On the projective geometry of sprays

J. Szilasi[1] and Sz. Vattamány
Institute of Mathematics and Informatics, Lajos Kossuth University, H-4010 Debrecen, P.O.B. 12, Hungary

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Abstract: After a careful study of the mixed curvatures of the Berwald-type (in particular, Berwald) connections, we present an axiomatic description of the so-called Yano-type Finsler connections. Using the Yano connection, we derive an intrinsic expression of Douglas' famous projective curvature tensor and we also represent it in terms of the Berwald connection. Utilizing a clever observation of Z. Shen, we show in a coordinate-free manner that a "spray manifold" is projectively equivalent to an affinely connected manifold iff its Douglas tensor vanishes. From this result we infer immediately that the vanishing of the Douglas tensor implies that the projective Weyl tensor of the Berwald connection "depends only on the position".

Keywords: Horizontal endomorphisms, Berwald endomorphisms, sprays, Finsler connections, Berwald-type connections, Yano-type connections, Douglas tensor, projective Weyl tensor.

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Introduction

In the context of affinely connected manifolds, i.e., in manifolds endowed with a linear connection, a "projective curvature tensor" with zero trace has already been constructed by H. Weyl [18]. The vanishing of this famous tensor, called Weyl tensor, characterizes the "projectively flat manifolds". The nonlinear generalizations of the affinely connected manifolds, i.e., manifolds endowed with a nonlinear connection, have also been playing an increasing role in differential geometry and its applications since the twenties. The most important examples of these structures are the "manifolds with a spray", where the spray is not necessarily smooth on the zero section. Their local study is the subject of the classical "geometry of paths", developed by L. Berwald, J. Douglas, M.S. Knebelman, T.Y. Thomas, O. Veblen and others in the twenties-thirties. The counterpart of the Weyl tensor in this nonlinear context was intrinsically constructed by L. del Castillo [6] in 1976, using the Frölicher–Nijenhuis formalism, see also [12]. It is known, but not so widely known, that in the nonlinear theory there is another projectively invariant "curvature tensor" discovered by Jesse Douglas [8] and called *Douglas tensor* after him. This tensor always vanishes in affinely connected manifolds (hence, in particular, in Riemannian manifolds), so, in the phrase of L.P. Eisenhart, it has a typically "non-Riemannian" character.

The importance of the Douglas tensor can be readily realized. Those Finsler manifolds which have vanishing projective Weyl tensor and Douglas tensor are just the solutions of Hilbert's

[1]szilasi@math.klte.hu.

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fourth problem [11]. Note that a complete classification of these manifolds is clearly not an easy task, for a solution see Z.I. Szabó's wonderful paper [16]. Another immediate and important question also comes up at this point. *Can we hope for an elegant description of the* (*Finsler*) *manifolds with vanishing Douglas tensor, but non-vanishing projective Weyl tensor*? Finsler manifolds with vanishing Douglas tensor were baptized *Douglas manifolds* by S. Bácsó and M. Matsumoto. Their papers [1]–[3] seem to be promising first steps in attacking the problem.

In the present paper our aim is less ambitious. We want only to present an intrinsic construction of the Douglas tensor, using also the Frölicher–Nijenhuis formalism. Nevertheless we hope that this contribution makes available further, efficient modern tools for the study of Douglas manifolds. The achievement of this modest aim still remains troublesome: a quite long process will culminate in a compact and transparent presentation of the Douglas tensor. Now we are going to sketch the skeleton of our ideas.

We shall represent the nonlinear connections by *horizontal endomorphisms*, i.e., by projectors whose kernel is the vertical subbundle (differentiability on the zero section is not required!). Any horizontal endomorphism gives rise to a special Finsler connection, called a *Berwald-type connection*. A Berwald-type connection is said to be a *Berwald connection* if the horizontal endomorphism is generated by a spray. In this case the horizontal endomorphism will be mentioned as a *Berwald endomorphism*. Any Berwald-type connection has two surviving "partial curvatures", the horizontal and the mixed curvature. A careful analysis of the behavior of the mixed curvature of a Berwald connection under a projective change of the associated spray yields the Douglas tensor. However, in order to identify the Douglas tensor we shall follow a slightly different path. As a generalization of Berwald-type connections, we introduce the so-called *Yano-type connections*. If, in particular, we start from a Berwald endomorphism, then the construction results in a *Yano connection*, called also—unfortunately—a projective connection. Indeed, the parameters of this connection were originally found by K. Yano, motivated by purely technical considerations. (For the quite curious story of this connection, see Matsumoto's instructive remarks in [1].) Thus in our present approach the definition, as well as the proof of the projective invariance of the Douglas tensor will be given in terms of a Yano connection. Although the central object of this study is the Douglas tensor, a very brief section will also be devoted to the projective Weyl tensors. This includes a simple proof of the following, recently discovered (see [2]) important fact: the vanishing of the Douglas tensor implies that one of the Weyl tensors is a vertical lift, i.e., its components depend only on the position.

We tried to make this paper as readable as possible, so we hope: the outcome of our efforts will be an essentially self-contained work. As for basic sources of preparatory material, we refer to [7, 9, 13], and [14]. Chapters 13–17 of Z. Shen's recent preprint [15] also provides a good introduction to the projective geometry of sprays.

1. Notations and background

1.1. Throughout this paper, M will denote a connected, smooth (i.e., C^∞), sometimes orientable manifold of dimension $n \geqslant 2$. $C^\infty(M)$ is the ring of real-valued smooth functions on M, $10(M)$ denotes the $C^\infty(M)$-module of vector fields on M. For $(r, s) \in \mathbb{N} \times \mathbb{N}$, $\mathcal{T}^r_s(M)$ is the module over $C^\infty(M)$ of smooth tensor fields (briefly tensors) of type (r, s), contravariant of order r and covariant of order s. We define the symmetric product $\omega_1 \odot \omega_2$ of the covariant tensors

$\omega_1 \in \mathcal{T}^{\circ}_{s_1}(M)$, $\omega_2 \in \mathcal{T}^{\circ}_{s_2}(M)$ by the formula

$$\omega_1 \odot \omega_2 := \frac{(s_1+s_2)!}{s_1!\, s_2!}\,\mathrm{Sym}\,(\omega_1 \otimes \omega_2). \tag{1.1}$$

$\Omega^k(M)$ $(0 \leqslant k \leqslant n)$ is the module of differential forms on M, $\Omega^{\circ}(M) := C^{\infty}(M)$. The differential forms constitute the graded algebra $\Omega(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$, with multiplication given by the wedge product.

1.2. Vector forms and derivations. A vector k-form on the manifold M is a skew-symmetric $C^{\infty}(M)$-multilinear map $[10(M)]^k \to 10(M)$ if $k \in \mathbb{N}^+$, and a vector field on M if $k = 0$. The set of all vector k-forms on M is a $C^{\infty}(M)$-module, denoted by $\Psi^k(M)$. In particular, the elements of $\Psi^1(M)$ are just the $(1, 1)$ tensor fields on M. To any vector k-form $K \in \Psi^k(M)$ two derivations of $\Omega(M)$, denoted by i_K and d_K, are associated. Now we briefly recall some of their basic properties.

(i) i_K is a derivation of degre $k-1$;

$$i_K \restriction C^{\infty}(M) = 0, \quad i_K\omega := \omega \circ K, \quad \text{if } \omega \in \Omega^1(M).$$

(ii) d_K is a derivation of degree k given by

$$d_K := [i_K, d] := i_K \circ d - (-1)^{k-1} d \circ i_K,$$

where d is the operator of the exterior derivative. In particular,

$$\forall f \in C^{\infty}(M) : d_K f = i_K df \overset{\text{(i)}}{=} df \circ K,$$

and d_K is uniquely determined by this formula.

(iii) If $K \in \Psi^1(M)$, then we define the endomorphism

$$K^* : \Omega(M) \to \Omega(M), \quad \omega \in \Omega^{\ell}(M) \mapsto K^*\omega$$

by the formula

$$K^*\omega(X_1, \dots, X_\ell) := \omega\big(K(X_1), \dots, K(X_\ell)\big), \quad (X_i \in 10(M),\ 1 \leqslant i \leqslant \ell).$$

If $\ell = 1$, $K^*\omega = i_K\omega$. If $\ell > 1$,

$$i_K\omega\,(X_1, \dots, X_\ell) = \sum_{i=1}^{\ell} \omega\,(X_1, \dots, K(X_i), \dots, X_\ell)$$

and $i_K\omega \neq K^*\omega$.

(iv) In case of a vector 0-form $X \in \Psi^{\circ}(M) = 10(M)$ i_X means the usual insertion operator, while d_X reduces to the Lie derivative $\mathcal{L}_X$.

1.3. The Frölicher–Nijenhuis bracket. Suppose that $K \in \Psi^k(M)$, $L \in \Psi^{\ell}(M)$. The graded commutator of d_K and d_L is defined by the formula

$$[d_K, d_L] = d_K \circ d_L - (-1)^{k\ell} d_L \circ d_K.$$

A substantial result of the *Frölicher–Nijenhuis theory* [9] states that there exits a unique vector form $[K, L] \in \Psi^{k+\ell}(M)$ such that

$$[d_K, d_L] = d_{[K,L]}.$$

$[K, L]$ is said to be the *Frölicher–Nijenhuis bracket* of K and L. If K and L are vector 0-forms, i.e., vector fields on M, then $[K, L]$ reduces to the ususal Lie bracket of vector fields. In our considerations we shall need the evaluation of the Frölicher–Nijenhuis bracket only in some special cases.

$$\begin{aligned}&\text{If } K \in \Psi^1(M),\ Y \in \Psi^\circ(M) = 10(M), \text{then } \forall X \in 10(M):\\ &\quad [K, Y](X) = [K(X), Y] - K[X, Y].\end{aligned} \tag{1.3a}$$

$$\begin{aligned}&\text{If } K, L \in \Psi^1(M), \text{then } \forall X, Y \in 10(M):\\ &\quad [K, L](X, Y) = [K(X), L(Y)] + [L(X), K(Y)] + K \circ L[X, Y]\\ &\qquad\quad + L \circ K[X, Y] - K[X, L(Y)] - K[L(X), Y]\\ &\qquad\quad - L[X, K(Y)] - L[K(X), Y].\end{aligned} \tag{1.3b}$$

In particular, the *Nijenhuis torsion* $N_K := \frac{1}{2}[K, K]$ of a vector 1-from K can be given by the formula

$$N_K(X, Y) = [K(X), K(Y)] + K^2[X, Y] - K[X, K(Y)] - K[K(X), Y]. \tag{1.3c}$$

A quite complete list of useful identities concerning the Frölicher–Nijenhuis bracket can be found in [20]. We shall explicitly need only the following of them:

$$\begin{aligned}&\forall K \in \Psi^k(M), X \in 10(M),\ f \in C^\infty(M):\\ &\qquad [fX, K] = f[X, K] + df \wedge i_X K - d_K f \otimes X.\end{aligned} \tag{1.3d}$$

1.4. The tangent bundle of the manifold M will be denoted by $\pi : TM \to M$, while $\pi_0 : \mathcal{T}M \to M$ stands for the subbundle of the nonzero tangent vectors. The kernel of the tangent map $T\pi : TTM \to TM$ is a distinguished subbundle of TTM, the *vertical subbundle*, whose total space will be denoted by $T^{\mathrm{v}}TM$. The sections of this bundle consitute the $C^\infty(TM)$-module $10^{\mathrm{v}}(TM)$ of the *vertical vector fields*. In our calculations we shall frequently use the *vertical lift* X^{v} and the *complete lift* X^{c} of a vector field $X \in 10(M)$. Their usefulness is established by the following simple observation.

Local basis property. *If* $(X_i)_{i=1}^n$ *is a local basis for the module* $10(M)$, *then* $(X_i^{\mathrm{v}}, X_i^{\mathrm{c}})_{i=1}^n$ *is a local basis for* $10(TM)$.

We also recall that

$$\forall X, Y \in 10(M): \ [X^{\mathrm{v}}, Y^{\mathrm{v}}] = 0, \quad [X^{\mathrm{v}}, Y^{\mathrm{c}}] = [X, Y]^{\mathrm{v}}, \quad [X^{\mathrm{c}}, Y^{\mathrm{c}}] = [X, Y]^{\mathrm{c}}. \tag{1.4}$$

1.5. Tangent bundle geometry is dominated by two canonical objects: the *Liouville vector field* $C \in 10^{\mathrm{v}}(TM)$ and the *vertical endomorphism* $J \in \Psi^1(TM) \cong \mathcal{T}_1^1(TM) \cong \mathrm{End}\, 10(TM)$. The following well-known relations will also be useful in our considerations.

$$\operatorname{Im} J = \operatorname{Ker} J = 10^{\mathrm{v}}(TM), \qquad J^2 = 0; \tag{1.5a}$$

$$[J, C] = J, \qquad [J, J] = 0; \tag{1.5b}$$

$$\forall\, X \in 10(M) : JX^{\mathrm{c}} = X^{\mathrm{v}}, \qquad [J, X^{\mathrm{v}}] = [J, X^{\mathrm{c}}] = 0; \tag{1.5c}$$

$$\forall\, X \in 10(M) : [C, X^{\mathrm{v}}] = -X^{\mathrm{v}}, \qquad [C, X^{\mathrm{c}}] = 0. \tag{1.5d}$$

1.6. Semibasic tensors, semibasic trace. A symmetric or skew-symmetric tensor $A \in \mathcal{T}_s^0(TM)$ $(s \neq 0)$ is called *semibasic* if

$$\forall\, X \in 10(TM) : i_{JX} A = 0.$$

Analogously, a symmetric or skew-symmetric tensor $L \in \mathcal{T}_s^1(TM)$ $(s \neq 0)$ is said to be semibasic if

$$\forall\, X \in 10(TM) : i_{JX} L = 0 \qquad \text{and} \qquad J \circ L = 0.$$

Now let us suppose that F is an almost complex structure on TM, i.e., $F \in \mathcal{T}_1^1(TM) \cong \operatorname{End} 10(TM)$ and $F^2 = -1_{10(TM)}$. Consider a semibasic tensor $L \in \mathcal{T}_s^1(TM)$. We define the *semibasic trace* $\widetilde{L}$ of L by recurrence as follows:

$$\text{if } L \text{ is a } (1,1)\text{-tensor, then } \widetilde{L} := \operatorname{tr}(F \circ L); \tag{1.6a}$$

$$\text{if } L \in \mathcal{T}_s^1(M),\ s \geqslant 2,\ \text{then } \forall\, X \in 10(TM) : i_X \widetilde{L} := \widetilde{i_X L}. \tag{1.6b}$$

It can easily be seen that $\widetilde{L}$ does not depend on the choice of the almost complex structure F.

We shall need the following technical results.

1.7. Lemma. *Let $\omega \in \mathcal{T}_s^0(TM)$ be a symmetric semibasic tensor and $L \in \mathcal{T}_1^1(TM)$ a semibasic tensor. Then*

$$\widetilde{J} = n, \quad \widetilde{\omega \otimes J} = n\omega. \tag{1.7a}$$

$$\begin{gathered} \forall\, X \in 10(TM) : \widetilde{\omega \otimes L X} = i_{FLX}\omega, \\ \textit{where } F \in \operatorname{End} 10(TM) \textit{ is an arbitrary almost complex structure.} \end{gathered} \tag{1.7b}$$

1.8. Definition and lemma. *The symmetric product of tensors $\omega \in \mathcal{T}_s^0(TM)$, $L \in \mathcal{T}_r^1(TM)$ is the $(1, s+r)$ tensor $\omega \odot L$ given by the formula*

$$\omega \odot L(X_1, \dots, X_{s+r}) := \frac{1}{s!\,r!} \sum_{\sigma \in S_{s+r}} \omega\big(X_{\sigma(1)}, \dots, X_{\sigma(s)}\big) L\big(X_{\sigma(s+1)}, \dots, X_{\sigma(s+r)}\big).$$

If $\omega \in \mathcal{T}_s^0(TM)$ $(s \geqslant 1)$ is a symmetric, semibasic tensor and $L \in \mathcal{T}_1^1(TM)$ is a semibasic tensor, then

$$\widetilde{\omega \odot L} = \widetilde{L}\omega + i_{FL}\omega \tag{1.8}$$

(*F is an arbitrary almost complex structure on TM*).

Using mathematical induction, the proofs of 1.7 and 1.8 are quite straightforward; we omit the tedious calculations. We remark that analogous results concerning the semibasic trace of *vector forms* can be found in [20].

2. Horizontal endomorphisms and semisprays

2.1. In our approach the role of a "nonlinear connection" is played by the *horizontal endomorphisms*. A vector 1-form $h \in \Psi^1(TM) \cong \operatorname{End} 10(TM)$, *smooth only on* $\mathcal{T}M$, is said to be a horizontal endomorphism on M if it is a *projector* (i.e., $h^2 = h$) and $\operatorname{Ker} h = 10^{\mathrm{v}}(TM)$. $\nu := 1_{10(TM)} - h$ is the *vertical projector* belonging to h. $10^{\mathrm{h}}(TM) := \operatorname{Im} h$ is called the module of *horizontal vector fields*. It is a direct summand, namely

$$10(TM) = 10^{\mathrm{v}}(TM) \oplus 10^{\mathrm{h}}(TM).$$

The mapping

$$X \in 10(M) \mapsto X^{\mathrm{h}} := hX^{\mathrm{c}} \in 10^{\mathrm{h}}(TM)$$

is called the *horizontal lifting* by h. We have the following basic relations among the vertical endomorphism J, a horizontal endomorphism h and the horizontal lifting:

$$h \circ J = 0, \quad J \circ h = J; \tag{2.1a}$$

$$\forall\, X \in 10(M) : JX^{\mathrm{h}} = X^{\mathrm{v}}. \tag{2.1b}$$

2.2. Definition. Suppose that h is a horizontal endomorphism on the manifold M. The vector forms

$$H := [h, C] \in \Psi^1(TM), \tag{2.2a}$$

$$t := [J, h] \in \Psi^2(TM), \tag{2.2b}$$

$$R := -N_h := -\tfrac{1}{2}[h, h] \in \Psi^2(TM) \tag{2.2c}$$

are called the *tension*, the *torsion* and the *curvature* of h, respectively. A horizontal endomorphism is said to be *homogeneous* if its tension vanishes.

2.3. Remark. H, t and R were introduced by Grifone [10]. It is easy to check that all of these tensors are semibasic, so they are completely determined by the following mappings:

$$\eta : X \in 10(M) \mapsto \eta(X) := H(X^{\mathrm{c}}) = [X^{\mathrm{h}}, C]; \tag{2.3a}$$

$$\begin{aligned} \tau : (X, Y) \in 10(M) \times 10(M) \mapsto \tau(X, Y) &:= t(X^{\mathrm{c}}, Y^{\mathrm{c}}) \\ &= [X^{\mathrm{h}}, Y^{\mathrm{v}}] - [Y^{\mathrm{h}}, X^{\mathrm{v}}] - [X, Y]^{\mathrm{v}}; \end{aligned} \tag{2.3b}$$

$$\varrho : (X, Y) \in 10(M) \times 10(M) \mapsto \varrho(X, Y) := R(X^{\mathrm{c}}, Y^{\mathrm{c}}) = -\nu[X^{\mathrm{h}}, Y^{\mathrm{h}}]. \tag{2.3c}$$

For details we refer to [17].

2.4. We recall (see [10]) that any horizontal endomorphism $h \in \operatorname{End} 10(TM)$ gives rise to a unique almost complex structure $F \in \operatorname{End} 10(TM)$, smooth over $\mathcal{T}M$, characterized by the relations

$$F \circ J = h, \quad F \circ h = -J. \tag{2.4a}$$

From these one can easily deduce the useful identities

$$J \circ F = \nu, \quad F \circ \nu = h \circ F; \tag{2.4b}$$

$$\nu \circ F = F - F \circ \nu = F - h \circ F = -J. \tag{2.4c}$$

2.5. Definition. A *semispray* on the manifold M is a mapping

$$S : TM \to TTM, \quad v \mapsto S_v \in T_vTM$$

satisfying the following conditions:

(i) S is smooth on $\mathcal{T}M$,

(ii) $JS = C$.

A semispray S is called a *spray* if

(iii) S is of class C^1 on TM, and

(iv) $[C, S] = S$ (i.e., S is homogeneous of degree 2).

2.6. Lemma. *If S is a semispray on the manifold M, then*

$$\forall\, Z \in 10(TM) : J[JZ, S] = JZ. \tag{2.6}$$

Proof. See [10, p. 295]. □

2.7. The fundamental relation between the horizontal endomorphisms and the semisprays was discovered, independently, by M. Crampin and J. Grifone [4, 5, 10]. Their main result can be summarized as follows.

(i) If $h \in \operatorname{End} 10(TM)$ is a horizontal endomorphism and S' is an arbitrary semispray on M, then $S := hS'$ is also a semispray on M. This semispray does not depend on the choice of S', it is horizontal with respect to h and satisfies the relation $h[C, S] = S$. S is called the *semispray associated to h*.

(ii) Any semispray $S : TM \to TTM$ generates in a canonical way a horizontal endomorphism which can be given by the formula

$$h := \tfrac{1}{2}\big(1_{10(TM)} + [J, S]\big). \tag{2.7}$$

Then h is torsion free (i.e., $t = 0$) and the semispray associated to h is $\frac{1}{2}(S + [C, S])$. If, in addition, S is a spray, then h is homogeneous and its associated semispray is just the starting spray S.

(iii) A horizontal endomorphism is generated by a semispray according to (2.7) if and only if it is torsion free.

2.8. In the sequel we shall call a horizontal endomorphism a *Berwald endomorphism* if it is generated by a *spray*. We emphasise that the Berwald endomorphisms are homogeneous and torsion free, but the converse is not true. It follows immediately that if h is a Berwald endomorphism with associated spray S, then the horizontal lifting with respect to h can be given by the formula

$$X \in 10(M) \mapsto X^{\mathrm{h}} = \tfrac{1}{2}\big(X^{\mathrm{c}} + [X^{\mathrm{v}}, S]\big). \tag{2.8}$$

2.9. Two sprays S and $\bar{S}$ are said to be *projectively equivalent* if there is a function $\lambda : TM \to \mathbb{R}$ satisfying the conditions

(i) λ is smooth on $\mathcal{T}M$, and C^1 on TM;

(ii) $\bar{S} = S + \lambda C$.

Then λ is automatically 1-homogeneous (i.e., $C\lambda = \lambda$). Conversely, if a spray S and a 1-homogeneous function λ, satisfying (i), are given, then $\bar{S} = S + \lambda C$ is also a spray. In this case we speak of a *projective change* of the spray.

2.10. Lemma. *Suppose that h is a Berwald endomorphism with associated spray S. Consider a projective change $S \to \bar{S} := S + \lambda C$ of S. Then h changes by the formula*

$$h \to \bar{h} = h + \tfrac{1}{2}\lambda J + \tfrac{1}{2} d_J \lambda \otimes C. \tag{2.10a}$$

$\bar{h}$ remains homogeneous and torsion free. The horizontal lifting with respect to $\bar{h}$ is the mapping

$$X \in 10(M) \mapsto X^{\bar{h}} = X^{\mathrm{h}} + \tfrac{1}{2}\lambda X^{\mathrm{v}} + \tfrac{1}{2}(X^{\mathrm{v}}\lambda)C. \quad \square \tag{2.10b}$$

3. Finsler connections of Berwald type

3.1. Suppose that h is a horizontal endomorphism on the manifold M and let F be the almost complex structure determined by h. A pair (D, h) is said to be a *Finsler connection* on M, if D is a linear connection on the manifold TM or $\mathcal{T}M$ and the following conditions are satisfied:

$$Dh = 0 \qquad (D \text{ is reducible}); \tag{3.1a}$$

$$DF = 0 \qquad (D \text{ is almost complex}). \tag{3.1b}$$

Then the mapping

$$h^* DC : X \in 10(TM) \mapsto DC(hX) = D_{hX} C$$

is called the h-*deflection* of (D, h).

An easy, but important consequence of (3.1b) is that D is completely determined by the restricted map $D \restriction 10(TM) \times 10^{\mathrm{v}}(TM)$. Explicitly, for any vector fields X, Y on TM,

$$D_{vX} hY = F D_{vX} JY, \tag{3.1c}$$

$$D_{hX} hY = F D_{hX} JY. \tag{3.1d}$$

3.2. Suppose (D, h) is a Finsler connection on the manifold M. Denote by $\mathbf{T}$ and $\mathbf{K}$ the (classical) torsion and the curvature of D, respectively. In view of (3.1a) it is readily verified that the mappings

$$\mathbf{A} : (X, Y) \mapsto h\mathbf{T}(hX, hY) \qquad h\text{-}\textit{horizontal torsion},$$

$$\mathbf{B} : (X, Y) \mapsto h\mathbf{T}(hX, JY) \qquad h\text{-}\textit{mixed torsion},$$

$$\mathbf{R}^1 : (X, Y) \mapsto v\mathbf{T}(hX, hY) \qquad v\text{-}\textit{horizontal torsion},$$

$$\mathbf{P}^1 : (X, Y) \mapsto v\mathbf{T}(hX, JY) \qquad v\text{-}\textit{mixed torsion},$$

$$\mathbf{S}^1 : (X, Y) \mapsto v\mathbf{T}(JX, JY) \qquad v\text{-}\textit{vertical torsion}$$

determine $\mathbf{T}$ completely. Similarly, $\mathbf{K}$ can be described by the following three mappings:

$$\mathbf{R} : (X, Y, Z) \mapsto \mathbf{K}(hX, hY)JZ \qquad \textit{horizontal curvature},$$

$$\mathbf{P} : (X, Y, Z) \mapsto \mathbf{K}(hX, JY)JZ \qquad \textit{mixed curvature},$$

$$\mathbf{Q} : (X, Y, Z) \mapsto \mathbf{K}(JX, JY)JZ \qquad \textit{vertical curvature}.$$

The tensors $\mathbf{R}, \mathbf{P}, \mathbf{Q} \in \mathcal{T}^1_3(TM)$ are semibasic.

3.3. Definition. Let (D, h) be a Finsler connection on M. If $L \in \mathcal{T}^1_3(TM)$ is one of the horizontal, the mixed or the vertical curvatures of (D, h), then the $(0, 2)$-tensor

$$\widetilde{L} : (X, Y) \in 10(TM) \times 10(TM) \mapsto \widetilde{L}(X, Y) := \operatorname{tr}[F \circ (Z \mapsto L(Y, Z)X)]$$

is said to be the *horizontal*, the *mixed*, or the *vertical Ricci tensor* of the Finsler connection, respectively (F is an almost complex structrure, e.g., the almost complex structure induced by h).

3.4. Berwald-type connections. We set out from a horizontal endomorphism $h \in \Psi^1(TM)$ and define the mapping

$$D^\circ : 10(TM) \times 10(TM) \to 10(TM), \qquad (X, Y) \mapsto D^\circ_X Y$$

by the following rules:

$$D^\circ_{JX} JY := J[JX, Y], \tag{3.4a}$$

$$D^\circ_{hX} JY := v[hX, JY], \tag{3.4b}$$

$$D^\circ_{vX} hY := h[vX, Y], \tag{3.4c}$$

$$D^\circ_{hX} hY := hF[hX, JY] \tag{3.4d}$$

and

$$D^\circ_X Y := D^\circ_{vX} vY + D^\circ_{hX} vY + D^\circ_{vX} hY + D^\circ_{hX} hY.$$

It is straightforward to prove that (D°, h) is a Finsler connection on M; this Finsler connection is said to be the *Berwald-type Finsler connection induced by* h. If, in particular, h is a Berwald endomorphism, then we call (D°, h) a *Berwald connection*.

Replacing the vector fields in (3.4a)–(3.4d) by complete lifts $X^{\mathrm{c}}, Y^{\mathrm{c}}$, and taking into account (1.5c), 2.1 and (2.4b), we get the useful formulas

$$D^\circ_{X^{\mathrm{v}}} Y^{\mathrm{v}} = 0, \tag{3.4e}$$

$$D^\circ_{X^{\mathrm{h}}} Y^{\mathrm{v}} = [X^{\mathrm{h}}, Y^{\mathrm{v}}], \tag{3.4f}$$

$$D^\circ_{X^{\mathrm{v}}} Y^{\mathrm{h}} = 0, \tag{3.4g}$$

$$D^\circ_{X^{\mathrm{h}}} Y^{\mathrm{h}} = F[X^{\mathrm{h}}, Y^{\mathrm{v}}]. \tag{3.4h}$$

3.5. Lemma. *Suppose (D°, h) is a Berwald-type Finsler connection on M.*

(i) *The h-deflection of (D°, h) coincides with the tension of h.*

(ii) *The torsion $\mathbf{T}^\circ$ of D° can be represented in the form*

$$\mathbf{T}^\circ = F \circ t + R, \tag{3.5}$$

where t and R are the torsion and the curvature of h, and F is the almost complex structure induced by h.

Proof. (i) Let S be an arbitrary semispray. For each vector field X on TM,

$$\begin{aligned} H(hX) &:= [h, C]hX = [hX, C] - h[hX, C] = v[hX, C] = v[hX, JS] \\ &\overset{(3.4b)}{=} D^{\circ}_{hX} JS = h^{*}\big(D^{\circ}C\big)(X) = h^{*}\big(D^{\circ}C\big)(hX). \end{aligned}$$

This proves our assertion since H and $h^{*}\big(D^{\circ}C\big)$ are semibasic tensors.

(ii) Let $X, Y \in 10(M)$. Then, using (3.4e)–(3.4h) we get

$$\begin{aligned} \mathbf{T}^{\circ}(X^{\mathrm h}, Y^{\mathrm v}) &= 0, \\ \mathbf{T}^{\circ}(X^{\mathrm v}, Y^{\mathrm v}) &= 0; \\ \mathbf{T}^{\circ}(X^{\mathrm h}, Y^{\mathrm h}) &= F\big([X^{\mathrm h}, Y^{\mathrm v}] - [Y^{\mathrm h}, X^{\mathrm v}]\big) - [X^{\mathrm h}, Y^{\mathrm h}] \\ &= F\big([X^{\mathrm h}, Y^{\mathrm v}] - [Y^{\mathrm h}, X^{\mathrm v}]\big) - h[X^{\mathrm h}, Y^{\mathrm h}] - v[X^{\mathrm h}, Y^{\mathrm h}] \\ &\overset{(2.4a),\,(2.3c)}{=} F\big([X^{\mathrm h}, Y^{\mathrm v}] - [Y^{\mathrm h}, X^{\mathrm v}] - [X, Y]^{\mathrm v}\big) + R(X^{\mathrm h}, Y^{\mathrm h}) \\ &\overset{(2.3b)}{=} (F \circ t + R)(X^{\mathrm h}, Y^{\mathrm h}), \end{aligned}$$

thus (3.5) is also true. □

3.6. Corollary. *If (D°, h) is a Berwald-type Finsler connection then*

(i) *the h-deflection of (D°, h) vanishes if and only if h is homogeneous*;

(ii) *the h-horizontal torsion $\mathbf{A}^{\circ}$ of D° is related to the torsion of h by the formula $\mathbf{A}^{\circ} = F \circ t$, therefore $\mathbf{A}^{\circ} = 0 \Leftrightarrow t = 0$, and then $\mathbf{T}^{\circ} = R$.* □

3.7. Lemma. *Suppose (D°, h) is a Berwald-type Finsler connection and $A \in \mathcal{T}^1_s(TM)$ is a semibasic tensor. Then*

$$\forall\, X \in 10(M) : D^{\circ}_{X^{\mathrm v}} A = \mathcal{L}_{X^{\mathrm v}} A, \tag{3.7a}$$

more precisely,

$$\begin{aligned} \forall\, X_1, \dots, X_s \in 10(M) : (D^{\circ}_{X^{\mathrm v}} A)(X_1^{\mathrm c}, \dots, X_s^{\mathrm c}) &= (\mathcal{L}_{X^{\mathrm v}} A)\,(X_1^{\mathrm c}, \dots, X_s^{\mathrm c}) \\ = \big[X^{\mathrm v}, A(X_1^{\mathrm c}, \dots, X_s^{\mathrm c})\big]. \end{aligned} \tag{3.7b}$$

Proof. Since $\forall\, X \in 10(M) : [X^{\mathrm v}, X_i^{\mathrm h}] \in 10^{\mathrm v}(TM)$, $1 \leqslant i \leqslant s$, and A is semibasic, it follows that

$$(\mathcal{L}_{X^{\mathrm v}} A)(X_1^{\mathrm h}, \dots, X_s^{\mathrm h}) = \big[X^{\mathrm v}, A(X_1^{\mathrm h}, \dots, X_s^{\mathrm h})\big].$$

Here

$$\begin{aligned} \big[X^{\mathrm v}, A(X_1^{\mathrm h}, \dots, X_s^{\mathrm h})\big] &\overset{(2.4b)}{=} -\big[JFA\big(X_1^{\mathrm h}, \dots, X_s^{\mathrm h}\big), X^{\mathrm v}\big] \\ &\overset{(1.3a)}{=} -[J, X^{\mathrm v}]\big(FA(X_1^{\mathrm h}, \dots, X_s^{\mathrm h})\big) - J\big[FA(X_1^{\mathrm h}, \dots, X_s^{\mathrm h}), X^{\mathrm v}\big] \\ &\overset{(1.5c)}{=} J\big[JX^{\mathrm c}, FA(X_1^{\mathrm h}, \dots, X_s^{\mathrm h})\big] \overset{(3.4a)}{=} D^{\circ}_{X^{\mathrm v}} A\,\big(X_1^{\mathrm h}, \dots, X_s^{\mathrm h}\big) \\ &\overset{(3.4g)}{=} \big(D^{\circ}_{X^{\mathrm v}} A\big)\,\big(X_1^{\mathrm h}, \dots, X_s^{\mathrm h}\big), \end{aligned}$$

so we obtain the result. □

3.8. Lemma. *Suppose that h is a homogeneous horizontal endomorphism with vanishing torsion, and let F be the almost complex structure associated to h. If $A \in \mathcal{T}_s^1(TM)$ is semibasic and homogeneous of degree k (i.e., $\mathcal{L}_C A = (k-1)A$) then $F \circ A$ is homogeneous of degree $k+1$.*

Proof. By the assumption, for any vector fields $X_1, \ldots, X_s \in \mathfrak{X}(M)$,

$$(k-1)A(X_1^c, \ldots, X_s^c) = (\mathcal{L}_C A)(X_1^c, \ldots, X_s^c) = \big[C, A(X_1^c, \ldots, X_s^c)\big],$$

since $[C, X_i^c] \overset{(1.5d)}{=} 0$, $1 \leqslant i \leqslant s$. Now we consider the Berwald-type connection (D°, h). It follows by the preceding arguments that

$$\big(D_C^\circ A\big)(X_1^c, \ldots, X_s^c) = D_C^\circ A(X_1^c, \ldots, X_s^c).$$

The right-hand side of this equation can be formed as follows:

$$\begin{aligned} D_C^\circ A(X_1^c, \ldots, X_s^c) &= D_{JS}^\circ JFA(X_1^c, \ldots, X_s^c) \overset{(3.4a)}{=} J\big[C, FA(X_1^c, \ldots, X_s^c)\big] \\ &\overset{(1.3a)}{=} [J, C]\big(FA(X_1^c, \ldots, X_s^c)\big) - [JFA(X_1^c, \ldots, X_s^c), C] \\ &\overset{(1.5b)}{=} JFA(X_1^c, \ldots, X_s^c) + \big[C, JFA(X_1^c, \ldots, X_s^c)\big] \\ &\overset{(2.4b)}{=} A(X_1^c, \ldots, X_s^c) + (\mathcal{L}_C A)\,(X_1^c, \ldots, X_s^c) \\ &= kA(X_1^c, \ldots, X_s^c). \end{aligned}$$

Thus

$$\begin{aligned} kFA(X_1^c, \ldots, X_s^c) &= F\big[D_C^\circ A(X_1^c, \ldots, X_s^c)\big] \overset{(3.4a)}{=} FJ\big[C, FA(X_1^c, \ldots, X_s^c)\big] \\ &\overset{(2.4a)}{=} h\big[C, FA(X_1^c, \ldots, X_s^c)\big] \\ &\overset{(1.3a)}{=} [h, C]\big(FA(X_1^c, \ldots, X_s^c)\big) - \big[hFA(X_1^c, \ldots, X_s^c), C\big] \\ &\overset{2.2,\,(2.4b)}{=} \big[C, FA(X_1^c, \ldots, X_s^c)\big] = \big[\mathcal{L}_C(F \circ A)\big](X_1^c, \ldots, X_s^c), \end{aligned}$$

which proves our assertion. □

4. The mixed curvature and the mixed Ricci tensor of a Berwald-type connection

4.1. If (D°, h) is a Finsler connection of Berwald-type, then we denote the horizontal, the mixed and the vertical curvature of D° by $\mathbf{R}^\circ$, $\mathbf{P}^\circ$ and $\mathbf{Q}^\circ$, respectively. In the centre of interest in this work is the mixed curvature, but for completeness we recall that $\mathbf{Q}^\circ$ *always vanishes*, while $\mathbf{R}^\circ$ can be given by the following formula:

$$\forall\, X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \mathbf{R}^\circ(X, Y)Z = \big[J, R(X, Y)\big]Z. \tag{4.1}$$

For a good study of the surviving curvatures (in the Finslerian case) we refer to [19].

4.2. Suppose D is a linear connection on the manifold TM (or $\mathcal{T}M$). It will be convenient to introduce the operator

$$D_J : A \in \mathcal{T}_s^r(TM) \mapsto D_J A \in \mathcal{T}_{s+1}^r(TM)$$

by the rule

$$i_X D_J A := D_{JX} A \qquad (X \in 10(TM)).$$

Then, for example, (3.4a) can be written in the more compact form

$$\forall\, Y \in 10(TM) : D^\circ_J JY = [J, JY].$$

4.3. Lemma. *The mixed curvature* $\mathbf{P}^\circ$ *of a Berwald-type Finsler connection* (D°, h) *can be given by*

$$\forall\, X, Y, Z \in 10(M) : \mathbf{P}^\circ(X^c, Y^c)Z^c = [[X^h, Y^v], Z^v]. \tag{4.3}$$

$\mathbf{P}^\circ$ *is symmetric in its second and third arguments. If, in addition,* h *is torsion free, then* $\mathbf{P}^\circ$ *is totally symmetric.*

Proof.

$$\begin{aligned}
\mathbf{P}^\circ(X^c, Y^c)Z^c &:= \mathbf{K}(hX^c, JY^c)JZ^c = \mathbf{K}(X^h, Y^v)Z^v \\
&:= D^\circ_{X^h} D^\circ_{Y^v} Z^v - D^\circ_{Y^v} D^\circ_{X^h} Z^v - D^\circ_{[X^h, Y^v]} Z^v \\
&\overset{(3.4e,f)}{=} - D^\circ_{Y^v}[X^h, Z^v] - D^\circ_{[X^h, Y^v]} Z^v.
\end{aligned}$$

Since $[X^h, Y^v]$ is vertical, it can be combined from vertically lifted vector fields, so (3.4e) implies that $D^\circ_{[X^h, Y^v]} Z^v = 0$. The remainder term can be formed as follows:

$$\begin{aligned}
-D^\circ_{Y^v}[X^h, Z^v] &= -D^\circ_{JY^c} JF[X^h, Z^v] \overset{(3.4a)}{=} -J[Y^v, F[X^h, Z^v]] \\
&= J[F[X^h, Z^v], Y^v] \overset{(1.3a)}{=} -[J, Y^v]F[X^h, Z^v] + [JF[X^h, Z^v], Y^v] \\
&\overset{(1.5c),\,(2.4b)}{=} [[X^h, Z^v], Y^v] \overset{\text{Jacobi-identity}}{=} -[[Z^v, Y^v], X^h] - [[Y^v, X^h], Z^v] \\
&\overset{(1.4)}{=} [[X^h, Y^v], Z^v].
\end{aligned}$$

Thus (4.3) is verified, and, at the same time, we have got the relation

$$\mathbf{P}^\circ(X^c, Z^c)Y^c = [[X^h, Z^v], Y^v] = \mathbf{P}^\circ(X^c, Y^c)Z^c.$$

If h is torsion free and hence, in view of (2.3b),

$$[X^h, Y^v] - [Y^h, X^v] - [X, Y]^v = 0,$$

it follows that

$$\begin{aligned}
[[X^h, Y^v], Z^v] &= [[Y^h, X^v], Z^v] + [[X, Y]^v, Z^v] \overset{(1.4)}{=} [[Y^h, X^v], Z^v] \\
&= \mathbf{P}^\circ(Y^c, X^c)Z^c,
\end{aligned}$$

thus $\mathbf{P}^\circ$ is totally symmetric. $\square$

4.4. Proposition. *If h is a homogeneous and torsion free horizontal endomorphism, then the mixed curvature of the Berwald-type connection (D°, h) has the following properties*:

$$i_S\mathbf{P}^{\circ} = 0 \text{ for any semispray } S; \tag{4.4a}$$

$$\mathcal{L}_C\mathbf{P}^{\circ} = -2\mathbf{P}^{\circ}, \text{ i.e., } \mathbf{P}^{\circ} \text{ is homogeneous of degree } -1; \tag{4.4b}$$

$$D^{\circ}_J\mathbf{P}^{\circ} \text{ is totally symmetric.} \tag{4.4c}$$

Proof. (a) Owing to Lemma 4.3, it is enough to check that

$$\forall\, X, Y \in 10(M):\ \mathbf{P}^{\circ}(X^{c}, S)Y^{c} = 0.$$

Calculating as before, we get

$$\begin{aligned}
\mathbf{P}^{\circ}(X^{c}, S)Y^{c} &= D^{\circ}_{X^{h}}D^{\circ}_{JS}Y^{v} - D^{\circ}_{JS}D^{\circ}_{X^{h}}Y^{v} - D^{\circ}_{[X^{h}, JS]}Y^{c}\\
&= D^{\circ}_{X^{h}}J[C, Y^{c}] - D^{\circ}_{JS}[X^{h}, Y^{v}] - D^{\circ}_{[X^{h}, C]}Y^{c}\\
&\overset{(1.5d),\,2.2,\,(2.4b)}{=} - D^{\circ}_{JS}JF[X^{h}, Y^{v}] \overset{(3.4a)}{=} -J[C, F[X^{h}, Y^{v}]]\\
&= J[F[X^{h}, Y^{v}], C] \overset{(1.3a)}{=} -[J, C]F[X^{h}, Y^{v}] + [JF[X^{h}, Y^{v}], C]\\
&\overset{(1.5b),\,(2.4b)}{=} -[X^{h}, Y^{v}] + [[X^{h}, Y^{v}], C]\\
&\overset{\text{Jacobi identity}}{=} - [X^{h}, Y^{v}] - [[Y^{v}, C], X^{h}] - [[C, X^{h}], Y^{v}]\\
&\overset{(1.5d),\,2.2}{=} -[X^{h}, Y^{v}] - [Y^{v}, X^{h}]\\
&= 0.
\end{aligned}$$

(b) Since the Lie brackets of C and the complete lifts vanish, for any vector fields X, Y, Z on M we have

$$\left(\mathcal{L}_C\mathbf{P}^{\circ}\right)(X^{c}, Y^{c}, Z^{c}) = [C, \mathbf{P}^{\circ}(X^{c}, Y^{c}, Z^{c})] \overset{(4.3)}{=} [C, [[X^{h}, Y^{v}], Z^{v}]].$$

Using the Jacobi identity repeatedly, in view of (1.5d) and the homogeneity of h it follows that

$$\begin{aligned}
[C, [[X^{h}, Y^{v}], Z^{v}]] &= -[[X^{h}, Y^{v}], [Z^{v}, C]] - [Z^{v}, [C, [X^{h}, Y^{v}]]]\\
&= -[[X^{h}, Y^{v}], Z^{v}] - [Z^{v}, [C, [X^{h}, Y^{v}]]]\\
&= -[[X^{h}, Y^{v}], Z^{v}] + [Z^{v}, [X^{h}, [Y^{v}, C]]] + [Z^{v}, [Y^{v}, [C, X^{h}]]]\\
&= -[[X^{h}, Y^{v}], Z^{v}] - [[X^{h}, Y^{v}], Z^{v}]\\
&\overset{(4.3)}{=} -2\mathbf{P}^{\circ}(X^{c}, Y^{c})Z^{c}.
\end{aligned}$$

Thus $\mathcal{L}_C\mathbf{P}^{\circ} = -2\mathbf{P}^{\circ}$.

(c) For any vector fields $X, Y, Z, U \in 10(M)$,

$$\left(D^{\circ}_J\mathbf{P}^{\circ}\right)(X^{c}, Y^{c}, Z^{c}, U^{c}) \overset{4.2}{=} \left(D^{\circ}_{JX^{c}}\mathbf{P}^{\circ}\right)(Y^{c}, Z^{c}, U^{c}) = D^{\circ}_{JX^{c}}\mathbf{P}^{\circ}(Y^{c}, Z^{c})U^{c},$$

since $\mathbf{P}^{\circ}$ is semibasic, and, for example,

$$D^{\circ}_{JX^{c}}Y^{c} = D^{\circ}_{JX^{c}}(\nu Y^{c} + hY^{c}) = D^{\circ}_{JX^{c}}\nu Y^{c} + D^{\circ}_{X^{v}}Y^{h} \overset{(3.4g)}{=} D^{\circ}_{JX^{c}}\nu Y^{c} \in 10^{v}(TM).$$

Thus in view of Lemma 4.3 and the calculation in its proof, we get

$$\begin{aligned}(D^\circ_J\mathbf{P}^\circ)(X^c, Y^c, Z^c, U^c) = {} & D^\circ_{JX^c}[[Y^h, Z^v], U^v] = D^\circ_{JX^c}JF[[Y^h, Z^v], U^v] \\ & \overset{(3.4a)}{=} J\big[X^v, F[[Y^h, Z^v], U^v]\big] = \big[X^v, [[Y^h, Z^v], U^v]\big] \\ & = -\big[[Y^h, Z^v], [U^v, X^v]\big] - \big[U^v, [X^v, [Y^h, Z^v]]\big] \\ & = \big[U^v, [[Y^h, Z^v], X^v]\big] = [U^v, \mathbf{P}^\circ(Y^c, Z^c)X^c] \\ & \overset{(3.7b)}{=} \big(D^\circ_{U^v}\mathbf{P}^\circ\big)(Y^c, Z^c, X^c) = \big(D^\circ_J\mathbf{P}^\circ\big)(U^c, Y^c, Z^c, X^c).\end{aligned}$$

This symmetry property together with the total symmetry of $\mathbf{P}^\circ$ implies the total symmetry of $D^\circ_J\mathbf{P}^\circ$. □

4.5. Corollary. *Under the hypothesis of* 4.4, *the mixed Ricci tensor* $\widetilde{\mathbf{P}}^\circ$ *of the Berwald-type connection* (D°, h) *has the following properties*:

$$\mathcal{L}_C\widetilde{\mathbf{P}}^\circ = D^\circ_C\widetilde{\mathbf{P}}^\circ = -\widetilde{\mathbf{P}}^\circ \text{ (i.e., } \widetilde{\mathbf{P}}^\circ \text{ is } (-1)\text{-homogeneous);} \tag{4.5a}$$

$$D^\circ_J\widetilde{\mathbf{P}}^\circ \text{ is totally symmetric.} \tag{4.5b}$$

Proof. (4.5a) is a consequence of (4.4b) and Lemma 3.8. Since the semibasic trace operator clearly commutes with the tensor derivations, (4.5b) follows from (4.4c). □

5. Yano-type connections

We start with a slight generalization of the Berwald-type connections.

5.1. Proposition. *Suppose* h *is a horizontal endomorphism on the manifold* M *with associated almost complex structure* F. *Let* $\beta \in \mathcal{T}^0_2(\mathcal{T}M)$ *be a symmetric tensor, satisfying the condition*

$$\text{for any semispray } S, \qquad i_S\beta = 0. \tag{$*$}$$

Let, finally, a vertical vector field $U \in 10(TM)$ *be given. Then there exists a unique Finsler connection* (D, h) *on* M *such that*

(i) *the* v*-mixed torsion of* D *is* $\mathbf{P}^1 := \beta \otimes U$;

(ii) *the* h*-mixed torsion* $\mathbf{B}$ *of* D *vanishes.*

The table of rules for calculation of the covariant derivatives with respect to D *is*

$$D_{JX}JY = J[JX, Y] = D^\circ_{JX}JY, \tag{5.1a}$$

$$D_{hX}JY = v[hX, JY] + \beta(X, Y)U = D^\circ_{hX}JY + \beta(X, Y)U, \tag{5.1b}$$

$$D_{vX}hY = h[vX, Y] = D^\circ_{vX}hY, \tag{5.1c}$$

$$D_{hX}hY = hF[hX, JY] + \beta(X, Y)FU = D^\circ_{hX}hY + \beta(X, Y)FU. \tag{5.1d}$$

If, in addition,

(iii) *the* h*-deflection of* (D, h) *vanishes,*

(iv) *the* h*-horizontal torsion of* D *vanishes,*

then the horizontal endomorphism h *is homogeneous and torsion free.*

Proof. (a) *Unicity*. We show that (i) and (ii) imply (5.1a)–(5.1d).

(i) $\Rightarrow$ (5.1b) $\quad \forall X, Y \in 10(TM)$:

$$\beta(X,Y)U \overset{(i)}{=} \mathbf{P}^1(X,Y) \overset{3.2}{=} \nu\,(D_{hX}JY - D_{JY}hX - [hX, JY]) = D_{hX}JY - \nu[hX, JY],$$

so (5.1b) is valid.

(ii) $\Rightarrow$ (5.1c) $\quad$ For any vector fields $X, Y \in 10(TM)$,

$$0 \overset{(ii)}{=} \mathbf{B}(Y,X) \overset{3.2}{=} h\,(D_{hY}JX - D_{JX}hY - [hY, JX]) = -D_{JX}hY - h[hY, JX],$$

thus $D_{JX}hY = h[JX, hY]$. Replacing X by FX, we get $D_{\nu X}hY = D_{JFX}hY = h[\nu X, hY] = h[\nu X, Y] - h[\nu X, \nu Y] = h[\nu X, Y]$. This means that (5.1c) also holds. Now (5.1a) and (5.1d) follow from (5.1b) and (5.1c) by (3.1c) and (3.1d).

(b) *Existence*. Having the data h, β, U, we define a linear connection D on TM by the rules (5.1a)–(5.1d). A straightforward calculation shows that (D, h) is a Finsler connection satisfying (i) and (ii).

(c) The remainder is an immediate consequence of Lemma 3.5.

5.2. Corollary. *A Finsler connection (D°, h) on M is a Berwald-type connection if and only if the following axioms are satisfied*:

(i) *The* υ*-mixed torsion of D vanishes.*

(ii) *The* h*-mixed torsion of D vanishes.* $\quad\square$

5.3. Corollary and definition. *Suppose h is a homogeneous and torsion free horizontal endomorphism on M. Let $\widetilde{\mathbf{P}}^\circ$ be the mixed Ricci tensor of the Berwald-type connection (D°, h). There exists a unique Finsler connection (D, h) on M such that*

(i) *the* υ*-mixed torsion of D is*

$$\mathbf{P}^1 := \frac{1}{n+1}(\widetilde{\mathbf{P}}^\circ \otimes C);$$

(ii) *the* h*-mixed torsion of D vanishes.*

This Finsler connection is said to be the Yano-type connection *induced by h. If, in particular, h is a Berwald endomorphism, then we speak of a* Yano connection.

Proof. Recall that $\widetilde{\mathbf{P}}^\circ$ is symmetric by Lemma 4.3, while the validity of the condition $(*)$ in 5.1 is assured by (4.4a). $\quad\square$

5.4. Remark. If (D°, h) and (D, h) are the Berwald-type and the Yano-type connections induced by h, then we see from (5.1a) and (5.1c) that $D^\circ_J = D_J$. In view of Lemma 3.7 and Proposition 4.4 this implies that

$$\text{for any vector field } X \in 10(M) \text{ and semibasic tensor } A \in \mathcal{T}^1_s(TM),\quad D_{X^v}A = \mathcal{L}_{X^v}A; \tag{5.4a}$$

$$D_J\mathbf{P}^\circ \text{ is totally symmetric.} \tag{5.4b}$$

5.5. Proposition. *Suppose that h is a torsion free, homogeneous horizontal endomorphism. Let (D°, h) and (D, h) be the induced Berwald-type and Yano-type connections with mixed curvature and mixed Ricci tensors $\mathbf{P}^\circ$, $\mathbf{P}$ and $\widetilde{\mathbf{P}}^\circ$, $\widetilde{\mathbf{P}}$, respectively. Then for any vector fields X, Y, Z on M,*

$$\begin{aligned}\mathbf{P}(X^c, Y^c)Z^c = \left[\mathbf{P}^\circ - \frac{1}{n+1}(D_J\widetilde{\mathbf{P}}^\circ \otimes C)\right](X^c, Y^c, Z^c)\\ - \frac{1}{n+1}(\widetilde{\mathbf{P}}^\circ \otimes J)(X^c, Z^c, Y^c).\end{aligned} \tag{5.5a}$$

$$\widetilde{\mathbf{P}} = \frac{2}{n+1}\widetilde{\mathbf{P}}^\circ. \tag{5.5b}$$

Proof. (a) By the rules of calculation (5.1a)–(5.1c),

$$\begin{aligned}\mathbf{P}(X^c, Y^c)Z^c &= D_{X^h}D_{Y^v}Z^v - D_{Y^v}D_{X^h}Z^v - D_{[X^h, Y^v]}Z^v = -D_{Y^v}D_{X^h}Z^v\\ &= -D_{Y^v}\left([X^h, Z^v] + \frac{1}{n+1}\widetilde{\mathbf{P}}^\circ(X^c, Z^c)C\right).\end{aligned}$$

Thus, taking into account (4.3) and (5.4),

$$\mathbf{P}(X^c, Y^c)Z^c = \mathbf{P}^\circ(X^c, Y^c)Z^c - \frac{1}{n+1}D_{Y^v}(\widetilde{\mathbf{P}}^\circ(X^c, Z^c)C).$$

Now using the fact that $(D_J\widetilde{\mathbf{P}}^\circ)(X^c, Y^c, Z^c) = D_{JX^c}\big[\widetilde{\mathbf{P}}^\circ(Y^c, Z^c)\big]$ and, taking a semispray S,

$$D_JC(Y^c) = D_{JY^c}C = D_{JY^c}JS \overset{(5.1a)}{=} J[Y^v, S] \overset{(2.6)}{=} Y^v = J(Y^c),$$

it follows that

$$\begin{aligned}D_{Y^v}(\widetilde{\mathbf{P}}^\circ(X^c, Z^c)C) &= Y^v(\widetilde{\mathbf{P}}^\circ(X^c, Z^c))C + \widetilde{\mathbf{P}}^\circ(X^c, Z^c)D_{Y^v}C\\ &= (D_J\widetilde{\mathbf{P}}^\circ \otimes C)(Y^c, X^c, Z^c) + (\widetilde{\mathbf{P}}^\circ \otimes J)(X^c, Z^c, Y^c)\\ &\overset{(5.4b)}{=} (D_J\widetilde{\mathbf{P}}^\circ \otimes C)(X^c, Y^c, Z^c) + (\widetilde{\mathbf{P}}^\circ \otimes J)(X^c, Z^c, Y^c).\end{aligned}$$

This proves (5.5a).

(b) In view of Definition 3.3 and the symmetry of $\widetilde{\mathbf{P}}^\circ$ and $D_J\widetilde{\mathbf{P}}^\circ$,

$$\begin{aligned}\widetilde{\mathbf{P}}(X^c, Y^c) &= \operatorname{tr}[F \circ (Z^c \mapsto \mathbf{P}(Y^c, Z^c)X^c)]\\ &\overset{(5.5a)}{=} \operatorname{tr}\Bigg[F \circ \Bigg(Z^c \mapsto \Bigg(\mathbf{P}^\circ(Y^c, Z^c)X^c - \frac{1}{n+1}(D_J\widetilde{\mathbf{P}}^\circ \otimes C)(Y^c, Z^c, X^c)\\ &\qquad - \frac{1}{n+1}(\widetilde{\mathbf{P}}^\circ \otimes J)(Y^c, X^c, Z^c)\Bigg)\Bigg)\Bigg]\\ &= \widetilde{\mathbf{P}}^\circ(X^c, Y^c) - \frac{1}{n+1}\widetilde{D_J\widetilde{\mathbf{P}}^\circ \otimes C}(X^c, Y^c) - \frac{1}{n+1}\widetilde{\widetilde{\mathbf{P}}^\circ \otimes J}(X^c, Y^c).\end{aligned}$$

Since $\widetilde{\widetilde{\mathbf{P}}^\circ \otimes J} = n\widetilde{\mathbf{P}}^\circ$, and (taking a semispray S)

$$\begin{aligned}\widetilde{D_J\widetilde{\mathbf{P}}^\circ \otimes C} &= \widetilde{D_J\widetilde{\mathbf{P}}^\circ \otimes JS} \overset{(1.7b)}{=} i_{FJS}D_J\widetilde{\mathbf{P}}^\circ = i_{hS}D_J\widetilde{\mathbf{P}}^\circ\\ &= D_{JhS}\widetilde{\mathbf{P}}^\circ = D_C\widetilde{\mathbf{P}}^\circ = \mathcal{L}_C\widetilde{\mathbf{P}}^\circ \overset{(4.5a)}{=} -\widetilde{\mathbf{P}}^\circ,\end{aligned}$$

we have

$$\widetilde{\mathbf{P}} = \widetilde{\mathbf{P}^\circ} - \frac{n}{n+1}\widetilde{\mathbf{P}^\circ} + \frac{1}{n+1}\widetilde{\mathbf{P}^\circ} = \frac{2}{n+1}\widetilde{\mathbf{P}^\circ}. \quad \square$$

5.6. Remark. Using the symmetric product (1.1), (5.5a) can also be written in the form

$$\mathbf{P} = \mathbf{P}^\circ - \frac{1}{n+1}\big[(D_J\widetilde{\mathbf{P}^\circ}) \otimes C + \widetilde{\mathbf{P}^\circ} \odot J - \widetilde{\mathbf{P}^\circ} \otimes J - J \otimes \widetilde{\mathbf{P}^\circ}\big]. \tag{5.6}$$

6. The Douglas tensor of a Berwald endomorphism

6.1. Definition. Suppose h is a Berwald endomorphism on the manifold M. If (D, h) is the Yano connection induced by h and $\mathbf{P}$ is the mixed curvature of D, then the tensor

$$\mathbf{D} := \mathbf{P} - \tfrac{1}{2}\big(\widetilde{\mathbf{P}} \otimes J + J \otimes \widetilde{\mathbf{P}}\big) \tag{6.1}$$

is said to be the *Douglas tensor* of the Berwald endomorphism.

6.2. Remark. (a) It is clear that $\mathbf{D}$ is semibasic and symmetric. By the definition, for any vector fields $X, Y, Z \in 10(M)$,

$$\mathbf{D}(X^c, Y^c)Z^c = \mathbf{P}(X^c, Y^c)Z^c - \tfrac{1}{2}\big(\widetilde{\mathbf{P}}(X^c, Y^c)Z^v + \widetilde{\mathbf{P}}(Y^c, Z^c)X^v\big). \tag{6.2a}$$

(b) It is easy to express $\mathbf{D}$ in terms of the Berwald connection (D°, h): in view of (6.2a), (5.5a) and (5.5b)

$$\begin{aligned}\mathbf{D}(X^c, Y^c)Z^c = {}& \mathbf{P}^\circ(X^c, Y^c)Z^c - \frac{1}{n+1}(D^\circ_J\widetilde{\mathbf{P}^\circ} \otimes C)(X^c, Y^c, Z^c)\\ &- \frac{1}{n+1}\widetilde{\mathbf{P}^\circ}(X^c, Z^c)Y^v - \frac{1}{1+n}\widetilde{\mathbf{P}^\circ}(X^c, Y^c)Z^v - \frac{1}{n+1}\mathbf{P}^\circ(Y^c, Z^c)X^v\\ &= \Big[\mathbf{P}^\circ - \frac{1}{n+1}\big(D^\circ_J\widetilde{\mathbf{P}^\circ} \otimes C + \widetilde{\mathbf{P}^\circ} \odot J\big)\Big](X^c, Y^c, Z^c).\end{aligned}$$

In a more compact form:

$$\mathbf{D} = \mathbf{P}^\circ - \frac{1}{n+1}\big(D^\circ_J\widetilde{\mathbf{P}^\circ} \otimes C + \widetilde{\mathbf{P}^\circ} \odot J\big). \tag{6.2b}$$

6.3. Proposition. *Let* $\mathbf{D}$ *be the Douglas tensor of a Berwald endomorphism. Then*

$$\textit{for any semispray } S, \quad i_S\mathbf{D} = 0; \tag{6.3a}$$

$$\widetilde{\mathbf{D}} = 0 \ (\textit{i.e., the semibasic trace of } \mathbf{D} \textit{ vanishes}). \tag{6.3b}$$

Proof. (a) Let us first observe that $i_S\widetilde{\mathbf{P}^\circ} := \widetilde{i_S\mathbf{P}^\circ} \overset{(4.4a)}{=} 0$, so for any vector fields $Y, Z \in 10(M)$,

$$\begin{aligned}(i_S\mathbf{P})\,(Y^c, Z^c) &\overset{(5.5a),\,(4.4a)}{=} -\frac{1}{n+1}\big[(D_C\widetilde{\mathbf{P}^\circ})(Y^c, Z^c) \otimes C + [i_S\widetilde{\mathbf{P}^\circ}(Z^c)]Y^v\big]\\ &= -\frac{1}{n+1}(D_C\widetilde{\mathbf{P}^\circ})(Y^c, Z^c) \otimes C \overset{(4.5a)}{=} \frac{1}{n+1}\widetilde{\mathbf{P}^\circ}(Y^c, Z^c)C.\end{aligned}$$

Thus

$$(i_S\mathbf{D})\,(Y^c, Z^c) = \mathbf{D}(S, Y^c, Z^c) = (i_S\mathbf{P})\,(Y^c, Z^c) - \tfrac{1}{2}\left[\widetilde{\mathbf{P}}(S, Y^c)Z^v + \widetilde{\mathbf{P}}(Y^c, Z^c)C\right]$$
$$= \frac{1}{n+1}\widetilde{\mathbf{P}^\circ}(Y^c, Z^c)C - \tfrac{1}{2}\left(i_S\widetilde{\mathbf{P}}(Y^c)Z^v + \widetilde{\mathbf{P}}(Y^c, Z^c)C\right)$$
$$\overset{(5.5b)}{=} \left(\frac{1}{n+1}\widetilde{\mathbf{P}^\circ} \otimes C - \frac{1}{n+1} i_S\widetilde{\mathbf{P}^\circ} \otimes J - \frac{1}{n+1}\widetilde{\mathbf{P}^\circ} \otimes C\right)(Y^c, Z^c)$$
$$\overset{(4.4a)}{=} 0.$$

(b) Since

$$D_J^\circ\widetilde{\widetilde{\mathbf{P}^\circ}\otimes C} = D_J^\circ\widetilde{\widetilde{\mathbf{P}^\circ}\otimes J S} \overset{(1.7b)}{=} i_{FJS}D_J^\circ\widetilde{\mathbf{P}^\circ} = i_{hS}D_J^\circ\widetilde{\mathbf{P}^\circ} = D_{JhS}^\circ\widetilde{\mathbf{P}^\circ} = D_C^\circ\widetilde{\mathbf{P}^\circ} \overset{(4.5a)}{=} -\widetilde{\mathbf{P}^\circ}$$

and

$$\widetilde{\widetilde{\mathbf{P}^\circ} \odot J} \overset{(1.8)}{=} \widetilde{J}\widetilde{\mathbf{P}^\circ} + i_{FJ}\widetilde{\mathbf{P}^\circ} \overset{(1.7a)}{=} n\widetilde{\mathbf{P}^\circ} + i_h\widetilde{\mathbf{P}^\circ} \overset{1.2}{=} (n+2)\widetilde{\mathbf{P}^\circ};$$

using (6.2b) we get

$$\widetilde{\mathbf{D}} = \widetilde{\mathbf{P}^\circ} - \frac{1}{n+1}\left(-\widetilde{\mathbf{P}^\circ} + (n+2)\widetilde{\mathbf{P}^\circ}\right) = \widetilde{\mathbf{P}^\circ} - \widetilde{\mathbf{P}^\circ} = 0. \quad \square$$

6.4. Theorem. *The Douglas tensor of a Berwald endomorphism is invariant under the projective changes of the associated spray.*

Proof. Suppose h is a Berwald endomorphism on M with associated spray S. Denote by $\mathbf{D}$ the Douglas tensor of h. Consider a projective change

$$S \to \bar{S} := S + \lambda C, \quad C\lambda = \lambda.$$

Then $\bar{S}$ generates a Berwald endomorphism $\bar{h}$ (2.10); let $\bar{\mathbf{D}}$ be the Douglas tensor of $\bar{h}$. First we express the mixed curvature $\overline{\mathbf{P}^\circ}$ of the Berwald connection $\left(\overline{D^\circ}, \bar{h}\right)$ in terms of the mixed curvature $\mathbf{P}^\circ$ of the Berwald connection (D°, h).

For any vector fields $X, Y, Z \in 10(M)$,

$$\overline{\mathbf{P}^\circ}(X^c, Y^c)Z^c \overset{(4.3)}{=} [[X^{\bar{h}}, Y^v], Z^v] \overset{(2.10b)}{=} [[X^h + \tfrac{1}{2}\lambda X^v + \tfrac{1}{2}(X^v\lambda)C, Y^v], Z^v]$$
$$= [[X^h, Y^v], Z^v] + \tfrac{1}{2}[[\lambda X^v, Y^v], Z^v] + \tfrac{1}{2}[[(X^v\lambda)C, Y^v], Z^v]$$
$$= \mathbf{P}^\circ(X^c, Y^c)Z^c + \tfrac{1}{2}[Z^v(Y^v\lambda)]X^v + \tfrac{1}{2}[Z^v[Y^v(X^v\lambda)]]C$$
$$+ \tfrac{1}{2}[Z^v(X^v\lambda)]Y^v + \tfrac{1}{2}[Y^v(X^v\lambda)]Z^v.$$

Consider the tensor

$$\alpha : (X, Y) \in 10(TM) \times 10(TM) \mapsto \alpha(X, Y) := dd_J\lambda(JX, Y).$$

α is obviously symmetric and semibasic, and it makes possible to abbreviate the above expression as follows:

$$\overline{\mathbf{P}^\circ} = \mathbf{P}^\circ + \tfrac{1}{2}\alpha \odot J + \tfrac{1}{2}\,(D_J\alpha) \otimes C. \tag{6.4a}$$

In the next step we derive the relation between the semibasic traces of $\overline{\mathbf{P}^\circ}$ and $\mathbf{P}^\circ$. By (1.7a), (1.7b) and (1.8) we obtain from (6.4a):

$$\widetilde{\overline{\mathbf{P}^\circ}} = \widetilde{\mathbf{P}^\circ} + \tfrac{1}{2}n\alpha + \tfrac{1}{2}i_{FJ}\alpha + \tfrac{1}{2}i_{FJS}D_J\alpha = \widetilde{\mathbf{P}^\circ} + \tfrac{1}{2}n\alpha + \alpha + \tfrac{1}{2}D_C\alpha.$$

Now we calculate the covariant derivative $D_C\alpha$. For any vector fields $X, Y \in 10(M)$

$$\begin{aligned}
(D_C\alpha)(X^{\mathrm{c}}, Y^{\mathrm{c}}) &= (\mathcal{L}_C\alpha)(X^{\mathrm{c}}, Y^{\mathrm{c}}) = C\alpha(X^{\mathrm{c}}, Y^{\mathrm{c}}) = C[X^{\mathrm{v}}(Y^{\mathrm{v}}\lambda)] \\
&= [C, X^{\mathrm{v}}](Y^{\mathrm{v}}\lambda) + X^{\mathrm{v}}[C(Y^{\mathrm{v}}\lambda)] \\
&\overset{(1.5\mathrm{d})}{=} -X^{\mathrm{v}}(Y^{\mathrm{v}}\lambda) + X^{\mathrm{v}}([C, Y^{\mathrm{v}}]\lambda + Y^{\mathrm{v}}(C\lambda)) \\
&\overset{(1.5\mathrm{d})}{=} -X^{\mathrm{v}}(Y^{\mathrm{v}}\lambda) - X^{\mathrm{v}}(Y^{\mathrm{v}}\lambda) + X^{\mathrm{v}}(Y^{\mathrm{v}}\lambda) = -X^{\mathrm{v}}(Y^{\mathrm{v}}\lambda) \\
&= -\alpha(X^{\mathrm{c}}, Y^{\mathrm{c}}).
\end{aligned}$$

(using the 1-homogeneity of λ). Thus $D_C\alpha = -\alpha$, therefore

$$\widetilde{\overline{\mathbf{P}^\circ}} = \widetilde{\mathbf{P}^\circ} + \tfrac{1}{2}(n+1)\alpha \tag{6.4b}$$

and, in view of (5.5b),

$$\widetilde{\overline{\mathbf{P}}} = \widetilde{\mathbf{P}} + \alpha. \tag{6.4c}$$

On the other hand, by (5.6),

$$\begin{aligned}
\overline{\mathbf{P}} &= \overline{\mathbf{P}^\circ} - \frac{1}{n+1}\big[(D_J\widetilde{\overline{\mathbf{P}^\circ}}) \otimes C + \widetilde{\overline{\mathbf{P}^\circ}} \odot J - \widetilde{\overline{\mathbf{P}^\circ}} \otimes J - J \otimes \widetilde{\overline{\mathbf{P}^\circ}}\big] \\
&\overset{(6.4\mathrm{a}),\,(6.4\mathrm{b})}{=} \mathbf{P}^\circ + \tfrac{1}{2}\alpha \odot J + \tfrac{1}{2}(D_J\alpha) \otimes C - \frac{1}{n+1}(D_J\widetilde{\mathbf{P}^\circ}) \otimes C - \tfrac{1}{2}\,(D_J\alpha) \otimes C \\
&\quad - \frac{1}{n+1}\widetilde{\mathbf{P}^\circ} \odot J - \tfrac{1}{2}\alpha \odot J + \frac{1}{n+1}\widetilde{\mathbf{P}^\circ} \otimes J + \tfrac{1}{2}\alpha \otimes J \\
&\quad + \frac{1}{n+1} J \otimes \widetilde{\mathbf{P}^\circ} + \tfrac{1}{2} J \otimes \alpha \overset{(5.6)}{=} \mathbf{P} + \tfrac{1}{2}(\alpha \otimes J + J \otimes \alpha) \\
&= \mathbf{P} + \tfrac{1}{2}\big[(\widetilde{\overline{\mathbf{P}}} - \widetilde{\mathbf{P}}) \otimes J + J \otimes (\widetilde{\overline{\mathbf{P}}} - \widetilde{\mathbf{P}})\big],
\end{aligned}$$

thus

$$\overline{\mathbf{P}} - \tfrac{1}{2}(\widetilde{\overline{\mathbf{P}}} \otimes J + J \otimes \widetilde{\overline{\mathbf{P}}}) = \mathbf{P} - \tfrac{1}{2}(\widetilde{\mathbf{P}} \otimes J + J \otimes \widetilde{\mathbf{P}}),$$

i.e., $\overline{\mathbf{D}} = \mathbf{D}$, as was to be shown. □

6.5. Now we are going roughly to clarify the meaning of the Douglas tensor. Suppose (M, ∇) is an affinely connected manifold. Then ∇ determines a horizontal endomorphism h (smooth on the whole tangent manifold TM!) which is related to h by

$$[X^{\mathrm{h}}, Y^{\mathrm{v}}] = (\nabla_X Y)^{\mathrm{v}}, \qquad (X, Y \in 10(M)). \tag{6.5}$$

Of course, h is a Berwald endomorphism, and the Berwald connection (D°, h) is just (∇^{h}, h), where ∇^{h} is the horizontal lift ([7]) of ∇. Then for any vector fields $X, Y, Z \in 10(M)$,

$$\mathbf{P}^\circ(X^{\mathrm{c}}, Y^{\mathrm{c}})Z^{\mathrm{c}} \overset{4.3}{=} [[X^{\mathrm{h}}, Y^{\mathrm{v}}], Z^{\mathrm{v}}] = [(\nabla_X Y)^{\mathrm{v}}, Z^{\mathrm{v}}] \overset{(1.4)}{=} 0,$$

thus the mixed curvature of D° vanishes, hence *the Douglas tensor of h also vanishes*. As for the converse and the whole story, we have the following important result.

6.6. Theorem (J. Douglas, Z. Shen**).** *Suppose M is an orientable manifold and let h be a Berwald endomorphism on M. The associated spray of h is projectively equivalent with the spray determined by a linear connection on M if and only if the Douglas tensor of h vanishes.*

Proof. The "if" part is an immediate consequence of the preceding remark and Theorem 6.4. Now we prove the "only if" part. Following Shen's idea, we consider an arbitrary volume form μ on M and extend it to a volume form $\widetilde{\mu}$ on TM as follows:

$$\widetilde{\mu} := \mu_1 \wedge \mu_2; \qquad \mu_1 := \pi^*\mu, \qquad \mu_2 : v \in TM \mapsto (\mu_2)_v \in \wedge^n T_v^{\mathrm{v}} TM,$$
$$\forall z_1, \ldots, z_n \in T_v^{\mathrm{v}} TM : \ (\mu_2)_v(z_1, \ldots, z_n) := \mu_{\pi(v)}\big(j_v(z_1), \ldots, j_v(z_n)\big),$$

where $j_v : T_v^{\mathrm{v}} TM \to T_{\pi(v)} M$ is the well-known canonical isomorphism. Having this volume form, one can speak of the *divergence* of any vector field $X \in 10(TM)$ in the usual manner: $\operatorname{div}_{\widetilde{\mu}} X$ is defined by the relation

$$\mathcal{L}_X \widetilde{\mu} = (\operatorname{div}_{\widetilde{\mu}} X)\widetilde{\mu}.$$

It is easy to check that, in particular,

$$\operatorname{div}_{\widetilde{\mu}} C = n.$$

Now we turn to the associated spray S of h. Observe that $\operatorname{div}_{\widetilde{\mu}} S$ is 1-*homogeneous*, i.e.,

$$\mathcal{L}_C \operatorname{div}_{\widetilde{\mu}} S = \operatorname{div}_{\widetilde{\mu}} S.$$

Indeed,

$$\begin{aligned}\mathcal{L}_C \mathcal{L}_S \widetilde{\mu} &= \mathcal{L}_C\big[(\operatorname{div}_{\widetilde{\mu}} S)\widetilde{\mu}\big] = \big[\mathcal{L}_C(\operatorname{div}_{\widetilde{\mu}} S)\big]\widetilde{\mu} + (\operatorname{div}_{\widetilde{\mu}} S)\mathcal{L}_C \widetilde{\mu} \\ &= \big[\mathcal{L}_C \left(\operatorname{div}_{\widetilde{\mu}} S\right)\big]\, \widetilde{\mu} + \left(n \operatorname{div}_{\widetilde{\mu}} S\right) \widetilde{\mu}.\end{aligned}$$

The left-hand side can also be written as follows:

$$\begin{aligned}\mathcal{L}_C \mathcal{L}_S \widetilde{\mu} &= \mathcal{L}_{[C,S]} \widetilde{\mu} + \mathcal{L}_S \mathcal{L}_C \widetilde{\mu} = \mathcal{L}_S \widetilde{\mu} + n \mathcal{L}_S \widetilde{\mu} \\ &= (\operatorname{div}_{\widetilde{\mu}} S)\widetilde{\mu} + (n \operatorname{div}_{\widetilde{\mu}} S)\widetilde{\mu}.\end{aligned}$$

Comparing the right-hand sides, we get the results.

Now let

$$\lambda := -\frac{1}{n+1} \operatorname{div}_{\widetilde{\mu}} S, \qquad \bar{S} := S + \lambda C = S - \frac{1}{n+1}\left(\operatorname{div}_{\widetilde{\mu}} S\right) C.$$

Then $\bar{S}$ is divergence-free:

$$\begin{aligned}\operatorname{div}_{\widetilde{\mu}} \bar{S} &= \operatorname{div}_{\widetilde{\mu}} S - \frac{1}{n+1}\left(\operatorname{div}_{\widetilde{\mu}} S \operatorname{div}_{\widetilde{\mu}} C + \mathcal{L}_C \operatorname{div}_{\widetilde{\mu}} S\right) \\ &= \operatorname{div}_{\widetilde{\mu}} S - \frac{1}{n+1}\left(n \operatorname{div}_{\widetilde{\mu}} S + \operatorname{div}_{\widetilde{\mu}} S\right) \\ &= 0.\end{aligned}$$

After some calculation, for the tensor α constructed in the proof of 6.4 now we obtain the relation

$$\alpha = -\frac{2}{n+1}\widetilde{\mathbf{P}}^{\circ}.$$

Substituting this into 6.4(a), we get:

$$\overline{\mathbf{P}^\circ} = \mathbf{P}^\circ - \frac{1}{n+1}\big(D_J \widetilde{\mathbf{P}^\circ} \otimes C + \widetilde{\mathbf{P}^\circ} \odot J\big) \overset{(6.2b)}{=} \mathbf{D}. \tag{6.6a}$$

From this follows (c.f. 6.5) that the vanishing of **D** implies the desired projective equivalence. □

6.7. Remark. The local version of 6.6 was proved by J. Douglas [8], the global result is due to Z. Shen [15]. Our proof is a more conceptual and coordinate-free realization of Shen's ingenious thought. Pay attention to the remarkable relation (6.6a): the mixed curvature of the Berwald endomorphism induced by the projectively deformed (divergence-free) spray is just the Douglas tensor!

7. Remarks on the Weyl tensors

7.1. First we recall del Castillo's result mentioned in the Introduction. If h is a horizontal endomorphism on M, R is the curvature tensor of h and

$$\sigma := \frac{1}{n+1}\left(\widetilde{R} + \frac{1}{n-1} d_J i_S \widetilde{R}\right) \tag{7.1a}$$

(S is a semispray on M), then the tensor

$$W := R - \sigma \wedge J + d_J \sigma \otimes C \tag{7.1b}$$

is invariant under the projective changes of the associated spray of h. W is said to be the *Weyl tensor of the Berwald endomorphism* h.

7.2. Proposition and definition. *Assume h is a Berwald endomorphism on the manifold M, and $\mathbf{R}^\circ$ is the horizontal curvature of the Berwald connection (D°, h). Let for any vector fields $X, Y \in 10(M)$ and for a semispray S*

$$A(X^c) := \frac{1}{n^2-1}\big(n i_S \widetilde{\mathbf{R}}(X^c) + \widetilde{\mathbf{R}}(X^c, S)\big),$$

$$B(X^c, Y^c) := \frac{1}{n+1}\big(\widetilde{\mathbf{R}}(X^c, Y^c) - \widetilde{\mathbf{R}}(Y^c, X^c)\big).$$

Then the tensor W^* defined by

$$W^*(X^c, Y^c)Z^c := \mathbf{R}^\circ(X^c, Y^c)Z^c - \big(D_J^\circ A \otimes J - D_J^\circ B \otimes C\big)(Z^c, X^c, Y^c)$$
$$+ \big(D_J^\circ A \otimes J\big)(Z^c, Y^c, X^c) + (B \otimes J)(X^c, Y^c, Z^c)$$

$(X, Y, Z \in 10(M))$ *is invariant under the projective changes of the associated spray of h. We call W^* the Weyl tensor of the Berwald connection (D°, h). W^* and the Weyl tensor W of h are related by the formula*

$$D_J^\circ W(Z^c, X^c, Y^c) = W^*(X^c, Y^c, Z^c), \qquad X, Y, Z \in 10(M). \tag{7.2}$$

The proof is a very long and tedious computation, so we have to omit it.

Now we are in a position to derive an essential result of [2], without any calculation.

7.3. Proposition. *Suppose h is a Berwald endomorphism with vanishing Douglas tensor. Then the Weyl tensor of the Berwald connection (D°, h) is a vertical lift, i.e., roughly speaking, its components depend only on the position.*

Proof. In view of 6.6 the associated spray of h is projectively equivalent with the associated spray of a linear connection ∇. Since the Weyl tensor W^* is a projective invariant by (7.2), it follows that W^* is the Weyl tensor of ∇. More preciesely, W^* is the Weyl tensor of the Berwald connection $(\nabla^{\bar{h}}, \bar{h})$, where $\bar{h}$ is the Berwald endomorphism determined by ∇ (c.f., 6.5). □

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