ and $\mathbf{v}$ induce naturally $C^\infty(\mathring{T}M)$-homomorphisms at the level of sections, the induced maps will also be denoted by the same symbols. The direct decomposition of TTM gives rise to the decomposition $\mathfrak{X}(TM) = \mathfrak{X}^h(TM) \oplus \mathfrak{X}^v(TM)$, where $\mathfrak{X}^h(TM) := \mathcal{H}(\mathfrak{X}(\tau)) = \mathbf{h}(\mathfrak{X}(TM))$ is the module of *horizontal vector fields* on TM. If $X \in \mathfrak{X}(M)$ then the *horizontal lift of* X *by* $\mathcal{H}$ is the horizontal vector field $X^h := \mathcal{H}(\widehat{X}) \overset{(5)}{=} \mathbf{h}(X^c)$. Notice that $JX^h = \mathbf{i} \circ \mathbf{j} \circ \mathcal{H}(\widehat{X}) = \mathbf{i}(\widehat{X}) \overset{(5)}{=} X^v$.

4. Covariant derivative operators in $\tau^*\tau$

4.1. By a covariant derivative operator in $\tau^*\tau$ we mean, as usual, an $\mathbb{R}$-bilinear map

$$D : \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \to \mathfrak{X}(\tau), \qquad (\xi, \widetilde{Y}) \mapsto D(\xi, \widetilde{Y}) =: D_\xi \widetilde{Y}$$

satisfying the conditions

$$D_{F\xi}\widetilde{Y} = F D_\xi \widetilde{Y} \quad \text{and} \quad D_\xi F\widetilde{Y} = (\xi F)\widetilde{Y} + F D_\xi \widetilde{Y}$$

concerning the multiplication with a smooth function F on TM. We define the curvature R^D of D by

$$R^D(\xi, \eta)\widetilde{Z} := D_\xi D_\eta \widetilde{Z} - D_\eta D_\xi \widetilde{Z} - D_{[\xi,\eta]}\widetilde{Z}; \qquad \xi, \eta \in \mathfrak{X}(TM), \qquad \widetilde{Z} \in \mathfrak{X}(\tau).$$

Specifying a horizontal map $\mathcal{H}$ for τ, R^D may be decomposed into a horizontal, a vertical and a mixed part (see e.g. [3, 15]). We shall need only the *mixed curvature* $\mathbf{P}$ given by

$$\mathbf{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := R^D(\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y})\widetilde{Z} = D_{\mathcal{H}\widetilde{X}} D_{\mathbf{i}\widetilde{Y}} \widetilde{Z} - D_{\mathbf{i}\widetilde{Y}} D_{\mathcal{H}\widetilde{X}} \widetilde{Z} - D_{[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]}\widetilde{Z}.$$

4.2. Any horizontal map $\mathcal{H}$ for τ gives rise to a covariant derivative operator ∇ in $\tau^*\tau$ by the rule

$$\nabla_\xi \widetilde{Y} := \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] + \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}] \tag{7}$$

(using the notation of Subsection 3.3). ∇ is said to be the *Berwald derivative* induced by $\mathcal{H}$. From (7) we immediately get the more convenient formulae

$$\nabla_{X^v}\widehat{Y} = 0, \qquad \nabla_{X^h}\widehat{Y} = \mathcal{V}[X^h, Y^v] \quad \text{for all } X, Y \in \mathfrak{X}(M). \tag{8}$$

LEMMA 2. *Let ∇ be the Berwald derivative induced by a horizontal map $\mathcal{H}$, and let $\mathring{\mathbf{P}}$ denote the mixed curvature of ∇. For any vector fields X, Y, Z on M we have*

$$\mathring{\mathbf{P}}(\widehat{X}, \widehat{Y})\widehat{Z} = \mathcal{V}[[X^h, Y^v], Z^v]. \tag{9}$$

$\mathring{\mathbf{P}}$ is symmetric in its second and third arguments.

For a proof (in terms of the 'tangent bundle formalism') the reader is referred to [19]; see also [18].

With the help of the mixed curvature $\mathring{\mathbf{P}}$, we define an important change of the Berwald derivative ∇ by the formula

$$D_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{n+1}(\operatorname{tr} \mathring{\mathbf{P}}(\mathbf{j}\xi, \widetilde{Y}))\delta. \tag{10}$$

The covariant derivative operator D so obtained is called the *Yano derivative* induced by $\mathcal{H}$. The idea of this modification of ∇ can be found in Yano's book [24], as for our modern interpretation, see also [19].

4.3. Let ∇ be the Berwald derivative induced by a horizontal map $\mathcal{H}$. For each vector field $\widetilde{X}$ along τ, we define the *v-covariant derivative* operator $\nabla^v_{\widetilde{X}}$ in $\mathfrak{X}(\tau)$ by $\nabla^v_{\widetilde{X}}\widetilde{Y} := \nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y}$. Actually, this operator does not depend on the choice of $\mathcal{H}$. Indeed, representing $\widetilde{Y}$ in the form $\mathbf{j}\eta$, $\eta \in \mathfrak{X}(TM)$, we find $\nabla^v_{\widetilde{X}}\mathbf{j}\eta = \nabla_{\mathbf{i}\widetilde{X}}\mathbf{j}\eta \overset{(7)}{:=} \mathbf{j}[\mathbf{v}\circ\mathbf{i}(\widetilde{X}), \mathcal{H}\circ\mathbf{j}(\eta)] + \mathcal{V}[\mathbf{h}\circ\mathbf{i}(\widetilde{X}), \mathbf{i}\circ\mathbf{j}(\eta)] = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathbf{h}\eta] = \mathbf{j}[\mathbf{i}\widetilde{X}, \eta - \mathbf{v}\eta] = \mathbf{j}[\mathbf{i}\widetilde{X}, \eta]$.

Let us define the action of $\nabla^v_{\widetilde{X}}$ on functions by the formula

$$\nabla^v_{\widetilde{X}}F := (\mathbf{i}\widetilde{X})F, \qquad F \in C^\infty(TM). \tag{11}$$

Then, by a reasonable product rule, the operators $\nabla^v_{\widetilde{X}}$ can be extended to operate on arbitrary tensor fields along τ. Moreover, it is possible to define a single operator ∇^v that associates to a tensor field $\widetilde{A} \in \mathcal{T}^k_l(\tau)$ the tensor field $\nabla^v\widetilde{A} \in \mathcal{T}^k_{l+1}(\tau)$ such that

$$(\nabla^v\widetilde{A})(\widetilde{X}, \widetilde{\alpha}^1, \ldots, \widetilde{\alpha}^k, \widetilde{X}_1, \ldots, \widetilde{X}_l) := (\nabla^v_{\widetilde{X}}\widetilde{A})(\widetilde{\alpha}^1, \ldots, \widetilde{\alpha}^k, \widetilde{X}_1, \ldots, \widetilde{X}_l) \tag{12}$$

for all $\widetilde{X}, \widetilde{X}_i \in \mathfrak{X}(\tau)$ and $\widetilde{\alpha}^j \in \mathcal{A}^1(\tau)$.

4.4. Let F be a smooth function on TM (or on $\mathring{T}M$). F is called *1-homogeneous* if $\nabla^v_\delta F = F$, i.e. if $CF = F$. By the *Hessian* of F we mean the type $(0, 2)$ tensor field $\nabla^v\nabla^v F := \nabla^v(\nabla^v F)$ along τ. Applying (11) and (12), we immediately get

$$\nabla^v\nabla^v F(\widehat{X}, \widehat{Y}) = X^v(Y^v F) \qquad \text{for all } X, Y \in \mathfrak{X}(M). \tag{13}$$

From this it follows that $\nabla^v\nabla^v F$ is actually symmetric.

LEMMA 3. *If $F : \mathring{T}M \to \mathbb{R}$ is a 1-homogeneous smooth function, then its Hessian has the homogeneity property $\nabla^v_\delta(\nabla^v\nabla^v F) = -\nabla^v\nabla^v F$.*

Proof: Let, temporarily, $\widetilde{A} := \nabla^v\nabla^v F$. For any two vector fields X and Y on M we have $(\nabla^v_\delta\widetilde{A})(\widehat{X}, \widehat{Y}) = (\nabla_C\widetilde{A})(\widehat{X}, \widehat{Y}) = C(\widetilde{A}(\widehat{X}, \widehat{Y})) - \widetilde{A}(\nabla_C\widehat{X}, \widehat{Y}) - \widetilde{A}(\widehat{X}, \nabla_C\widehat{Y}) \overset{(8),(13)}{=} C(X^v(Y^v F))$. On the other hand, by the homogeneity of F, and using repeatedly the second relation of (6), we obtain

$$\begin{aligned} -\widetilde{A}(\widehat{X}, \widehat{Y}) &= -X^v(Y^v F) = [C, X^v](Y^v F) = C(X^v(Y^v F)) - X^v(C(Y^v F)) \\ &= C(X^v(Y^v F)) - X^v([C, Y^v]F + Y^v(CF)) \\ &= C(X^v(Y^v F)) - X^v(-Y^v F + Y^v F) = C(X^v(Y^v F)). \end{aligned}$$

This concludes the proof. □

4.5. Let $\mathcal{H}$ be a horizontal map for τ. In terms of the Berwald derivative induced by $\mathcal{H}$ we define the *torsion* $\mathbf{T}$ of $\mathcal{H}$ by

$$\mathrm{T}(\widetilde{X}, \widetilde{Y}) := \nabla\mathbf{j}(\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}) = \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]. \tag{14}$$

Using basic vector fields, this formula takes the more convenient form

$$\mathrm{T}(\widehat{X}, \widehat{Y}) = [X^h, Y^v] - [Y^h, X^v] - [X, Y]^v. \tag{15}$$

Another important concept, the homogeneity of a horizontal map, may also be expressed by means of the Berwald derivative: $\mathcal{H}$ is called *homogeneous* if $\nabla_{\mathcal{H}\widetilde{X}}\delta = 0$ for all $\widetilde{X} \in \mathfrak{X}(\tau)$. $\mathcal{H}$ has this property if and only if $[X^h, C] = 0$ for all $X \in \mathfrak{X}(M)$.

LEMMA 4. *If a horizontal map $\mathcal{H}$ has vanishing torsion, or if it is homogeneous, then the mixed curvature of the Berwald derivative induced by $\mathcal{H}$ is totally symmetric. In the homogeneous case we also have*

$$\mathring{\mathbf{P}}(\widetilde{X}, \widetilde{Y})\delta = \mathring{\mathbf{P}}(\widetilde{X}, \delta)\widetilde{Y} = \mathring{\mathbf{P}}(\delta, \widetilde{X})\widetilde{Y} = 0 \qquad \text{for all } \widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\tau), \tag{16}$$

and, consequently,

$$\operatorname{tr} \mathring{\mathbf{P}}(\widetilde{X}, \delta) = \operatorname{tr} \mathring{\mathbf{P}}(\delta, \widetilde{X}) = 0 \qquad \text{for all } \widetilde{X} \in \mathfrak{X}(\tau). \tag{17}$$

For a proof the reader is referred to [18] or [19].

5. Projectively related sprays

5.1. A *semispray* over the manifold M is a C^1 vector field ξ on TM which is smooth on $\mathring{T}M$ and has the property $J\xi = C$. We define a *spray* to be a semispray ξ which satisfies the *homogeneity condition* $[C, \xi] = \xi$. A spray is called *affine* if it is of class C^2 on the whole tangent manifold TM. A smooth path $c : I \to M$ in M, defined on an interval I, is said to be a *geodesic* of a semispray ξ if $\ddot{c} = \xi \circ \dot{c}$.

LEMMA 5. *If ξ is a semispray over M, then*

$$J[J\eta, \xi] = J\eta \qquad \text{for all } \eta \in \mathfrak{X}(TM). \tag{18}$$

Proof: Since $\mathfrak{X}^v(TM)$ is generated by vertical lifts X^v ($X \in \mathfrak{X}(M)$) and $J[FX^v, \xi] = FJ[X^v, \xi]$ for all $F \in C^\infty(TM)$, it is enough to show that $J[X^v, \xi] = X^v$ for all $X \in \mathfrak{X}(M)$. Recall that the Lie derivatives of the (1,1) tensor field J on TM may be computed by using the formula

$$(\mathcal{L}_{\eta_1} J)(\eta_2) = [\eta_1, J\eta_2] - J[\eta_1, \eta_2]; \qquad \eta_1, \eta_2 \in \mathfrak{X}(TM).$$

It is now easy to check that the Lie derivative of J by a vertical lift vanishes. Hence $0 = (\mathcal{L}_{X^v} J)(\xi) = [X^v, J\xi] - J[X^v, \xi] \overset{(6)}{=} X^v - J[X^v, \xi]$ for all $X \in \mathfrak{X}(M)$. This proves (18). □

5.2. Any semispray ξ over M generates a horizontal map $\mathcal{H}_\xi$ for τ by

$$\mathcal{H}_\xi(\widehat{X}) = \frac{1}{2}(X^c + [X^v, \xi]) \qquad \text{for all } X \in \mathfrak{X}(M) \tag{19}$$

(see e.g. [2, 18]). The torsion of $\mathcal{H}_\xi$ vanishes, so by Lemma 4 the mixed curvature of the Berwald derivative induced by $\mathcal{H}_\xi$ is totally symmetric. If, in particular, ξ is a spray, then $\mathcal{H}_\xi$ is homogeneous, and ξ itself is a horizontal vector field with respect to $\mathcal{H}_\xi$. Conversely, if $\mathcal{H}$ is a horizontal map with vanishing torsion, then $\mathcal{H}$ is generated by a semispray according to (19) (a theorem of Crampin). If, in addition, $\mathcal{H}$ is homogeneous, then it can always be generated by a spray.

LEMMA 6. *If ξ is a spray and ∇ is the Berwald derivative induced by $\mathcal{H}_\xi$, then $\nabla_\xi \delta = 0$.*

Proof: Since ξ is horizontal with respect to $\mathcal{H}_\xi$, $\xi = (\mathcal{H}_\xi \circ \mathbf{j})\xi = \mathcal{H}_\xi(\delta)$, and therefore $\nabla_\xi \delta = \nabla_{\mathcal{H}_\xi(\delta)}\delta \overset{(8)}{=} \mathcal{V}_\xi[\mathcal{H}_\xi \delta, C] = \mathcal{V}_\xi[\xi, C] = -\mathcal{V}_\xi(\xi) = -\mathcal{V}_\xi \circ \mathcal{H}_\xi(\delta) = 0$, as was to be checked. □

5.3. Two sprays ξ and $\bar{\xi}$ over M are said to be *(pointwise) projectively related* if there exists a function f which is of class C^1 on TM, smooth on $\mathring{T}M$, such that $\bar{\xi} = \xi + fC$. Then f is automatically 1-homogeneous. On the other hand, if ξ is a spray and f is a 1-homogeneous function (of class C^1 on TM, smooth on $\mathring{T}M$), then $\bar{\xi} := \xi + fC$ is a spray again, called a *projective change* of ξ.

PROPOSITION 1. *If $\bar{\xi} = \xi + fC$ is a projective change of the spray ξ, then the horizontal maps generated by ξ and $\bar{\xi}$ are related by*

$$\mathcal{H}_{\bar{\xi}} = \mathcal{H}_\xi + \frac{1}{2}(f\mathbf{i} + \nabla^v f \otimes C), \tag{20}$$

and the relations between the corresponding vertical maps and horizontal projectors are given by

$$\mathcal{V}_{\bar{\xi}} = \mathcal{V}_\xi - \frac{1}{2}f\mathbf{j} - \frac{1}{2}(\nabla^v f \circ \mathbf{j}) \otimes \delta, \tag{21}$$

$$\mathbf{h}_{\bar{\xi}} = \mathbf{h}_\xi + \frac{1}{2}fJ + \frac{1}{2}(\nabla^v f \circ \mathbf{j}) \otimes C. \tag{22}$$

If $\mathring{\mathbf{P}}$ and $\overline{\mathring{\mathbf{P}}}$ are the mixed curvatures of the Berwald derivatives induced by $\mathcal{H}_\xi$ and $\mathcal{H}_{\bar{\xi}}$ respectively, then we have

$$\overline{\mathring{\mathbf{P}}} = \mathring{\mathbf{P}} + \frac{1}{2}(\nabla^v\nabla^v f) \odot 1_{\mathfrak{X}(\tau)} + \frac{1}{2}(\nabla^v\nabla^v\nabla^v f) \otimes \delta; \tag{23}$$

in particular

$$\operatorname{tr}\overline{\mathring{\mathbf{P}}} = \operatorname{tr}\mathring{\mathbf{P}} + \frac{1}{2}(n+1)\nabla^v\nabla^v f. \tag{24}$$

Proof: (a) The routine gives: $\mathcal{H}_{\bar{\xi}}(\widehat{X}) := \frac{1}{2}(X^c + [X^v, \bar{\xi}]) = \frac{1}{2}(X^c + [X^v, \xi]) + \frac{1}{2}[X^v, fC] \overset{(6)}{=} \mathcal{H}_\xi(\widehat{X}) + \frac{1}{2}(fX^v + (X^v f)C) = (\mathcal{H}_\xi + \frac{1}{2}(f\mathbf{i} + \nabla^v f \otimes C))(\widehat{X})$, which proves formula (20). This implies immediately relations (21) and (22).

(b) Now we turn to the change of the mixed curvature. The result comes from applying (20) and (21) to (9):

$$\begin{aligned}\overline{\mathring{\mathbf{P}}}(\widehat{X},\widehat{Y})\widehat{Z} &= \mathcal{V}_{\bar{\xi}}[[\mathcal{H}_{\bar{\xi}}(\widehat{X}),Y^v],Z^v] = \mathcal{V}_{\xi}[[\mathcal{H}_{\xi}(\widehat{X})+\frac{1}{2}(fX^v+(X^vf)C),Y^v],Z^v]\\ &\quad -\frac{1}{2}(f\mathbf{j}+(\nabla^v f\circ\mathbf{j})\otimes\delta)([[\mathcal{H}_{\bar{\xi}}(\widehat{X}),Y^v],Z^v])\\ &\overset{(*)}{=} \mathring{\mathbf{P}}(\widehat{X},\widehat{Y})\widehat{Z}+\frac{1}{2}\mathcal{V}_{\xi}[-(Y^vf)X^v-(X^vf)Y^v-Y^v(X^vf)C,Z^v]\\ &= \mathring{\mathbf{P}}(\widehat{X},\widehat{Y})\widehat{Z}+\frac{1}{2}\mathcal{V}_{\xi}(Z^v(Y^vf)\mathbf{i}\widehat{X}+Z^v(X^vf)\mathbf{i}\widehat{Y}\\ &\quad +Z^v(Y^v(X^vf))\mathbf{i}\delta+Y^v(X^vf)\mathbf{i}\widehat{Z})\\ &= (\mathring{\mathbf{P}}+\frac{1}{2}(\nabla^v\nabla^v f\odot 1_{\mathfrak{X}(\tau)}+\nabla^v\nabla^v\nabla^v f\otimes\delta))(\widehat{X},\widehat{Y},\widehat{Z}).\end{aligned}$$

(At step (*) we used the fact that $[[\mathcal{H}_{\bar{\xi}}(\widehat{X}),Y^v],Z^v]$ is vertical, hence it is in the kernel of the homomorphism $\mathbf{j}$.)

(c) Finally, we act by the trace operator on both sides of (23). By (1), (2) and Lemma 3 we obtain

$$\begin{aligned}\operatorname{tr}\overline{\mathring{\mathbf{P}}} &= \operatorname{tr}\mathring{\mathbf{P}}+\frac{1}{2}\operatorname{tr}(\nabla^v\nabla^v f\odot 1_{\mathfrak{X}(\tau)})+\frac{1}{2}\operatorname{tr}(\nabla^v\nabla^v\nabla^v f\otimes\delta)\\ &= \operatorname{tr}\mathring{\mathbf{P}}+\frac{1}{2}n\nabla^v\nabla^v f+\nabla^v\nabla^v f+\frac{1}{2}i_\delta\nabla^v(\nabla^v\nabla^v f)\\ &= \operatorname{tr}\mathring{\mathbf{P}}+\frac{1}{2}n\nabla^v\nabla^v f+\nabla^v\nabla^v f+\frac{1}{2}\nabla^v_\delta(\nabla^v\nabla^v f)\\ &= \operatorname{tr}\mathring{\mathbf{P}}+\frac{1}{2}(n+1)\nabla^v\nabla^v f.\end{aligned}$$

This concludes the proof. □

6. The main result

THEOREM 2. *Let ξ and $\bar{\xi}$ be two sprays over the manifold M. The following conditions are equivalent:*

(i) *ξ and $\bar{\xi}$ are projectively related: there is a function f, of class C^1 on TM, smooth on $\mathring{T}M$ such that $\bar{\xi}=\xi+fC$.*

(ii) *The geodesics of ξ and $\bar{\xi}$ differ only in a strictly increasing parameter transformation.*

(iii) *The Yano derivatives D and $\bar{D}$ induced by $\mathcal{H}_\xi$ and $\mathcal{H}_{\bar{\xi}}$ respectively, are related by*

$$\bar{D}=D-\frac{1}{2}((\nabla^v f\circ\mathbf{j})\otimes 1_{\mathfrak{X}(\tau)}+\mathbf{j}\otimes\nabla^v f). \tag{25}$$

Proof: The equivalence of (i) and (ii) is easy and known (see e.g. [8] or [16]), but it is stated for the record.

(i) $\Rightarrow$ (iii): Since the vertical parts of the Berwald and Yano derivatives coincide ($\nabla^v_{\widetilde{X}} = D^v_{\widetilde{X}} := D_{\mathbf{i}\widetilde{X}}$ for all $\widetilde{X} \in \mathfrak{X}(\tau)$) and the second term of (25) kills the pairs of form $(J\eta, \widetilde{Y})$, it is enough to check that (25) holds for the pairs of form $(X^c, \widehat{Y})$; $X, Y \in \mathfrak{X}(M)$. Let $\bar{\nabla}$ be the Berwald derivative induced by $\bar{\xi}$. Then

$$\begin{aligned}
\bar{D}_{X^c}\widehat{Y} &= \bar{D}_{\mathcal{H}_{\bar{\xi}}(\widehat{X})}\widehat{Y} + \bar{D}_{\mathbf{i}\mathcal{V}_{\bar{\xi}}(X^c)}\widehat{Y} \\
&= \bar{\nabla}_{\mathcal{H}_{\bar{\xi}}(\widehat{X})}\widehat{Y} + \frac{1}{n+1}(\operatorname{tr}\bar{\mathring{\mathbf{P}}}(\widehat{X}, \widehat{Y}))\delta + \bar{D}_{\mathbf{i}\mathcal{V}_{\bar{\xi}}(X^c)}\widehat{Y} \\
&\overset{(8)}{=} \mathcal{V}_{\bar{\xi}}[\mathcal{H}_{\bar{\xi}}(\widehat{X}), Y^v] + \frac{1}{n+1}(\operatorname{tr}\bar{\mathring{\mathbf{P}}}(\widehat{X}, \widehat{Y}))\delta \\
&\overset{(21),(24)}{=} \mathcal{V}_{\xi}[\mathcal{H}_{\bar{\xi}}(\widehat{X}), Y^v] - \frac{1}{2}f\mathbf{j}[\mathcal{H}_{\bar{\xi}}(\widehat{X}), Y^v] + \frac{1}{n+1}(\operatorname{tr}\mathring{\mathbf{P}}(\widehat{X}, \widehat{Y}))\delta \\
&\quad + \frac{1}{2}(\nabla^v\nabla^v f)(\widehat{X}, \widehat{Y})\delta \\
&= \mathcal{V}_{\xi}[\mathcal{H}_{\xi}(\widehat{X}), Y^v] + \frac{1}{n+1}(\operatorname{tr}\mathring{\mathbf{P}}(\widehat{X}, \widehat{Y}))\delta \\
&\quad + \frac{1}{2}\mathcal{V}_{\xi}[fX^v + (X^v f)C, Y^v] + \frac{1}{2}(X^v(Y^v f))\delta \\
&= D_{X^c}\widehat{Y} + \frac{1}{2}(-(Y^v f)\widehat{X} - Y^v(X^v f)\delta - (X^v f)\widehat{Y} + X^v(Y^v f)\delta) \\
&= D_{X^c}\widehat{Y} - \frac{1}{2}((\nabla^v f \circ \mathbf{j})(X^c)\widehat{Y} + (\nabla^v f(\widehat{Y}))\mathbf{j}X^c) \\
&= (D - \frac{1}{2}((\nabla^v f \circ \mathbf{j}) \otimes 1_{\mathfrak{X}(\tau)} + \mathbf{j} \otimes \nabla^v f))(X^c, \widehat{Y}),
\end{aligned}$$

thus proving implication (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i): We evaluate both sides of (25) on the pair $(\bar{\xi}, \delta)$. Applying Lemma 6 and (17), we have on the one hand

$$\bar{D}_{\bar{\xi}}\delta := \bar{\nabla}_{\bar{\xi}}\delta + \frac{1}{n+1}(\operatorname{tr}\bar{\mathring{\mathbf{P}}}(\delta, \delta))\delta = 0.$$

On the other hand,

$$\begin{aligned}
(D - \frac{1}{2}((\nabla^v f \circ \mathbf{j}) \otimes 1_{\mathfrak{X}(\tau)} + \mathbf{j} \otimes \nabla^v f))(\bar{\xi}, \delta) &= D_{\bar{\xi}}\delta - \frac{1}{2}((Cf)\delta + (Cf)\delta) \\
&= \nabla_{\bar{\xi}}\delta + \frac{1}{n+1}(\operatorname{tr}\bar{\mathring{\mathbf{P}}}(\delta, \delta))\delta - f\delta = \nabla_{\mathcal{H}_{\xi}(\mathbf{j}\bar{\xi})}\delta + \nabla_{\mathbf{i}\mathcal{V}_{\xi}(\bar{\xi})}\delta - f\delta \\
&= \nabla_{\xi}\delta + \mathbf{j}[\mathbf{i}\mathcal{V}_{\xi}(\bar{\xi}), \mathcal{H}_{\xi}(\delta)] - f\delta = \mathbf{j}[\mathbf{i}\mathcal{V}_{\xi}(\bar{\xi}), \xi] - f\delta \\
&\overset{(18)}{=} \mathcal{V}_{\xi}(\bar{\xi}) - f\delta,
\end{aligned}$$

thus we obtain $\mathcal{V}_\xi(\bar{\xi}) - f\delta = 0$. Acting on both sides of this relation by the homomorphism $\mathbf{i}$, it follows that

$$\begin{aligned} 0 &= \mathbf{i} \circ \mathcal{V}_\xi(\bar{\xi}) - fC = \bar{\xi} - \mathcal{H}_\xi \circ \mathbf{j}\bar{\xi} - fC \\ &= \bar{\xi} - \mathcal{H}_\xi(\delta) - fC = \bar{\xi} - \xi - fC \end{aligned}$$

i.e., $\bar{\xi} = \xi + fC$, as was to be shown. □

Discussion

Assume, in particular, that ξ and $\bar{\xi}$ are *affine sprays*. Then the mixed curvatures of the Berwald derivatives induced by ξ and $\bar{\xi}$ vanish (see e.g. [16] or [18]), therefore $D = \nabla$, $\bar{D} = \overline{\nabla}$, and by (8), (25) specializes to

$$\mathcal{V}_{\bar{\xi}}[X^{\bar{h}}, Y^v] = \mathcal{V}_\xi[X^h, Y^v] - \frac{1}{2}((\nabla^v f)(\widehat{X})Y^v + (\nabla^v f)(\widehat{Y})X^v)$$

for all $X, Y \in \mathfrak{X}(M)$, where $X^{\bar{h}} := \mathcal{H}_{\bar{\xi}}(\widehat{X})$. The vanishing of the mixed curvatures also implies that there exist unique, necessarily torsion-free covariant derivative operators D_M and $\bar{D}_M$ on M such that

$$[X^h, Y^v] = ((D_M)_X Y)^v, \qquad [X^{\bar{h}}, Y^v] = ((\bar{D}_M)_X Y)^v.$$

The geodesics of D_M and $\bar{D}_M$ are just the geodesics of ξ and $\bar{\xi}$, respectively. Observe now that the 1-homogeneity of f implies $\nabla^v_\delta(\nabla^v f) = 0$. This means that $\nabla^v f$ is homogeneous of degree 0. Then there is a 1-form α on M such that $\widehat{\alpha} = -\frac{1}{2}\nabla^v f$. Since $\widehat{\alpha}(\widehat{X}) = (\alpha(X))^v$ for all $X \in \mathfrak{X}(M)$, we deduce that (25) takes the form

$$((\bar{D}_M)_X Y)^v = ((D_M)_X Y)^v + (\alpha(X)Y + \alpha(Y)X)^v,$$

which yields immediately the Weyl relation. Thus, for affine sprays, our Theorem reduces to Weyl's classical theorem. Notice finally that there is an important class of Finsler manifolds in which the *canonical spray* ξ arising from the energy $E : TM \to M$ by $i_\xi d(dE \circ J) = -dE$ is an affine spray. Finsler manifolds with this property are called *Berwald manifolds*. By our preceding discussion the projective relatedness of the canonical sprays of two Berwald manifolds with common base manifold can also be characterized by the Weyl relation.

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